HIGH RESOLUTION 3D SPECTRAL METHOD ESTIMATION

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ABSTRACT

In this paper, we investigate the problem of three-dimensional (3D) frequency estimation. We propose a new approach based on the shift invariance property in the data structure. The data are modeled as a sum of 3D complex exponential (SCE) embedded in white noise. In 1 and 2D cases, the approaches based on invariance property have shown efficiency, the purpose of this paper is to take advantage of this feature in the 3D framework. Indeed the special structure of the model permits a decomposition of the autocorrelation matrix into a linear subspace called signal subspace and its orthogonal complement, the noise subspace. The method operates in two steps, firstly one estimates the autocorrelation matrix which is defined and performed from a subset of data. Secondly the estimation of the frequencies is involved by the existence of an invertible matrix mapping between the signal subspace basis and an exact 3D Vandermonde matrix.

Index Terms

3D SCE Model, 3D Frequencies, Autocorrelation Matrix, EigenValue Decomposition, Signal Subspace, Invariance Property, 3D Vandermonde Matrix.

1 INTRODUCTION

In the framework of image data analysis, the full 3D nature of the data must be taken into account so that the model fits well the data according to error modeling criterion. Processing a 3D block of images as a sequence of independent 2D image usually leads to a heavily lost in the effectiveness of data modeling and processing.

For two decades, many algorithms have been developed for frequency estimation. Several recent approaches are based on the invariance property such as 1 and 2D ESPRIT [1] [2], MP (Matrix Pencil) [3] and recently [4].

In 1 and 2D cases, the spectral and frequency estimation can be classified into two classes: namely scanning and analytical methods. In the first class, the methods are based on the spectrum or pseudo spectrum such as Fast Fourier Transform, Prony method, autoregressive and MUSIC approaches [5]. All these methods consist in scanning the frequency space with a discrete finite lag frequency and looking for the value which maximizes an appropriate pseudo spectrum. In this case, the resolution is poor and a tradeoff between the resolution and the variance of the estimated frequencies must be made. The common feature of the second family is to exploit an analytical formulation of the frequency content. It is based on the decomposition of the space spanned by the eigenvectors of the autocorrelation (or data) matrix into two orthogonal subspaces namely noise and signal subspace. The first method using this concept is the Pisarenko method. After on, this notion is well exploited in root-MUSIC, ESPRIT, MP approaches. These methods are commonly called high resolution (HR) techniques.

For RADAR, seismic data, and in the framework of multi-components array processing, the 3D data are usually a sum of complex exponential embedded in noise. For this reason, we propose in this paper to use the model based on the sum of 3D complex exponential (SCE). The task consists on estimating the 3D frequencies and to achieve their correct pairing. In addition, if one needs to estimate amplitude and phase information, the use of Least Mean Squares (LMS) algorithm is of most importance.

When dealing with 3D spectral estimation, the scanning methods fail because it's not easy to check visually whenever the spectral estimation has reached a maximum. Hence, only analytical methods are appropriate to solve this 3D problem. In the framework of 3D and mD spectral analysis, ESPRIT, unitary ESPRIT, and Total Least Squares (TLS) phased averaged ESPRIT have been proposed for joint angle-carrier estimation in array processing [6]-[8]. However, these methods have some drawbacks, Unitary ESPRIT do not address the case of damped modes. Moreover, the TLS phased averaged ESPRIT uses implicit orthogonal iteration technique for computing low-rank approximation to overcame the Eigenvalue decompositions, so, this method needs the number of modes to be a priori known.

The 3D MP method developed in this paper can deals with damped or undamped modes. Indeed, the use of Eigenvalue or Singular value decomposition helps to estimate the number of modes prior to any frequencies estimation.

The rest of the paper is organized as follows. In Section 2, we present the 3D SCE model and develop the 3D MP method for 3D frequencies estimation. Simulation examples are presented in Section 3. Finally Section 4 summarizes our conclusions.

2 3D DATA SCE MODEL AND 3D MP METHOD

2.1 Preliminaries

Let $y(m,n,t)$ be a sampled function of 3D variable, where m, n, and t range over a finite parallelepiped (or cubic) grid. A 3D dataset can be represented by a MxNxT cube.

In the 2D case, we refer each point as a pixel (m,n). Here, we refer each point in a 3D image as a pixel (m,n,t). We suppose that each pixel can be modeled as follows:

$$y(m,n,t) = \sum_{i=1}^{K} a_i \exp(j2\pi(f_{1i}(m-1)+f_{2i}(n-1)+f_{3i}(t-1))+j\varphi_i) + b(m,n,t) \quad (1)$$

where (f_{1i}, f_{2i}, f_{3i}) are the 3D normalized frequencies. The additive noise b is assumed to be zero-mean circular complex white noise with variance σ_b^2. The real and imaginary parts of the noise are supposed to be statistically independent and uncorrelated with the signal. The parameters a_i and φ_i are respectively the amplitude and the phase of the i^{th} wave.

K is the number of waves i.e. model order. In this paper, it is assumed to be known otherwise its estimation can be performed via AIC or MDL information criteria [9] or SVD algebraic methods [10]. It should be noted that the model order is a crucial first key to get an efficient frequency estimation.

2.2 Exact Autocorrelation Matrix Computation

Let us consider a $PxQxL$ sub cube from the $MxNxT$ dataset. Our aim is to compute the autocorrelation matrix of this $PxQxL$ data. For such a purpose, we concatenate the data in a $PQLx1$ column vector denoted Y_{PQL}. It exist six ways to scan such 3D volume data. One way is to scan the data column by column in each image i. e. layer as follows

$$Y_{PQL}^T=[y_0^T,y_1^T,\cdots,y_{L-1}^T], \tag{4}$$

where

$$\begin{gathered} y_i^T=[y_{0,i}^T,y_{1,i}^T,\cdots,y_{Q-1,i}^T] \\ y_{j,i}^T=[y(m,n+j,t+i),\cdots,y(m+P-1,n+j,t+i)] \\ i\in[0,L-1],\ j\in[0,\mathrm{Q}-1] \end{gathered} \tag{5}$$

where $(.)^T$ is the transposition of matrix and vectors.

The data autocorrelation matrix is given by

$$Ry=E\{\ Y_{PQL}\ Y_{PQL}^{\oplus}\ \} \tag{6}$$

where $E\{.\}$ stands for the expectation operator and the symbol $\oplus$ is the transposition and conjugate for complex matrices.

Ry is a LxL block Toeplitz matrix defined as follows:

$$Ry=\begin{bmatrix} R_0 & R_{-1} & \cdots & R_{-L+1} \\ R_1 & R_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & R_{-1} \\ R_{L-1} & \cdots & R_1 & R_0 \end{bmatrix} \tag{7}$$

in which each block is QxQ Toeplitz block Toeplitz matrix.

$$R_i=\begin{bmatrix} r_{0,i} & r_{-1,i} & \cdots & r_{-Q+1,i} \\ r_{1,i} & r_{0,i} & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_{-1,i} \\ r_{Q-1,i} & \cdots & r_{1,i} & r_{0,i} \end{bmatrix} \tag{8}$$

$$r_{j,i}=\begin{bmatrix} r_y(0,j,i) & r_y(-1,j,i) & \cdots & r_y(-P+1,j,i) \\ r_y(1,j,i) & r_y(0,j,i) & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_y(-1,j,i) \\ r_y(P-1,j,i) & \cdots & r_y(1,j,i) & r_y(0,j,i) \end{bmatrix} \tag{9}$$

$$i\in[-(L-1),L-1],\ j\in[-(Q-1),\mathrm{Q}-1].$$

For the 3D process (1), the exact autocorrelation function is

$$ry(k,l,h)=\sum_{i=1}^{K}a_i^2\exp(j2\pi(f_{1i}k+f_{2i}l+f_{3i}h)+\sigma_b^2\delta(k,l,h) \tag{10}$$

Then the exact autocorrelation matrix is:

$$R_y=S_{[PQL,K]}^1\Psi S_{[PQL,K]}^{1\oplus}+\sigma_b^2I \tag{11}$$

where

$$S_{[PQL,K]}^1=[s_{31,L}\otimes s_{21,Q}\otimes s_{11,P},\cdots,s_{3K,L}\otimes s_{2K,Q}\otimes s_{1K,P}] \tag{12}$$

$$s_{mn,H}=[1\ \exp(j2\pi f_{mn}),\cdots,\exp(j2\pi f_{mn}(H-1))]^T \tag{13}$$

$m=1,2,3$; $n=1,2,\cdots K$; $H\in\{P,Q,L\}$. $\otimes$ is the Kronecker product.

Ψ is the diagonal matrix containing the power of each wave

$$\Psi=diag(a_i^2)_{i=1}^K \tag{14}$$

We refer to the $S_{[PQL,K]}^1$ matrix as the 3D Vandermonde matrix. The signal subspace is then spanned by the column of this 3D Vandermonde matrix. The eigenvalue decomposition of Ry is:

$$Ry=UDU^{\oplus} \tag{15}$$

where

$$\begin{gathered} U=[u_1,\cdots,u_K,u_{K+1},\cdots,u_{PQL}] \\ D=diag(\lambda_1,\lambda_2,\cdots,\lambda_K,\lambda_{K+1},\cdots,\lambda_{PQL}) \end{gathered} \tag{16}$$

λ_i are the eigenvalues ordered in a decreasing order $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_K>\sigma_b^2$, and $\lambda_{K+1}=\cdots=\lambda_{PQL}=\sigma_b^2$; where σ_b^2 is the white noise variance. Based on the multiplicity of the noise variance, one can split the eigenvector basis into two orthogonal subspaces i.e. signal and noise subspaces spanned respectively by

$$Us1=[u_1,u_2,\cdots,u_K],\text{ and } Un=[u_{K+1},\cdots,u_{PQL}]. \tag{17}$$

2.3 Estimated Autocorrelation Matrix

Given a 3D subset $MxNxT$ of data, the exact autocorrelation matrix can be approximated, under the stationary and ergodicity assumptions, by:

$$\hat{R}y=\frac{1}{(M-P+1)(N-Q+1)(T-L+1)}\cdot\sum_{m=1}^{M-P+1}\sum_{n=1}^{N-Q+1}\sum_{t=1}^{T-L+1}(Y_{PQL}\ Y_{PQL}^{\oplus}) \tag{18}$$

The eigenvalue decomposition of this autocorrelation matrix gives approximate basis of signal and noise subspaces.

Note that, in order to get an estimated autocorrelation matrix allowing the best modes estimation, the size $PxQxL$ of the sub block must satisfy the following inequalities

$$\begin{cases} P\ge K \\ Q\ge K \\ L\ge K \end{cases} \tag{19}$$

The 3D MP based 3D frequencies estimation will be built on some partitions of the signal subspace approximation.

The signal subspace is spanned by $Us1$ and $S_{[PQL,K]}^1$, i.e. $\Re\{Us1\}=\Re\{S_{[PQL,K]}^1\}$, $\Re\{.\}$ represents the range of the subspace. Hence, there exist a mapping KxK nonsingular matrix Θ between these two basis.

$$Us1=S_{[PQL,K]}^1\,\Theta \tag{20}$$

Indeed, the orthogonality property states that the 3D Vandermonde matrix and the noise subspace satisfy the following relationship i.e. $\Re\{Un\}\perp\Re\{S_{[PQL,K]}^1\}$.

In the following, we provide with some background for the derivation of 3D MP method. Let us rewrite the 3D Vandermonde matrix in a well organized form

$$S^1_{[PQL,K]}=\begin{bmatrix} S^1_{[PQ,K]} \\ \hdashline S^1_{[PQ,K]}\Phi_3 \\ \hdashline \vdots \\ \hdashline S^1_{[PQ,K]}\Phi_3^{L-1} \end{bmatrix}, \text{ and } S^1_{[PQ,K]}=\begin{bmatrix} S^1_{[P,K]} \\ \hdashline S^1_{[P,K]}\Phi_2 \\ \hdashline \vdots \\ \hdashline S^1_{[P,K]}\Phi_2^{Q-1} \end{bmatrix} \qquad (21)$$

where $S^1_{[PQ,K]}$ is the 2D Vandermonde matrix associated with the frequencies in the first and second dimensions. The 1D Vandermonde matrix $S^m_{[H,K]}$ are defined by

$$S^m_{[H,K]}=[s_{m1,H}, s_{m2,H}, \cdots, s_{mK,H}] \qquad (22)$$

$s_{mn,H}$ is given by (13) for $m=1,2,3$; $n=1,2,\cdots K$; $H\in\{P,Q,L\}$.

Moreover, the matrices Φ_m, for $m=1,2,3$ are given by

$$\Phi_m=diag(\exp(j2\pi f_{mi}))_{i=1}^{K} \qquad (23)$$

2.4 Shift Invariance Property and 3D Frequency Estimation

Based on the structure of the 3D Vandermonde matrix and in order to retrieve the shift invariance property operating on the analytical formulation of the frequency estimation we split the signal subspace and the 3D Vandermonde matrix as follows

$$S^1_{[PQL,K]}=\begin{bmatrix} S^1_{[PQ,K]} \\ \hdashline EM1 \end{bmatrix}\updownarrow PQ=\begin{bmatrix} EM1\Phi_3^{-1} \\ \hdashline S^1_{[PQ,K]}\Phi_3^{L-1} \end{bmatrix}\updownarrow PQ \qquad (24)$$

The symbol $\updownarrow$ indicates the number of rows to extract from one matrix. Similarly, the signal subspace can be partitioned as follows:

$$Us1=\begin{bmatrix} xxx \\ \hdashline \overline{Us}1 \end{bmatrix}\updownarrow PQ \quad =\begin{bmatrix} \underline{Us}1 \\ \hdashline xxx \end{bmatrix}\updownarrow PQ \qquad (25)$$

The mapping Θ matrix can be used as follows :

$$\begin{cases} EM1\Theta=\overline{U}s1 \\ EM1\Phi_3^{-1}\Theta=\underline{U}s1 \end{cases} \Rightarrow \begin{cases} EM1=\overline{U}s1\Theta^{-1} \\ \overline{U}s1\Theta^{-1}\Phi_3^{-1}\Theta=\underline{U}s1 \end{cases} \qquad (26)$$

$$\Rightarrow \Theta^{-1}\Phi_3^{-1}\Theta=\overline{U}s1^{\nabla}\underline{U}s1$$

Hence, the f_{3i} components are the angles of the eigenvalues of the matrix $\overline{U}s1^{\nabla}\underline{U}s1$, i.e. $f_{3i}=\frac{1}{2\pi}\angle\lambda_i(\overline{U}s1^{\nabla}\underline{U}s1)$ for $i=1,\cdots,K$.

Here $\angle$ stands for angle.

To extract the frequencies information in the two others dimension, we consider two other scanning forms.

For an impelling the PxQxL data, raw by raw from each layer, the corresponding 3D Vandermonde matrix which spans the signal subspace is now given by

$$S^2_{[QLP,K]}=E^2_1 S^1_{[PQL,K]} \qquad (27)$$

where E^2_1 is the permutation matrix operator given by

$$E^2_1=\sum_{i=1}^{P}\sum_{j=1}^{Q}\sum_{k=1}^{L}E^{P,Q}_{i,j}\otimes E^{Q,L}_{j,k}\otimes E^{L,P}_{k,i} \qquad (28)$$

and $E^{P,Q}_{i,j}$ is the elementary matrix of size PxQ having zeros entries excepted for the (i,j) location which is one.

The corresponding signal subspace of the autocorrelation matrix is

$$Us2=E^2_1 Us1 \qquad (29)$$

The $S^2_{[QLP,K]}$ and $Us2$ can be spilled into

$$S^2_{[QLP,K]}=\begin{bmatrix} S^2_{[QL,K]} \\ \hdashline EM2 \end{bmatrix}\updownarrow QL=\begin{bmatrix} EM2\Phi_1^{-1} \\ \hdashline S^2_{[QL,K]}\Phi_1^{P-1} \end{bmatrix}\updownarrow QL \qquad (30)$$

and

$$Us2=\begin{bmatrix} xxx \\ \hdashline \overline{Us}2 \end{bmatrix}\updownarrow QL \quad =\begin{bmatrix} \underline{Us}2 \\ \hdashline xxx \end{bmatrix}\updownarrow QL \qquad (31)$$

The f_{1i} components are the angles of the eigenvalues of the matrix $\overline{U}s2^{\nabla}\underline{U}s2$, i.e. $f_{1i}=\frac{1}{2\pi}\angle\lambda_i(\overline{U}s2^{\nabla}\underline{U}s2)$ for $i=1,\cdots,K$.

To get the frequencies f_{2i}, one has to scan data in order to get the 3D Vandermonde matrix as follows

$$S^3_{[LPQ,K]}=E^3_2 S^2_{[QLP,K]} \text{ and } Us3=E^3_2 Us2 \qquad (32)$$

where E^3_2 is given by

$$E^3_2=\sum_{i=1}^{Q}\sum_{j=1}^{L}\sum_{k=1}^{P}E^{Q,L}_{i,j}\otimes E^{L,P}_{j,k}\otimes E^{P,Q}_{k,i} \qquad (33)$$

The obtained $Us3$ and $S^3_{[LPQ,K]}$ can be partitioned as follows

$$S^3_{[LPQ,K]}=\begin{bmatrix} S^3_{[LP,K]} \\ \hdashline EM3 \end{bmatrix}\updownarrow LP=\begin{bmatrix} EM3\Phi_2^{-1} \\ \hdashline S^3_{[LP,K]}\Phi_2^{Q-1} \end{bmatrix}\updownarrow LP \qquad (34)$$

$$Us3=\begin{bmatrix} xxx \\ \hdashline \overline{Us}3 \end{bmatrix}\updownarrow LP \quad =\begin{bmatrix} \underline{Us}3 \\ \hdashline xxx \end{bmatrix}\updownarrow LP \qquad (35)$$

The f_{2i} components are the angles of the eigenvalues of the matrix $\overline{U}s3^{\nabla}\underline{U}s3$, i.e. $f_{2i}=\frac{1}{2\pi}\angle\lambda_i(\overline{U}s3^{\nabla}\underline{U}s3)$ for $i=1,\cdots,K$.

Until now, the 3D frequencies components are estimated, but not paired in the appropriate form. To this end, we use the orthogonality property between noise subspace and 3D Vandermonde matrix.

The best triplets minimize the projection into the noise subspace, or equivalently maximize the projection into the signal subspace according to the following criterion:

$$C\,(i,j,k)=\sum_{l=1}^{K}\left\| Us1(l)^{\oplus}(s_{3k,L}\otimes s_{2j,Q}\otimes s_{1i,P})\right\|^2 . \qquad (36)$$

The method proceeds by successive elimination until all triplets are recombined.

3 NUMERICAL EXAMPLES

In this section, we present some numerical examples. Our approach is tested with two signal to noise ratio (SNR) scenari i.e. $SNR=20\,dB$ and $SNR=0\,dB$. The data are generated according to the model in equation (1). In both cases we consider three waves i.e. $K=3$ with unit power $a_i=1$, the corresponding 3D frequencies are given in table 1. The data and the autocorrelation matrix sizes are respectively $(M,N,T):(7,7,7)$, and $(P,Q,L):(5,5,5)$. The estimated frequencies based on the 3D MP developed in this paper are given in table 2.

	f_{1i}	f_{2i}	f_{3i}
1st wave	f_{11}=0.2500	f_{21}=0.2000	f_{31}=–0.2500
2nd wave	f_{12}=0.3500	f_{22}=0.1000	f_{32}=0.4500
3rd wave	f_{13}=–0.1500	f_{23}=–0.3500	f_{33}=0.1500

Table 1: 3D frequencies used for simulation examples.

	f_{1i}		f_{2i}		f_{3i}	
	Estimated at 20dB	Estimated at 0dB	Estimated at 20dB	Estimated at 0dB	Estimated at 20dB	Estimated at 0dB
1st wave	f_{11}=0.2499	f_{11}=0.2536	f_{21}=0.2001	f_{21}=0.1995	f_{31}=–0.2507	f_{31}=–0.2472
2nd wave	f_{12}=0.3505	f_{12}=0.3559	f_{22}=0.1001	f_{22}=0.0992	f_{32}=0.4510	f_{32}=0.4522
3rd wave	f_{13}=–0.1509	f_{13}=–0.1512	f_{23}=–0.3498	f_{23}=–0.3487	f_{33}=0.1512	f_{33}=0.1483

Table 2: 3D frequencies estimated by 3D MP method for $SNR=20\ dB$ and $SNR=0\ dB$.

In order to propose a quantitative measure of the 3D estimated frequency, we evaluate the variance of the estimation error versus the SNR value in the range -10 to 30dB. For each SNR, 100 trials are used. The result is plotted in Fig. 1.

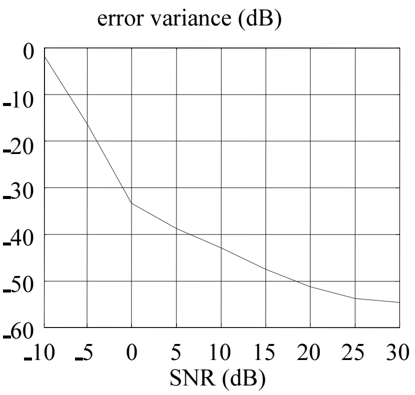

Fig. 1: estimation-error variance versus the SNR.

4 CONCLUSION

In this paper, we have addressed the 3D frequency problem. After formulating the background problem, we have developed a 3D MP method, which consists in building some partitions of principal components vectors known as the signal subspace. This signal subspace extracted from the eigenvectors of the autocorrelation matrix is exactly spanned by the columns vector of the 3D Vandermonde matrix corresponding to the scanning manner. Following this theoretical link, a mapping invertible matrix between these two basis is then found. Exploiting this relation, an algebraic development allows the estimation of the frequency in one dimension. In order to estimate the frequencies in the two other dimensions one has to permute the signal subspace using unitary matrices (28) and (33). To retrieval the 3D frequencies, we use a criterion based on the projection into the signal subspace to restore the correct triplets.

Due to the limited space, we give only two numerical examples. Indeed, the detailed results will be proposed for publication in an extended paper. Moreover, our further studies will be also focused on building some filters from the estimated frequencies for denosing 3D block of images. Such scenario occurs for RADAR and Seismic multicomponents data.

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