α-STABLE POSITIVE DISTRIBUTIONS : A NEW APPROACH BASED ON SECOND KIND STATISTICS

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ABSTRACT

Probability Density Functions defined on $\mathbb{R}^+$ can be successfully modeled with the help of "second kind statistics". This new approach, proposed in [4], is based on Mellin transform instead of Fourier transform so that classical probability density functions defined on $\mathbb{R}^+$ can be identified with the help of "second kind moments" and "second kind cumulants", the analytic expressions of which are oversimple.
In this article, we propose to analyse α-stable positive distributions. Indeed we know that classical moments of such distributions are generally not defined. As it is possible to derive their "second kind moments" and their "second kind cumulants", the estimation of the parameters of such laws is avalaible with the help of very simple expressions.

1 INTRODUCTION

Second kind statistics have been proposed recently to deal with Probability Density Functions (pdf) defined on $\mathbb{R}^+$ [4]. We propose a brief recall of the definitions of such an approach, and a new theorem for "second kind moments" and "second kind cumulants".

1.1 Definitions of second kind characteristic functions

A pdf $p_x(u)$ defined on $\mathbb{R}$ is traditionally linked to its *characteristic functions* $\Phi(\nu)$ via the Fourier Transform :

$$\Phi_x(\nu) \;=\; \mathcal{F}\left[p_x(u)\right](\nu) \;=\; \int_{-\infty}^{+\infty} \mathrm{e}^{j\nu u} p_x(u)\mathrm{d}u.$$

If $p_x(u)$ is a pdf defined on $\mathbb{R}^+$, it is possible to mimic the traditional approach by replacing the Fourier transform by the Mellin transform. By this way we obtain :

- the *second kind first characteristic function* (SKFCF) $\phi_x(s)$ defined as the Mellin transform of $p_x(u)$:

$$\phi_x(s) \triangleq \int_0^{+\infty} u^{s-1}\, p_x(u)\, \mathrm{d}u, \qquad (1)$$

- the *second kind second characteristic function* (SKSCF) $\psi_x(s)$ as the logarithm of the *second kind first characteristic function* :

$$\psi_x(s) \triangleq \log\left(\phi_x(s)\right), \qquad (2)$$

- the n-th *second kind moment*, defined by the relation

$$\tilde{\mathrm{m}}_{x(n)} \triangleq \left.\frac{\mathrm{d}^n \phi_x(s)}{\mathrm{d}s^n}\right|_{s=1},$$

- the n-th *second kind cumulant* (SKC), defined by the relation

$$\tilde{\kappa}_{x(n)} \triangleq \left.\frac{\mathrm{d}^n \psi_x(s)}{\mathrm{d}s^n}\right|_{s=1}.$$

Second kind cumulants are related to second kind moments with the same expressions as in classical statistics.

Second kind moments and second kind cumulants can be seen as *Log-moments* and *Log-cumulants*. Indeed, knowing the following property of the Mellin transform :

$$\mathcal{M}\left[f(u)\left(\log u\right)^n\right](s) \;=\; \frac{\mathrm{d}^n\left(\mathcal{M}\left[f(u)\right](s)\right)}{\mathrm{d}s^n}$$

the following relations can be derived :

$$\tilde{\mathrm{m}}_n \;=\; \left.\frac{\mathrm{d}^n \phi_x(s)}{\mathrm{d}s^n}\right|_{s=1} \qquad (3)$$

$$\;=\; \int_0^{+\infty} (\log u)^n\; p_x(u)\, \mathrm{d}u \qquad (4)$$

The first expression (equation 3) allows numerical approach and the second one (equation 4) can be approximate by a discrete sum when a set of data is avalaible, yielding estimations of second kind moments and second kind cumulants.

1.2 A new theorem for second kind cumulants

At this step of the presentation of second kind moments and second kind cumulants, an important question has to be asked about their existences. Indeed, there are some pdf for which classical moments are not defined, as α-stable distributions. Actually, what are the conditions for the existence of "log-moments" and "log-cumulants"?

A very interesting property of "log-moments" and "log-cumulants" is the fact that, under very weak conditions, they are already defined. A new theorem demonstrates this affirmation.

Theorem : if a pdf has a second kind characteristic function defined on a neighbourhood Ω of $s=1$, all its second kind moments and its second kind cumulants are defined.

Demonstration : Let a pdf so that its second kind characteristic function is defined on a neighbourhood Ω of $s=1$. We have to determine the existence of

$$\int_{a\to 0}^{b\to +\infty} (\log u)^n \; p(u) \mathrm{d}u.$$

both for $a \to 0$ and $b \to \infty$.

- $b \to \infty$: let $\alpha \in \Omega$ so that $\alpha > 1$. By assumption, as $\alpha \in \Omega$, $\phi(\alpha)$ is defined :

$$\phi(\alpha) \;=\; \int_0^{+\infty} u^{\alpha-1} \, p(u) \mathrm{d}u < \infty$$

  i.e. it possible to obtain (traditional) moments for all orders in $[1, \alpha]$. Let $n \geq 1$. Two cases have to be investigated :

  - $\forall x > 1 \;\; (\log x)^n < x^{\alpha-1}$. In this case, as $p(u)$ is a pdf and verifies $p(u) \geq 0$, we can write :

$$\begin{aligned}\lim_{b\to\infty} \int_1^b (\log u)^n \; p(u)\mathrm{d}u &\leq \lim_{b\to\infty} \int_1^b u^{\alpha-1} \, p(u) \mathrm{d}u \\ &\leq \phi(\alpha)\end{aligned}$$

    yielding the convergence of the improper integral for $x \to \infty$.

  - $\exists c > 1 \;\; (\log c)^n \;=\; c^{\alpha-1}$ so that $\forall x > c \;\; (\log x)^n \leq x^{\alpha-1}$. As previously, we can deduce :

$$\lim_{b\to\infty} \int_c^b (\log u)^n \; p(u)\mathrm{d}u \;\; \leq \phi(\alpha)$$

    yielding the convergence of the improper integral for $x \to \infty$.

- $a \to 0$: by putting $x' = \frac{1}{x}$, the convergence near 0 can be transform in the previous case (convergence near ∞), yielding the existence of the limit near 0.

1.3 An example : the Nakagami distribution

The Nakagami distribution, used in various topics [1, 3], is a two parameters pdf defined on $\mathbb{R}^+$

$$\mathcal{RN}\left[\mu, L\right](u) \;=\; \frac{2}{\mu} \frac{\sqrt{L}}{\Gamma(L)} \left(\frac{\sqrt{L} u}{\mu} \right)^{2L-1} \mathrm{e}^{-\left(\frac{\sqrt{L} u}{\mu}\right)^2}. \quad (5)$$

With $L=1$, the previous expression yields the Rayleigh distribution.

There is a fundamental relation linking the Nakagami distribution $\mathcal{RN}$ and the famous Gamma law $\mathcal{G}$ [4] :

$$\mathcal{RN}\left[\mu, L\right](u) \;=\; 2\, u \, \mathcal{G}\left[\mu_{\mathcal{G}}, L\right](u^2). \quad (6)$$

Knowing the following properties of Mellin transform :

$$\begin{aligned}\mathcal{M}\left[u^a f(u)\right](s) &= \mathcal{M}\left[f(u)\right](s+a) \\ \mathcal{M}\left[f(u^a)\right](s) &= \frac{1}{a} \mathcal{M}\left[f(u)\right]\left(\frac{s}{a}\right)\end{aligned}$$

and the second kind characteristic function of the Gamma law $\phi_{\mathcal{G},x}$, we can directly write :

$$\phi_{\mathcal{RN},x}(s) \;=\; \phi_{\mathcal{G},x}\left(\frac{s+1}{2}\right)$$

Knowing $\phi_{\mathcal{G},x}$:

$$\phi_{\mathcal{G},x}(s) \;=\; \mu_{\mathcal{G}}^{s-1} \, \frac{\Gamma(s-1+L)}{L^{s-1}\,\Gamma(L)}$$

and putting $\mu_{\mathcal{G}} = \mu^2$, we obtain directly :

$$\phi_x(s) \;=\; \mu^{s-1} \, \frac{\Gamma\left(\frac{s-1}{2} + L\right)}{L^{\frac{s-1}{2}}\,\Gamma(L)}.$$

Nakagami law second kind cumulants can be directly derived from Gamma law second kind cumulants. Indeed, we can write :

$$\begin{aligned}\tilde{\kappa}_{\mathcal{RN},x(r)} &= \left.\frac{\mathrm{d}^r \psi_{\mathcal{RN}}(s)}{\mathrm{d}s^r}\right|_{s=1} \\ &= \left.\frac{\mathrm{d}^r \, \log\left(\phi_{\mathcal{RN}}(s)\right)}{\mathrm{d}s^r}\right|_{s=1} \\ &= \left.\frac{\mathrm{d}^r \, \log\left(\phi_{\mathcal{G}}\left(\frac{s+1}{2}\right)\right)}{\mathrm{d}s^r}\right|_{s=1} \\ &= \left(\frac{1}{2}\right)^r \left.\frac{\mathrm{d}^r \, \log\left(\phi_{\mathcal{G}}\left(s'\right)\right)}{\mathrm{d}s'^r}\right|_{s'=1} \\ &= \left(\frac{1}{2}\right)^r \tilde{\kappa}_{\mathcal{G},x(r)}\end{aligned}$$

yielding :

$$\begin{aligned}\tilde{\kappa}_{x(1)} &= \log(\mu) + \frac{1}{2}\Psi(L) - \frac{1}{2}\log(L) \\ \tilde{\kappa}_{x(2)} &= \frac{1}{4}\,\Psi(1, L)\end{aligned}$$

and for all $r > 1$:

$$\tilde{\kappa}_{x(r)} \;=\; \left(\frac{1}{2}\right)^r \Psi(r-1, L).$$

By this way, L can be deduced from second SKC :

$$\Psi(1, L) \;=\; \tilde{\kappa}_{x(2)}$$

and, knowing L, μ is derived from :

$$\log(\mu) \;=\; \tilde{\kappa}_{x(1)} \;-\; \frac{1}{2}\Psi(L) \;+\; \frac{1}{2}\log(L)\,.$$

As Digamma and Polygamma functions are easily inverted by numerical techniques (these functions are monotonous on $]0, +\infty[$), we obtain an exact method to estimate the parameters (for example, [1] needs some approximations).

2 APPLICATIONS TO α-STABLE DISTRIBUTIONS

2.1 α-Stable Positive Distributions

An α-stable positive distribution is a two parameters pdf defined only by its characteristic function $\Phi(\nu)$ [6] :

$$\Phi \;=\; \mathrm{e}^{-\gamma|\nu|^{\alpha}\left(1+j\,\mathrm{sgn}(\nu)\tan\left(\frac{\alpha\pi}{2}\right)\right)}$$

with

$$\mathrm{sgn}(\nu) \;=\; \begin{cases} 1, & \nu > 0 \\ 0, & \nu = 0, \\ -1, & \nu < 0 \end{cases} \qquad 0 < \alpha < 1 \qquad \gamma > 0.$$

α is the characteristic exponent and γ is the dispersion or scale parameter.

The heavy tail shows the algebraic asymptote that is a characteristic of all α-stable distributions.

The moments can be directly derived for $0 \leq \nu \leq \alpha$:

$$m_{\nu} \;=\; \frac{\gamma^{\frac{\nu}{\alpha}}\sin(\pi\nu)\;\Gamma(\nu+1)\;\left(1+\left(\tan\left(\frac{\pi\alpha}{2}\right)\right)^2\right)^{\frac{\nu}{2\alpha}}}{\alpha\sin\left(\frac{\pi\nu}{\alpha}\right)\;\Gamma\left(1+\frac{\nu}{\alpha}\right)} \qquad (7)$$

For α-stable positive distributions, negative moments can be considered so that classical methods involving ratio of negative moments allow the determination of the parameters, except that only implicit expressions can be derived for these parameters.

Yet second kind characteristic function can be deduced from equation 7, yielding

$$\phi(s) \;=\; \frac{\gamma^{\frac{s-1}{\alpha}}\sin(\pi(s-1))\;\Gamma(s)\;\left(1+\left(\tan\left(\frac{\pi\alpha}{2}\right)\right)^2\right)^{\frac{s-1}{2\alpha}}}{\alpha\sin\left(\frac{\pi(s-1)}{\alpha}\right)\;\Gamma\left(1+\frac{s-1}{\alpha}\right)} \qquad (8)$$

As this expression is defined by analytic continuation on a neighbourhood of $s = 1$, second kind moments (log-moments) and second kind cumulants (log-cumulants) exist. More precisely, first and second log-cumulants can be deduced by logarithmic derivation of equation 8, yielding after some tedious simplifications easily processed with any symbolic solver :

$$\begin{aligned} \tilde{\kappa}_1 &= \frac{(1-\alpha)\Psi(1)}{\alpha} - \frac{\log\left(\cos\left(\frac{\pi\alpha}{2}\right)\right)}{\alpha} + \frac{\log\gamma}{\alpha} \\ \tilde{\kappa}_2 &= \frac{\left(1-\alpha^2\right)}{\alpha^2}\,\Psi(1,1) \\ \tilde{\kappa}_3 &= \frac{\alpha^3-1}{\alpha^3}\Psi(2,1) \end{aligned} \qquad (9)$$

with $\Psi(x)$ the Digamma function and $\Psi(N,x)$ the Polygamma function.

These expressions can be inverted, so that the parameters can be seen as functions of second kind cumulants. α can be written as :

$$\alpha \;=\; \sqrt{\frac{\Psi(1,1)}{\Psi(1,1)-\tilde{\kappa}_2}}$$

and γ as

$$\gamma = \mathrm{e}^{\sqrt{\frac{\Psi(1,1)}{\Psi(1,1)-\tilde{\kappa}_2}}\,\tilde{\kappa}_1 - \left(1-\sqrt{\frac{\Psi(1,1)}{\Psi(1,1)-\tilde{\kappa}_2}}\right)\Psi(1)+\log\left(\cos\left(\frac{\pi}{2}\sqrt{\frac{\Psi(1,1)}{\Psi(1,1)-\tilde{\kappa}_2}}\right)\right)}.$$

As log-cumulants can be easily estimated with the help of expression 4, this new approach allows an over-simple method to estimate the parameters of an α-stable positive distribution.

2.2 A generalization of Rayleigh distribution

An other example of α-stable distributions is proposed by Kuruoğlu and Zerubia in order to generalize Rayleigh distribution [5]. The proposed pdf defined by two parameters (α and γ) can be expressed as :

$$p(u) \;=\; u\int_0^{\infty} v\,\mathrm{e}^{-\gamma v^{\alpha}}\,J_0(uv)\,\mathrm{d}v \qquad (10)$$

with J_0 first kind Bessel function. m_ν, the ν-th moment of such a pdf, is defined except if $\nu \geq min(\alpha, 2)$ so that this pdf is also an heavy tailed distribution.

Second kind characteristic function can be directly derived from equation 10 by calculating its Mellin transform :

$$\begin{aligned} \phi(s) &= \int_0^{\infty} u^{s-1}p(u)\mathrm{d}u \\ &= \int_0^{\infty} u^s\int_0^{\infty} v\,\mathrm{e}^{-\gamma v^{\alpha}}\,J_0(uv)\,\mathrm{d}v\,\mathrm{d}u \\ &= \int_0^{\infty} v\,\mathrm{e}^{-\gamma v^{\alpha}}\left(\int_0^{\infty} u^s J_0(uv)\,\mathrm{d}u\right)\mathrm{d}v \\ &= \int_0^{\infty} v\,\mathrm{e}^{-\gamma v^{\alpha}}\left(\frac{1}{v^{s+1}}\int_0^{\infty} w^s J_0(w)\,\mathrm{d}w\right)\mathrm{d}v \\ &= \int_0^{\infty} v^{-s}\,\mathrm{e}^{-\gamma v^{\alpha}}\mathrm{d}v\;\int_0^{\infty} w^{s-1}\left(wJ_0(w)\right)\,\mathrm{d}w \end{aligned}$$

As this last expression is a product of second kind characteristic functions, and knowing the analytical expressions with the help of Mellin transform tables [2], we obtain :

$$\phi(s) \;=\; \frac{2^s\,\Gamma\left(\frac{s+1}{2}\right)\,\gamma^{\frac{s-1}{\alpha}}\,\Gamma\left(\frac{1-s}{\alpha}\right)}{\Gamma\left(\frac{1-s}{2}\right)\,\alpha}$$

As this expression is defined on a neighbourhood of $s = 1$ by analytic continuation, second kind moments (log-moments) and second kind cumulants (log-cumulants) exist. By this way, we can write :

$$\tilde{\kappa}_1 \;=\; -\Psi(1)\frac{1-\alpha}{\alpha} \;+\; \log\left(2\gamma^{\frac{1}{\alpha}}\right)$$

$$\tilde{\kappa}_2 = \frac{\Psi(1,1)}{\alpha^2}.$$

with $\Psi(x)$ the Digamma function and $\Psi(N,x)$ the Polygamma function.

The two parameters of such a law can be written as :

$$\alpha = \sqrt{\frac{\Psi(1,1)}{\tilde{\kappa}_2}}$$
$$\gamma = \mathrm{e}^{\sqrt{\frac{\Psi(1,1)}{\tilde{\kappa}_2}}\,(\tilde{\kappa}_1+\Psi(1))\,-\,\Psi(1)}$$

3 CONCLUSIONS

Second kind statistics seem to be a very useful tool to process data modeled by some probability density functions defined on $\mathbb{R}^+$. In a previous work [4], this approach has been validated on classical SAR images laws as Gamma law, K-law, yielding an alternative method to estimate the law parameters with a rather small variance of the estimators. More this approach well suit multiplicative noise as it can be modeled by a Mellin convolution, yielding a simple sum for second kind cumulants.

The present paper, devoted to pathological pdf for which positive moments are not defined, emphasize the interest of such an approach. "Log-moments" and "log-cumulants" allow an innovative way for such distributions. A direct estimation of the law parameters is derived and can be compared with traditional implicit expressions which generally need approximations or specific numerical scheme.

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