ACHIEVING SUPERRESOLUTION BY SUBSPACE EIGENANALYSIS IN MULTIDIMENSIONAL SPACES

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ABSTRACT

An extension of superresolution methods MUSIC (**MU**ltiple **SI**gnal **C**lassification) and ESPRIT (**E**stimation of **S**ignal **P**arameters by **R**otational **I**nvariance **T**echniques) to spaces of arbitrary dimension is proposed in the paper. Generalizations of signal model, spatial smoothing method and estimate equations are provided. Although many applications can be considered in the areas of radar and wireless communications, only one of them is considered for simulation results: high resolution 3D radar target imaging. The concluding remarks drawn in the final part of the paper are supported by simulation results performed on echo signals from a synthetic target. The discussed methods are also compared to the scattering center extraction using the Fourier transform.

1. INTRODUCTION

The subspace based methods represent a significant advance in achieving a very high resolution using the eigenanalysis of the data autocorrelation matrix.

The great interest in the subspace approach is mainly due to the introduction of the MUSIC algorithm [1], which led to many applications requiring superresolution, such as spectral analysis, direction of arrival and time delay estimation problems. Basically, this technique takes advantage of the autocorrelation matrix eigenstructure resulting in the decomposition of the observation space into two orthogonal subspaces, called signal and noise subspaces. The signal components are searched by means of a variable mode vector which is continuously projected onto the noise subspace.

Although this method is simple and robust, allowing even for non-uniformly spaced samples, it may sometimes become computationally too expensive because of the searching nature of the algorithm.

Another very effective superresolution technique introduced by Roy and Kailath [2] and called ESPRIT, overcomes this drawback by calculating the signal components as the solutions of a matrix eigenanalysis problem.

Extensions of the two algorithms, MUSIC and ESPRIT, to the 2D and more recently even to the 3D case [3,4] have been already proposed. This paper performs the last step generalizing them to the case of multidimensional hyperspaces. We will discuss them in the framework of the accurate position recovering of an electromagnetic source set using noisy observations.

2. ESTIMATOR EQUATION DERIVATION

2.1. Signal model

The superresolution techniques MUSIC and ESPRIT are well suited for analyzing signals expressed as a sum of weighted complex exponentials corrupted by additive, white, Gaussian noise. A general model of this type of signal is given bellow:

$$s(m_1,\dots,m_n)=\sum_{k=1}^{N_s} g_k \exp\left[jh\sum_{i=1}^{n} f_{m_i}^{(i)} x_i^{(k)}\right]+w(m_1,\dots,m_n) \quad (1)$$

From a physical point of view this model can be seen as representing a distribution of N_s sources in a n-dimensional space, each of them being characterized by a reflection coefficient g_k and a position vector $\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)$. We also denoted by $f_{m_i}^{(i)}$, $m_i=1..M_i$, the m_i^{th} sample of the $f^{(i)}$ spatial frequency corresponding to the dimension i of the space. The data are organized in the form of an $M_1\times M_2\times\dots\times M_n$ array, $s(m_1,m_2,\dots,m_n)$ and $w(m_1,m_2,\dots,m_n)$ representing an observation and a noise sample respectively. h is a constant, whose value depends on the problem nature.

Spatial frequencies are supposed to be uniformly spaced, so that:

$$f_{m_i}^{(i)} = f_0^{(i)} + (m_i - 1)\Delta f^{(i)},\ i = 1.n \qquad (2)$$

where $\Delta f^{(i)}$ is the spatial frequency step along the dimension i.

The final goal of the two investigated methods is to recover the source localisation with very high resolution from the available data. This problem becomes very difficult especially when the data volume is poor or the signal to noise ratio is low.

Both methods involved in our study require the eigenanalysis of the autocorrelation matrix of the data. Since it is not generally known it must be estimated from the acquired data samples. A common approach consists in averaging a set of observation vectors:

$$\mathbf{R_s} = \frac{1}{N_{obs}} \cdot \sum_{i=1}^{N_{obs}} \mathbf{s}_i \cdot \mathbf{s}_i^H \qquad (3)$$

The $\mathbf{R_s}$ matrix must be of full rank in order to properly separate the sources [5]. In order to restore the full rank of the autocorrelation matrix, even when one data vector is available, we firstly generalized the spatial smoothing method.

The $M_1 \times M_2 \times \ldots \times M_n$ data array is scanned by a $p_1 \times p_2 \times \ldots \times p_n$ window, with $1 \le p_k \le M_k,\ k = 1,..,n$. For an arbitrary fixed position of this sliding window, there are $n!$ different ways for scanning the data array inside this window, corresponding to the $\{1,2,..,n\}$ set permutations. Let $\{q_1,...,q_n\}$ be a given permutation. q_1 will be then the dimension the most often scanned, followed by q_2 and so on. We will call it *the sparse scanning* $\{q_n, q_{n-1},..., q_1\}$ to stress that the dimension q_n is the less often scanned dimension. We group these *sparse scanning orders* in n equivalence classes, according to the equivalence relationship:

$$\{q'_n,..., q'_1\} \equiv \{q''_n,..., q''_1\} \text{ if } q'_n = q''_n \qquad (4)$$

In other words two sparse scanning orders belong to the same class if the less often scanned dimension is the same. Each scanning yields a $p = p_1 \times p_2 \times \ldots \times p_n$ length vector, which includes all the window elements, in the order given by the scanning procedure.

Next, a representative of each sparse scanning class is chosen, which can be considered, without any loss of generality, the direct circular permutation included into the specified class, i.e.: $\{1,2,..,n-1,n\}$, $\{2,3,...,n,1\}$, $...,\{n,1,..,n-2,n-1\}$, denoted by $ord_1, ord_2,..., ord_n$. In the following the scanning order is considered fixed.

For a given window *f*, a vector s_f is formed with its elements, according the scanning order ord_l (this vector will be sometimes denoted by $\mathbf{s}_f^{ord_l}$ in order to integrate the associated information concerning the window index and the scanning order). A noise vector w_f can be defined in the same manner for every sliding window. Because $N_v = \prod_{k=1}^{n} (M_k - p_k + 1)$ windows can be considered, the same number of vectors can be defined. The vector obtained by concatenating the elements of each window is considered as an independent observation.

The overall autocorrelation matrix is then obtained by averaging the corresponding individual autocorelation matrices.

$$\mathbf{R}_\mathbf{s}^{ord_l} = \frac{1}{N_v} \cdot \sum_{f=1}^{N_v} \mathbf{s}_f^{ord_l} \cdot \left(\mathbf{s}_f^{ord_l}\right)^H \qquad (5)$$

For a given window and a fixed sparse scanning order, the following matrix relationship can be written:

$$\mathbf{s}_f^{(ord_l)} = \mathbf{A}^{(ord_l)} \mathbf{?} + \mathbf{w}_f \qquad (6)$$

where:

$$\mathbf{?} = \left[g_1 \quad g_2 \quad \cdots \quad g_{N_s} \right]^T \qquad (7)$$

$$\begin{aligned} \mathbf{A}^{(ord_l)} = \Big[& \mathbf{a}_1^{(ord_l)}(x_1^{(1)},..,x_n^{(1)}) \ldots \mathbf{a}_2^{(ord_l)}(x_1^{(2)},..,x_n^{(2)}) \\ & \ldots \mathbf{a}_{N_s}^{(ord_l)}(x_1^{(N_s)},..,x_n^{(N_s)}) \Big] \end{aligned} \qquad (8)$$

$$\begin{aligned} \mathbf{a}_k^{(ord_l)} &= \mathbf{a}_k^{(ord_l)} \left(x_1^{(k)} \quad \ldots \quad x_n^{(k)} \right) \\ &= \mathbf{a}_{l,k,p_l} \otimes \mathbf{a}_{l+1,k,p_{l+1}} \otimes \ldots \otimes \mathbf{a}_{l-1,k,p_{l-1}} \end{aligned} \qquad (9)$$

$$\mathbf{a}_{uvb} = \left[\exp\left(jh\, f_0^{(u)} x_u^{(v)} \right) \quad \ldots \quad \exp\left(jh\, f_{b-1}^{(u)} x_u^{(v)} \right) \right] \qquad (10)$$

where and the index u = 1,..,n stands for the spatial dimension, n = 1,..,N_s for the source number, and $b = p_u \in \{ p_1, \quad \ldots \quad p_n \}$ for the smoothing window length along the u dimension. The operator ⊗ is the Kronecker product. In order to make more comprehensive the notations above, a particular case corresponding to the sparse scanning order ord_1 is detailed below:

$$\begin{aligned} \mathbf{a}_k^{(ord_1)} &= \mathbf{a}_k^{(ord_1)} \left(x_1^{(k)} \quad \ldots \quad x_n^{(k)} \right) \\ &= \left[\exp\left(jh \left(\sum_{i=1}^{n-1} f_0^{(i)} x_i^{(k)} + f_0^{(n)} x_n^{(k)} \right) \right) \quad \ldots \right. \\ &\quad \left. \ldots \quad \exp\left(jh \left(f_{p_1-1}^{(1)} x_1^{(k)} + \sum_{i=2}^{n} f_{p_i-1}^{(i)} x_i^{(k)} \right) \right) \right]^T \end{aligned} \qquad (11)$$

As it can be readily seen in these equations the matrix $\mathbf{A}$ can be built by using different sparse orders ord_l for considering the blocks generated by the n-D spatial smoothing technique.

For each of them we can express the observation vector autocorrelation matrix using:

$$\mathbf{R}_\mathbf{s}^{(ord_l)} = \mathbf{A}^{(ord_l)} \mathbf{R}_? \left[\mathbf{A}^{(ord_l)} \right]^H + s^2 \mathbf{I} = \mathbf{S}^{(ord_l)} + \mathbf{W} \qquad (12)$$

where $\mathbf{R}_?$ is the autocorrelation matrix of the $\mathbf{?}$ vector and s^2 is the variance of the noise.

2.2. Multidimensional MUSIC algorithm

The main idea behind this method is to split the observation space, spanned by the eigenvectors of the autocorrelation matrix $\mathbf{R}_s$, into two orthogonal subspaces, usually named the signal subspace and the noise subspace. Note that for this algorithm any sparse scanning order ord_l can be considered to form the vectors $\boldsymbol{s_f}$ and therefore it will be ommitted in the following.

Let m_i and v_i be the eigenvalues and the eigenvectors of the $\mathbf{S}$ matrix. There will be only N_s non-zero eigenvalues because of the rank of the $\mathbf{S}$ matrix. Hence, the following relationship holds:

$$\mathbf{S} = \sum_{i=1}^{p} m_i \mathbf{v_i}\mathbf{v_i^H} = \sum_{i=1}^{N_s} m_i \mathbf{v_i}\mathbf{v_i^H} \tag{13}$$

The unit matrix can be put in the same form, because all its eigenvalues are equal to 1 and any vector may be considered as its eigenvector:

$$\mathbf{I} = \sum_{i=1}^{p} \mathbf{v_i}\mathbf{v_i^H} \tag{14}$$

The following equation is then obtained by replacing (13) and (14) in (12):

$$\mathbf{R_s} = \sum_{i=1}^{N_s} (m_i + s^2)\mathbf{v_i}\mathbf{v_i^H} + \sum_{i=N_s+1}^{p} s^2 \mathbf{v_i}\mathbf{v_i^H} \tag{15}$$

Consequently, the autocorrelation matrix eigenvectors corresponding to the largest N_s eigenvalues, known as the principal eigenvectors, span the same subspace as the signal vectors, while the other eigenvectors span the noise subspace.

Let us define the $p\times(p-N_s)$ matrix $\mathbf{V}_n$, whose columns are the $p-N_s$ eigenvectors corresponding to the noise subspace. The location of each scattering center can be then found by searching the maxima of the function:

$$P_{MUSIC-nD}(x_1,\ldots,x_n) = \frac{1}{\tilde{a}(x_1,\ldots,x_n)^H V_n V_n^H \tilde{a}(x_1,\ldots,x_n)} \tag{16}$$

where $\tilde{a}(x_1,\ldots,x_n)$ is the mode vector defined by the same relationship as the columns of the $\mathbf{A}$ matrix.

The mode vector becomes orthogonal to the noise subspace and on any linear combination of the eigenvectors that span this subspace for $(x_1,\ldots,x_n) = (x_1^{(k)},\ldots,x_n^{(k)}),\ k = 1..N_s$. That means that the estimate defined by Eq. (16) will theoretically have infinite value whenever it is evaluated at a location corresponding to a scattering center. In the equation above $\prod_n^{\perp} = \mathbf{V}_n\mathbf{V}_n^H$ stands for the projection operator on the noise subspace. In practice, this function will be finite because of the estimation errors, but it will exhibit very sharp peaks.

2.3. Multidimensional ESPRIT algorithm

Let us denote by $S\{\mathbf{R}\}$ the vector space spaned by the columns of the $\mathbf{R}$ matrix. In this case we can write: $S\left\{\mathbf{A}^{(ord_l)}\right\} = S\left\{\mathbf{V}_s^{(ord_l)}\right\}$, where $\mathbf{V}_s^{(ord_l)}$ is the matrix whose columns are the principal eigenvectors of the $\mathbf{R}_s$ matrix when the order used for its construction is ord_l. Hence, there is a full rank N_s matrix $\mathbf{T}^{(ord_l)}$ so that the following relationship holds:

$$\mathbf{A}^{(ord_l)}\mathbf{T}^{(ord_l)} = \mathbf{V}_s^{(ord_l)} \tag{17}$$

The $\mathbf{A}^{(ord_l)}$ matrix can be also writen as:

$$\begin{aligned}\mathbf{A}^{(ord_l)} &= \begin{bmatrix}\mathbf{A}_1^{(ord_l)} \\ \mathbf{A}_L^{(ord_l)}\end{bmatrix}\{ p_{l+1}\times p_{l+2}\times\ldots\times p_{l-1} \text{ rows} \\ &= \begin{bmatrix}\mathbf{A}_H^{(ord_l)} \\ \mathbf{A}_2^{(ord_l)}\end{bmatrix}\{ p_{l+1}\times p_{l+2}\times\ldots\times p_{l-1} \text{ rows}\end{aligned} \tag{18}$$

with the $\mathbf{A}_1$ and $\mathbf{A}_2$ matrices related by:

$$\begin{aligned}\mathbf{F}^{(ord_l)} = \operatorname{diag}\Bigg(&\exp\left(j\frac{4p}{c}\Delta f^{(l)} x_l^{(1)}\right) \\ &\ldots \exp\left(j\frac{4p}{c}\Delta f^{(l)} x_l^{(N_s)}\right)\ldots \exp\left(j\frac{4p}{c}\Delta f^{(l)} x_l^{(N_s)}\right)\Bigg)\end{aligned} \tag{19}$$

$$\mathbf{A}_2^{(ord_l)} = \mathbf{A}_1^{(ord_l)}\mathbf{F}^{(ord_l)} \tag{20}$$

Eq. (17) and (18) result in:

$$\begin{cases}\mathbf{A}_2^{(ord_l)}\mathbf{T}^{(ord_l)} = \overline{\mathbf{V}}_s^{(ord_l)} \\ \mathbf{A}_1^{(ord_l)}\mathbf{T}^{(ord_l)} = \underline{\mathbf{V}}_s^{(ord_l)} \\ \mathbf{A}_2^{(ord_l)} = \mathbf{A}_1^{(ord_l)}\mathbf{F}^{(ord_l)}\end{cases} \tag{21}$$

$$\Rightarrow \mathbf{F}^{(ord_l)} = \left(\underline{\mathbf{V}}_s^{(ord_l)}\right)^{\dagger}\overline{\mathbf{V}}_s^{(ord_l)} = \left[\mathbf{T}^{(ord_l)}\right]^{-1}\mathbf{F}^{(ord_l)}\mathbf{T}^{(ord_l)}$$

We have thus succedeed in computing the source coordinates on each of the n dimensions of the hyperspace. Nevertheless, since they have been found from independent equation systems, they cannot be associated to uniquely identify the source positions.

A very important property, which help us to solve this problem is defined by the relationship:

$$\mathbf{T}^{(ord_1)} = \mathbf{T}^{(ord_2)} = \ldots = \mathbf{T}^{(ord_n)} = \mathbf{T} \tag{24}$$

Futhermore, it can be shown that this matrix simultaneously diagonalizes the matrices $\mathbf{F}_l,\ l = 1..n$. Consequently, it will diagonalize also the matrix $\mathbf{F} = a_1\mathbf{F}_1 + \ldots + a_n\mathbf{F}_n$, where the coefficients $a_l,\ l = 1..n$ satisfy $\sum a_l = 1$.

Applying this transformation to all the matrices $\mathbf{F}_l,\ l = 1..n$, we can therefore obtain the diagonal matrices $\mathbf{F}^{(ord_l)},\ l = 1..n$, which is equivalent to find the couples of coordinates $(x_1^{(k)}, x_2^{(k)},\ldots,x_n^{(k)}),\ k = 1..N_s$.

3. SIMULATION RESULTS

Some simulation results will be presented nextly to illustrate the performances of the methods described in the previous sections for the 3-D case.

A frequency stepped signal, with a frequency band between 9880 MHz and 10000 MHz has been considered. The frequency step equals 15 MHz, which results in a slant range ambiguity window of $\Delta W_s = c/(2\Delta f) = 10\,\text{m}$ and a slant range resolution of $\Delta R_s = c/(2B) = 1.25\,\text{m}$. The cross range ambiguity window and resolution can be calculated based on the coherent integration angular sector $\Delta b = 0.68^0$. An increment $db = 0.085^0$ yields $\Delta W_c = l_m/(2db) \cong 10\,\text{m}$ and $\Delta R_c = l_m/(2\Delta b) \cong 1.27\,\text{m}$. The same values are considered for the vertical range ambiguity window and resolution.

In order to test the conceived algorithms a synthetic target defined by 8 scattering centers has been considered. Their coordinates are provided in table 1, while their spatial distribution is represented on figure 1. Note that there are distances between scattering centers under the Fourier resolution along all the three axis.

Table 1: Scattering centers coordinates

SC #	#1	#2	#3	#4	#5	#6	#7	#8
x	-3	-2	0.5	0.5	2	2	3.5	3.5
y	0	0	-3	3	-0.5	0.5	-2	2
z	0	1.5	-1	-1	2.5	2.5	1.5	1.5

FFT-3D transform has been used firstly to image the target (fig. 2). It can be readily seen that in the reconstructed image the scattering center that are closer than the Fourier resolution overlap and therefore cannot be correctly localized.MUSIC and ESPRIT algorithms have been then employed to image the same target with $p_1 = p_2 = p_3 = 5$. Note that these algorithms fully succeeded in resolving all the scattering centers (fig. 3) and therefore they can be easily localized now. The results obtained by using ESPRIT-3D method are not represented because they are identical with the scattering center positions given on figure 1.

4. CONCLUSION

The extension of two superresolution methods, MUSIC and ESPRIT, to multidimensional hyperspaces is demonstrated in the paper. Signal model, autocorrelation matrix eigenanalysis, subspace splitting, spatial smoothing method and estimation equations are discussed. The particular case of the 3D space is considered for simulations in the context of the radar target image reconstruction. The highly accurate estimates of the scattering center positions are excellent candidates as feature vectors for automatic radar target classification.

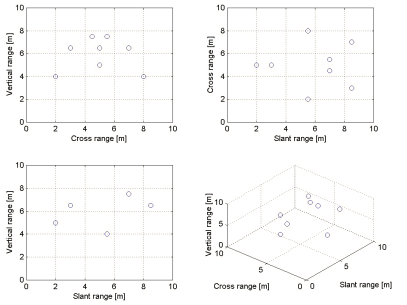

Fig. 1 Synthetic target scattering center localization

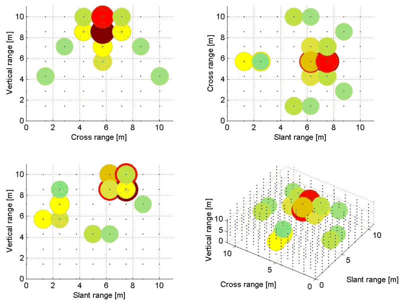

Fig. 2 Radar target imaging using FFT-3D transform

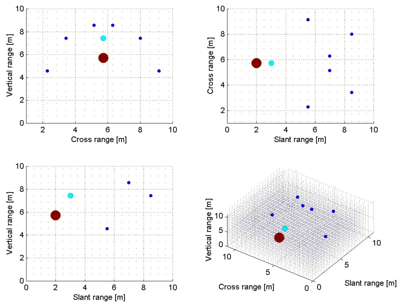

Fig. 3 Radar target imaging using MUSIC-3D

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