# AN AUDIO MOTIVATED HYBRID OF WARPING AND KAUTZ FILTER TECHNIQUES

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## ABSTRACT

In this paper we present novel filter design methods that combine well-known frequency warping techniques and a new procedure for the optimization of Kautz filter poles. The motivation, from an audio signal processing perspective, is as follows. Many audio related target responses can be well-modelled by a combination of distributed decaying exponential components, which is by definition what a Kautz filter does, but in an orthonormalized form, providing many favorable properties. The most essential part is then how to choose the Kautz filter poles, to which we propose a new effective and well-behaving iterative method. Utilizing an intermediate warping procedure, desired frequency resolution allocation is incorporated to the choosing of poles - the target signal is warped prior to the pole optimization and then the produced poles are mapped back to the original frequency domain using inverse warping.

## 1 INTRODUCTION

An old concept of *Kautz filters* [7, 2], or generalized transversal filters, has found renewed interest over the last ten years, mainly from the system identification point of view [5, 11]. The perspective has usually been to form generalizations to the well-established *Laguerre models* in system identification [8] and control [15]. In this context, the *Kautz filter or model* has often the meaning of a two-pole generalization of the Laguerre structure, whereupon further generalizations restrict typically, as well, to structures with identical blocks [5]. At least partly due to this, methods proposed for the Kautz filter pole optimization focus on constructions of identical blocks [3, 12]. A more profound reason is that the optimization with respect to the poles is highly non-linear and the use of a repeated substructure provides a somewhat "analytical" solution. In addition, the pole optimization and model order selection problems are thereby essentially separated.

To our knowledge, there has been only two prior attempts to directly optimize a distributed set of (complex) Kautz filter poles in the least-square (LS) sense, with respect to a given target response [9, 4]. These methods are verbally outlined in Sec. 4. With a slight modification, a method proposed originally to pure FIR-to-IIR filter conversion [1], has proven to be very suitable for the optimization of Kautz filter poles. Our adoption is named the *BU-method*, to reflect the connection to the work of Brandenstein and Unbehauen [1].

The BU-method genuinely optimizes the poles with respect to an orthonormal structure, i.e., not according to the general pole-zero configuration, which would be highly suboptimal. Furthermore, the Kautz filter is an orthonormalized counterpart of a superposition of decaying complex exponentials, i.e., *parallel resonators*, providing better modelling of typical audio responses than a connected pole-zero model. In the orthonormal configuration, the contribution of each pole (pair) to the approximation is explicitly at hand, enabling direct modifications of the produced pole set, e.g., pruning, tuning, clustering and appending poles. Additionally, the BU-method is capable of producing very large sets of unconditionally stable and accurate poles, which would be unachievable with standard pole-zero modelling methods. The fact that the algorithm operates on time domain responses makes it possible to utilize manipulations of the target response in the design phase. In this article, we address only a combined intermediate warping procedure, the *warped BU-method* presented in Sec. 5 that can be used to incorporate desired frequency resolution allocation to the modelling.

The concept of Kautz filters is introduced in Sec. 2, followed by a Section describing the particular utilization of Kautz filters in this paper. Audio oriented examples of the proposed methods are included in Sections 4 and 5. This is merely a presentation of the new warped-BU-method, and more comprehensive studies of the applicability of Kautz filters can be found in our related publications at http://www.acoustics.hut.fi, as well as, MATLAB scripts and demos.

## 2 KAUTZ FUNCTIONS AND FILTERS

For a given set of desired poles $\{z_i\}$ in the unit disk, the corresponding set of rational orthonormal functions is uniquely defined in the sense that the lowest order rational functions, square-integrable and orthonormal on the unit circle, analytic for $|z| > 1$, are of the form [13]

$$G_i(z) = \frac{\sqrt{1 - z_i z_i^*}}{1 - z_i z^{-1}} \prod_{j=0}^{i-1} \frac{z^{-1} - z_j^*}{1 - z_j z^{-1}}, \quad i = 0, 1, \ldots. \quad (1)$$

The meaning of orthonormality is most economically established using the definition for the time-domain *inner product*: for the impulse responses of (1), $(g_i, g_k) := \sum_{n=0}^{\infty} g_i(n) g_k^*(n) = 0$ for $i \neq k$, and $(g_i, g_i) = 1$.

Functions (1) form a recurrent structure: up to a given order, i.e., the number of poles, functions associated to the subsets $\{z_j\}_{j=0}^{i}$ of an ordered pole set $\{z_i\}_{i=0}^{N}$ are produced as intermediate substructures defining a tapped transversal system. In agreement with the continuous-time counterpart [7], a weighted sum of these functions is called a Kautz filter, depicted in Fig. 1.

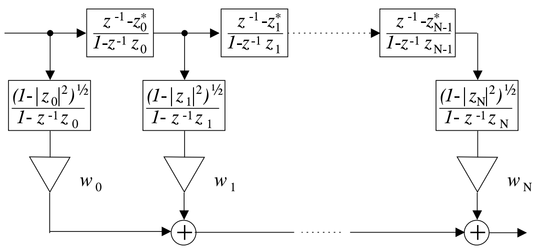


Figure 1: The Kautz filter. For $z_i = 0$ in (1) it degenerates to an FIR filter, for $z_i = a$, $-1 < a < 1$, it is a Laguerre filter where the tap filters can be replaced by a common pre-filter.

Defined in this manner, Kautz filters are merely a class of fixed-pole IIR filters, forced to produce orthonormal tap-output impulse responses. A particular Kautz filter is thus determined by a set of poles $\{z_i\}_{i=0}^{N}$, consequently defining the filter order, the all-pass filter backbone, and the corresponding tap-output filters, complemented with a set of somehow assigned filter weights $\{w_i\}_{i=0}^{N}$. However, more profoundly, defined by any set of points $\{z_i\}_{i=0}^{\infty}$ in the unit disk, functions (1) form an orthonormal set which is complete, or a *base*, with a moderate restriction on the poles [13]. That is, a basis representation (as a *generalized z-transform*) of any causal and finite-energy discrete-time signal is obtained as an orthonormal (*Fourier*) series expansion with respect to these basis functions. As impulse responses of causal and stable linear time-invariant systems form a subset of finite energy signals, this is also a valid model structure for input-output-data identification.

A reasonable presumption in modelling a real response would be that the poles should be real or occur in complex conjugate pairs. For complex conjugate poles, an equivalent *real Kautz filter* formulation [2], depicted in Fig. 2, prevents dealing with complex (internal) signals and filter weights, in contrast to the structure of Fig. 1. Normalization terms are given by $p_i = \sqrt{(1-\rho_i)(1+\rho_i-\gamma_i)/2}$ and $q_i = \sqrt{(1-\rho_i)(1+\rho_i+\gamma_i)/2}$, where $\gamma_i = -2RE\{z_i\}$ and $\rho_i = |z_i|^2$ can be recognized as corresponding second-order polynomial coefficients. In the case of both real

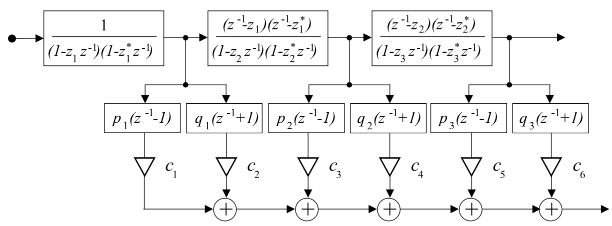


Figure 2: One possible realization of a real Kautz filter, corresponding to a sequence of complex conjugate pole pairs.

and complex poles we use a mixture of first- and second-order sections, i.e., a combination of Fig. 1 and Fig. 2.

## 3 KAUTZ FILTER SYNTHESIS

Kautz filters provide linear-in-parameter models for many types of system identification and approximation schemes, including adaptive filtering. In this paper we address only the "prototype" least-square approach to the approximation problem. The approximation of a given target response $h(n)$ (or $H(z)$) is obtained as its truncated Fourier series expansion with respect to the time or frequency domain basis functions, i.e., to the chosen fixed pole set:

$$\hat{h}(n) = \sum_{i=0}^{N} c_i g_i(n), \quad c_i = (h, g_i), \quad (2)$$

(or $\hat{H}(z) = \sum_{i=0}^{N} c_i G_i(z)$, $c_i = (H, G_i)$). Evaluation of the Fourier coefficients $c_i$ can be implemented by feeding the signal $h(-n)$ to the Kautz filter and reading the tap outputs $x_i(n) = G_i[h(-n)]$ at $n = 0$: $c_i = x_i(0)$. This implements convolutions by filtering and it can be seen as a generalization of the rectangular window FIR design. We use these true orthonormal expansion coefficients because they are easy to obtain, providing simultaneous time and frequency domain design, and powerful means to the Kautz filter pole position optimization, demonstrated in the following Sections. Moreover, the coefficients are independent of ordering and approximation order, which makes choosing of poles, approximation error evaluation, and model reduction efficient.

## 4 CHOOSING OF THE POLES

Even in the one-pole Laguerre case it is generally impossible to optimize the pole position analytically. Nevertheless, there are many methods that can be used in search for suitable poles, including all-pole or pole-zero modelling, sophisticated guesses, and random or iterative search. For structures with identical all-pass blocks, a relation between optimal model parameters and error energy surface stationary points with respect to the poles may be utilized [3], as well as, a classification of systems to associate systems and basis functions [12].

As a direct consequence of the orthonormality, the approximation error energy is given by

$$E = \sum_{i=N+1}^{\infty} |c_i|^2 = H - \sum_{i=0}^{N} |c_i|^2, \quad (3)$$

where $H = (h, h)$ is the energy of the target response $h(n)$ and $N$ is the approximation order. Hence, the energy of an infinite duration error signal is attained as a by-product from finite filtering operations, previously described for the evaluation of the filter weights. Moreover, the *all-pass operator* defined by the (transversal part of the) chosen Kautz filter induces an *complementary* division of the energy of the signal $h(-n)$, $n = 0, \ldots, M$, [14]. An all-pass filter $A(z)$ is *lossless* by definition and (from the above) it can be deduced that the portion of the response $a(n) = A[h(-n)]$ in the time interval $[-M, 0]$ corresponds to the approximation error energy $E$. In other words, the Kautz filter optimization problem reduces to minimization of the energy of an equivalent finite duration error signal.

McDonough and Huggins utilize a more constrained orthogonality principle: at the stationarity points of the error energy with respect to the poles, the filter coefficients $c_{i+N} = (h, g_{i+N})$, $i = 0, \ldots, N$, where the Kautz filter is extended with the same set of poles, must vanish [9]. This is precisely a special case of the optimality conditions in [3]. Starting from equations $(h, g_{i+N}) = 0$, $i = 0, \ldots, N$, McDonough and Huggins replace the all-pass numerator with a polynomial approximating the denominator mirror polynomial to produce linear equations for the new denominator polynomial coefficients [9]. By operating directly on the complementary error signal $a(n)$, Friedman constructs a network structure for parallel calculations of all partial derivatives of the approximation error with respect to the real second-order polynomial coefficient (see Fig. 2), to be used in a gradient algorithm [4]. At least for our purposes of use, these algorithms are interesting solely on a principal level.

The method proposed by Brandenstein and Unbehauen [1] for converting an FIR filter to an IIR filter is implicitly based on the concept of complementary signals. No connection is, however, made to the orthonormal filter structures, and after attaining a desired denominator polynomial, the numerator of the same order is solved in the LS sense. With a modification to include the $(N-1, N)$-order case implied by the Kautz filter, the (denominator part of the) BU-method is a very efficient method for the determination of Kautz filter poles. The method is effectively based on approximating an all-pass operator with

$$\hat{A}^{(k)}(z) = \frac{z^{-N} Q^{(k)}(z^{-1})}{Q^{(k-1)}(z)} \tag{4}$$

to iteratively generate a sequence of polynomials $Q^{(0)}(z), \ldots$, although (4) is never actually used to produce an approximative error signal. A matrix equation $\mathbf{A}^{(k)}\mathbf{q}^{(k)} = \mathbf{u}^{(k)} + \mathbf{b}^{(k)}$ is formed, where $\mathbf{q}^{(k)}$ and $\mathbf{u}^{(k)}$ are unknown, and where the elements of $\mathbf{A}^{(k)}$ and $\mathbf{b}^{(k)}$ are updated by (all-pole) filtering $h(-n)$ with $1/Q^{(k)}(z)$. The solution of the (over-determined) matrix equation $\mathbf{A}^{(k)}\mathbf{q}^{(k)} = \mathbf{b}^{(k)}$ minimizes the square norm of $\mathbf{u}^{(k)} = \mathbf{A}^{(k)}\mathbf{q}^{(k)} - \mathbf{b}^{(k)}$, which is the approximative error produced by (4). Elements of $\mathbf{q}^{(k)}$ are the polynomial coefficients of $Q^{(k)}(z)$ at iteration $k$. After a sufficient number of iterations, we choose $Q^{(k)}(z)$ that minimizes the true LS error, i.e., the energy of the complementary signal $a(n) = z^{-N}Q^{(k)}(z^{-1})/Q^{(k)}(z)[h(-n)]$, $n = -M+1, \ldots, 0$, where $M$ is the duration of the target signal. The optimal Kautz filter poles are the roots of $Q^{(k)}(z)$. A more detailed deduction and description of the algorithm is given in [1], bearing in mind the modifications implied by our particular case.

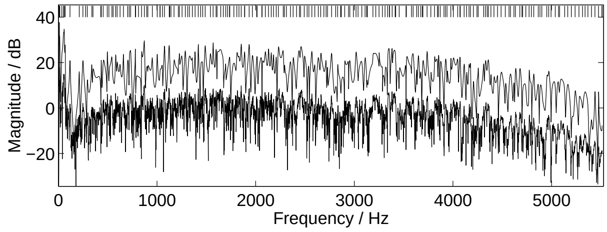


Figure 3: Magnitude response of a measured room impulse response (bottom),an 400th order Kautz model and vertical lines indicating complex conjugate pole positions.

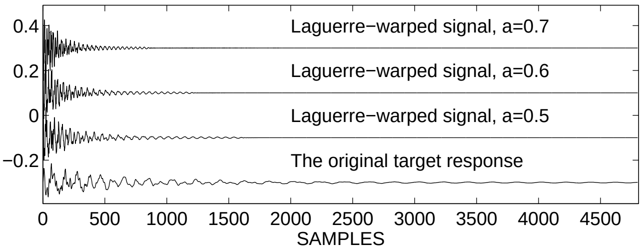


Figure 4: The measured guitar body impulse response and some Laguerre transformed signals.

To demonstrate the robustness of the algorithm, in Fig. 3 we have an 400th order Kautz filter approximating a measured room impulse response (sampled at 11025 Hz, 8192 samples). Vertical lines indicate complex conjugate pole pair positions. Using previous notations, $N = 400$ and $M = 8192$.

## 5 THE WARPED-BU-METHOD

*Frequency warping* is a widely used technique in filter design, and analysis, coding and synthesis of signals (see e.g. [6] for an overview). Here we use the "Laguerre-warping", which differs from the traditional *all-pass warping* by an optional orthonormalizing all-pole pre-filter. Displayed in Fig. 4 we have *Laguerre transformed* a measured acoustic guitar body impulse response for different Laguerre parameters $a$. That is, we have chosen $z_i = a$, $i = 0, \ldots, 4799$, in Fig. 1 and evaluated the corresponding Laguerre-Fourier coefficients. The interpretation of Fig. 4 is that the original signal is *compressed* to a smaller set of (Laguerre-Fourier) filter weights. For example, an 800th order Laguerre filter with $a = 0.7$ would (practically) exactly reproduce the original response. This implies a reduction in the computational complexity in the order of 10–20, depending on implementations, compared to a full-size FIR filter.

Kautz filters designed using the BU-method would give almost perfect matchings at further reduced filter

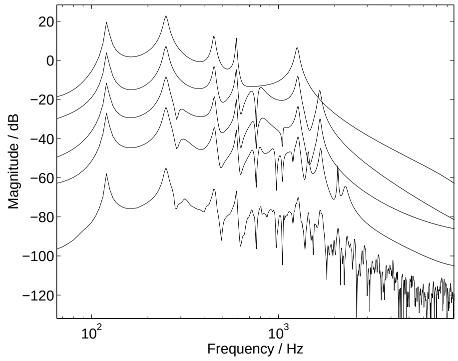


Figure 5: Displayed from top to bottom, Kautz models of orders 10, 16, 20 and 40, and the target magnitude response.

orders 100–200. However, the aim of this paper is to present a new *warped BU-method*, in which the BU-method is applied on the (Laguerre) warped target response and the produced poles are subsequently mapped back to the original frequency domain using the corresponding inverse all-pass mapping

$$z \mapsto \frac{1+az}{z+a}. \tag{5}$$

The warping parameter, $a$, can be, e.g., chosen to approximately match the auditorily motivated *Bark scale warping* [10], or on more technical grounds. We may also interlace pole sets produced with different warpings to achieve desired allocation of frequency resolution. By band-limiting the target response we can also utilize warping with higher-order all-pass blocks. A peg leg version of the latter would be to decimate the target and to map the poles according to the corresponding complex exponent function, producing detailed models for the low-frequency part. Fig. 5 demonstrates that in the case of the target response of Fig. 4 we may form really low-order Kautz models for the prominent resonances, where we have used $a = 0.7$. For example with respect to the LS approximation error, the 10th order Kautz filter is compatible with an 1000th order FIR filter (designed by truncation), and respectively, the 40th order Kautz filter with an 3000th order FIR filter.

## 6 DISCUSSION AND CONCLUSIONS

We have briefly presented a new *warped BU-method* for the optimization of Kautz filter poles. More generally, it can be seen as a technique for designing specific IIR filters. In some (low-order) cases we may actually return to more efficient traditional IIR filter structures. The main advantages of the method is that it efficiently models decaying resonant target responses, typical in audio applications, and that it works on very high filter order, often unattainable with traditional filter design means. Using the intermediate warping procedure we may incorporate desired frequency resolution allocations.

## 7 ACKNOWLEDGEMENTS


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