Robust non-Gaussian Matched Subspace Detectors *

Mukund Desai †, Rami Mangoubi

ABSTRACT

We address the problem of matched subspace detection in the presence of arbitrary noise and interferents, or interfering signals that may lie in a possibly unknown subspace, but that nevertheless corrupt the measurements. A hypothesis test that is robust to interferents yet sensitive to the signal of interest is formulated. The test is applicable to a large class of noise density functions. In addition, specific expressions for the generalized likelihood ratio (GLR) detectors are derived for the class of Generalized Gaussian noise. The detectors are generalizations of the χ^2, t, and F statistics used with Gaussian noise. For matched filter detection, these expressions are simpler and computationally efficient. ROC performance results based on simulation demonstrate the superior performance obtained with detectors based on the correct noise model. The results also demonstrate the improved performance robust detectors offer when interferents are present.

1 Introduction

We design matched subspace detectors that account for: 1) the presence of *non-Gaussian* noise, in particular *Generalized Gaussian* noise, and 2) the presence of interfering signals that may lie in an unknown subspace different from that of the signal of interest, but that nevertheless corrupt the measurements. We refer to these interfering signals as *interferents*. Pioneering and current work on matched subspace detection has been conducted by Scharf et al. in [8, 9], and in the references therein. This work deals with the case when the noise variance is known, and when it is unknown (CFAR). Subspace detection in the presence of spherically invariant noise density functions, has been considered for instance, in [11]. We note that *though in the presence of Gaussian noise, CFAR optimal and robust detectors are equivalent from the point of view of performance, such is not the case when the noise variance is known. Nor is it the case when the noise is non-Gaussian, whether the variance is known or unknown.* The main contributions of the paper are:

- *The derivation of a hypothesis test for subspace detection that is robust to unlearned interferents* (Eqs. 9 and 10).
- *When the noise is Gaussian and the variance is known*, we derive a robust detector (Eq. 12) that is more general than the well known χ^2-distributed matched subspace detector [8].
- For the class of *Generalized Gaussian density functions*, we derive expressions for the optimal and robust detectors that can be computed numerically. This rich family of density functions is used in applications ranging from random media [3] to underwater acoustics [7] to medical imaging [1] and video technology [10].
- In the particular case of *robust matched filter* detection, when the signal space is one-dimensional, we derive *computationally efficient expressions* (Eqs. 18 and 21); The expressions require computations in one-dimensional subspaces only.

We analyze through simulation the performance of the detectors under different scenarios. Moreover, we apply the robust non-Gaussian detectors to clinical functional magnetic resonance image (fMRI) data to characterize the brain's response to stimuli in [1].

We note that the term robust hypothesis test, or more generally, robust statistics, is used in other literature to designate robustness to statistical outliers [4]. In failure detection work related to dynamic plants, such as in [5], robustness is with respect to noise and plant model uncertainties. The relationship between the different concepts of robustness is not discussed here.

In the next section, we formulate the robust detection test, and in Section 3, we derive the robust detectors. Numerical results are in Section 4, while Conclusions are in Section 5.

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†The authors are with the C.S. Draper Laboratory, Cambridge, Massachusetts, USA. correspondence:mdesai@draper.com.

2 Problem Formulation

We describe the signal and measurement models in Section 2.1, before formulating the robust detection test in Section 2.2.

2.1 The signal and measurement models

Consider the general model

$$x = S\theta + U\psi + \omega v \tag{1}$$

where x is a measurement vector, S is the $K \times M$ matrix whose columns span the signal space, θ is the associated gain vector, U is the $K \times (K - M)$ matrix whose columns span the unlearned interferent space, and ψ is the associated vector gain. One can only unambiguously see this unlearned component in the space orthogonal to the signal subspace. Therefore, the presence of these effects can only be inferred from the component of the measurement that lies in the *null space* of the space spanned by the columns of S, so we assume the columns of U to span the null space of S. We deal with known subspace interference in [2]. The noise vector v is a unit-variance noise vector with known or unknown scalar width factor ω. Its elements are assumed to be *independent and identically distributed Generalized Gaussian random variables.* Specifically, For a K-dimensional random vector x, the Generalized Gaussian density function is defined as

$$f_p(x|m,\omega) = \left(\frac{p}{2\omega\Gamma(1/p)}\right)^K \exp\left(-\left(\frac{\|x-m\|_p}{\omega}\right)^p\right)$$

for $p \in (0,\infty)$, where $\Gamma(k) = \int_0^\infty t^{k-1}\exp(-t)\,dt$, is the Gamma function, and for an arbitrary vector y, $\|y\|_p \equiv \left(\sum_i |y_i|^p\right)^{\frac{1}{p}}$. Here m, ω, and p are respectively the location, width factor and shape or decay parameters of the density function. For any p, the *width* parameter ω is proportional to the standard deviation σ. Specifically, $\omega = \sigma\left(\Gamma(1/p)/\Gamma(3/p)\right)^{\frac{1}{2}}$.

2.2 The robust detection test

For hypotheses H_0 and H_1, let $\mathcal{U}_0$ and $\mathcal{U}_1$ denote the subspaces of unlearned effects spanned by the *columns* of U_0 and U_1, respectively, and, for any matrix W, let $\mathcal{N}(W)$ denote its *null* subspace. A hypothesis test that we can pose is

$$\begin{aligned} H_0: x &= U_0\psi_0 + \omega_0 v_0, & \mathcal{U}_0 \subseteq \mathcal{N}(S) \quad (2)\\ H_1: x &= S\theta + U_1\psi_1 + \omega_1 v_1, & \mathcal{U}_1 \subset \mathcal{N}(S) \quad (3)\end{aligned}$$

The conditions on the dimensions of the unlearned effects subspaces $\mathcal{U}_0$ and $\mathcal{U}_1$ are needed for the hypothesis test to be well posed. In terms of the matrices U_0 and U_1 the generalized likelihood ratio test for Eqs.(2-3) is

$$\Lambda(x; U_0, U_1) = \frac{\max_{\theta_1,\psi_1,\omega_1} f(x|H_1, U_1, \theta_1, \psi_1, \omega_1)}{\max_{\psi_0,\omega_0} f(x|H_0, U_0, \psi_0, \omega_0)} \tag{4}$$

To obtain robustness to unlearned interferents while maintaining sensitivity to the signal of interest, we choose U_{0_r}, U_{1_r} such that

$$\begin{aligned} U_{0_r} &= \arg\max_{U_0}\max_{\psi_0,\omega_0} f(x|H_0, U_0, \psi_0, \omega_0) \quad (5)\\ U_{1_r} &= \arg\min_{U_1}\max_{\theta_1,\psi_1,\omega_1} f(x|H_1, U_1, \theta_1, \psi_1, \omega_1) \quad (6)\end{aligned}$$

which, as derived in [2], leads to

$$\begin{aligned} \mathcal{U}_{0r} &= \mathcal{N}(S) \quad (7)\\ \mathcal{U}_{1r} &= \{0\} \quad (8)\end{aligned}$$

where $\mathcal{N}(S)$, the null space of the matrix S, is spanned by the columns of $U_{0r} = U_0$ (we drop the subscript r for convenience), and $\{0\}$ is the zero subspace. We then have the following *robust matched subspace detection test.*

$$\begin{aligned} H_0: x &= U_0\psi_0 + \omega_0 v_0, \quad \mathcal{U}_0 = \mathcal{N}(S) \quad (9)\\ H_1: x &= S\theta + \omega_1 v_1, \quad (10)\end{aligned}$$

We show in [2] that the solutions of Eqs.(7-8) are obtained for a large class of non-Gaussian noise that is not restricted to the Generalized Gaussian pdf. We only impose this condition to derive the generalized likelihood ratio statistics for these tests. Note that, when $U_0 = 0$, then we have an optimal or conventional matched subspace detection test.

3 The Robust Matched Subspace Detectors

We first consider the case where the width factor is known, and then the case where it is unknown (CFAR).

Known ω. For the test of Eqs.(9-10), we have, apart from a constant, the log-likelihood ratio

$$\begin{aligned} \lambda_{p,rk}(x) &= \log\frac{\exp\left(-\|x - S\hat{\theta}_p\|_p^p/\omega_1^p\right)}{\exp\left(-\|x - U\hat{\psi}_p\|_p^p/\omega_0^p\right)}\\ &= -\left(\frac{1}{\omega_1}\|x - S\hat{\theta}_p\|_p\right)^p + \left(\frac{1}{\omega_0}\|x - U\hat{\psi}_p\|_p\right)^p \end{aligned} \tag{11}$$

where the subscripts r and k are for robust detector and known ω respectively, and $\hat{\theta}_p$ and $\hat{\psi}_p$ are the maximum likelihood estimates of θ and ψ, respectively. When there are no interferents and $U \equiv 0$, we have an optimal matched filter detector. For the Gaussian case, when $p = 2$ and with $\omega_0 = \omega_1$, Eq.(11) becomes in terms of the common variance σ^2

$$\lambda_{2,rk}(x) = \frac{1}{2\sigma^2}x'\left(P_S - P_U\right)x \tag{12}$$

where for an arbitrary matrix W, the projection matrix is given by $P_W \equiv W\left(W'W\right)^{-1}W'$. When $U \equiv 0$, we have the well known χ^2-distributed matched subspace detector [8]

$$\lambda_{2,ok}(x) = \frac{1}{2\sigma^2}x'P_S x \tag{13}$$

When the signal space is one dimensional, we have a *matched filter* detector. In that case, it is possible to eliminate the need for parameter computations in higher dimensional spaces even though the interferent space $\mathcal{U}$ is not one dimensional. For this, we need the following lemma.

Lemma 1 *Let h and η be column vectors and define $J_p(\eta) = \|\eta\|_p^p, \quad p \in (0,\infty)$ The optimization problem*

$$\min_{\eta} J_p(\eta) \tag{14}$$

$$\text{subject to} \quad h'\eta = b \tag{15}$$

has as a solution $J_p^* = (|b|/\|h\|_q)^p$ where $q = (p/p-1)$ for $p > 1$, and $q \equiv \infty$ otherwise. The optimal η is

$$\eta_i = b\frac{\operatorname{sgn}(h_i)\,|h_i|^{(1/(p-1))}}{\|h\|_q^q} \qquad p > 1 \tag{16}$$

$$\eta_i = \begin{cases} b/h_i & if \quad |h_i| = \max_j |h_j| \\ 0 & \text{otherwise} \end{cases} \qquad p \le 1 \tag{17}$$

See [2] for the proof. Using this lemma (See [2]), we obtain a simplied expression for $\lambda_{p,rk}$ when the signal space is one-dimensional, in which case we set $S = s$,

$$\lambda_{p,rk}(x) = -\left(\frac{1}{\omega_1}\|x - s\hat{\theta}_p\|_p\right)^p + \left(\frac{1}{\omega_0}\frac{|s'x|}{\|s\|_q}\right)^p \tag{18}$$

The advantage of this simplified form is that we do not need to be concerned with any residual computation in the larger dimensional subspace $\mathcal{U}$ spanned by the columns of U, while the computation of $\hat{\theta}_p$ takes place in a one-dimensional subspace.

Unknown ω (CFAR). After performing the necessary algebra, which we omit, the generalized likelihood ratio (GLR) taken to the power of $1/K$ for convenience, leads to the *robust CFAR matched subspace detector*

$$\lambda_{p,ru}(x) = \frac{\|x - U\hat{\psi}_p\|_p}{\|x - S\hat{\theta}_p\|_p} \tag{19}$$

Here the subscripts r and u are for robust detector and unknown parameter ω respectively. Note that, unlike in the previous section, we do not need to take the log of the likelihood ratio to obtain an expression in terms of the ratio of residuals. In the absence of unlearned interferents, i.e. when $U \equiv 0$, we have the *optimal CFAR matched subspace detector*

$$\lambda_{p,ou}(x) = \frac{\|x\|_p}{\|x - S\hat{\theta}_p\|_p} \tag{20}$$

When S is one-dimensional ($S = s$), we can eliminate the need for a search in the generally multidimensional space spanned by the columns of U in the CFAR robust detector of Eq.(19), by applying Lemma 1 to get

$$\lambda_{p,ru}(x) = \frac{|s'x|}{\|s\|_q\|x - s\hat{\theta}_p\|_p} \tag{21}$$

In the Gaussian case, Eqs.(21 and 20) become, respectively

$$\begin{aligned} \lambda_{2,ru}(x) &= \frac{x'P_s x}{x'P_U x} \\ &= \cot(<x,s>) \end{aligned} \tag{22}$$

$$\begin{aligned} \lambda_{2,ou}(x) &= \frac{x'x}{x'P_U x} \\ &= \csc(<x,s>) \end{aligned} \tag{23}$$

where $<x,s>$ denotes the angle between x and s. Thus, in the Gaussian case, as is well known [8], the underlying statistic is the angle between x and s, and the *two detectors thus provide the same performance.* For an arbitrary p, we have

$$\frac{\lambda_{p,ru}}{\lambda_{p,ou}} = \frac{\|s\|_2\|x\|_2}{\|s\|_q\|x\|_p}\cos(\langle x,s\rangle) \tag{24}$$

From the above discussion, we observe the following:

- *Performance.* the performance of the robust and optimal CFAR detectors are not necessarily the same when $p \neq 2$, as shown in Section 4.
- *Invariance properties.* When $p = 2$, as Eqs.(22 and 23) show, the two CFAR detectors are rotation and scale invariant. By contrast, for a general p, they are only scale invariant. They would also be invariant to transformations that leave the ratio of the p-norm of residuals unchanged.

4 Results and Discussion

Figure 1 compares the optimal and robust CFAR Laplacian detectors ($p = 1$), or $\lambda_{1,ou}$ and $\lambda_{1,ru}$ respectively, in the presence of Laplacian noise with unit standard deviation under two different scenarios: interferent absent and interferent present. The thick solid curve represents the performance of the optimal detector $\lambda_{1,ou}$ in the absence of unlearned interferents, while the the thick dashed curve represents the performance of the same detector in the presence of an unlearned interferent signal of magnitude equal to twice that of the signal magnitude, meaning $\psi = 2\theta$. The thin curve represents the performance of the robust detector $\lambda_{1,ru}$ in the absence (solid) and presence (dashed) of interferents. Comparison of the four curves indicates that the robust detector, whose performance is suboptimal when there is no interferent, provides superior performance when compared to the suboptimal performance of the optimal detector in the presence of interferents. Since the magnitude of the interferent signal is generally unknown apriori, the robust detector's insensitivity to interferents implies a more predictable ROC performance.

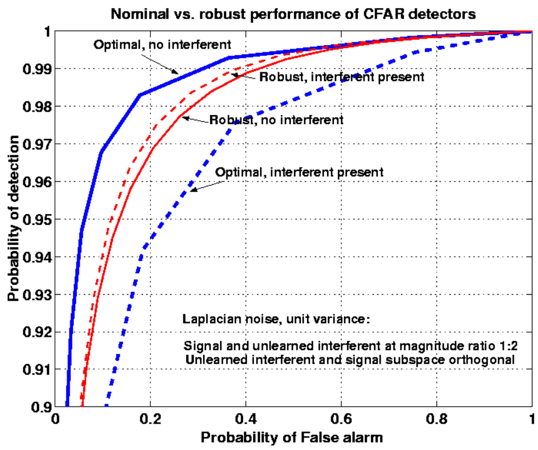

Figure 1: *Nominal vs. robust performance* for the CFAR optimal or $\lambda_{1,ou}$ (thick curves) and CFAR robust or $\lambda_{1,ru}$ (thin curves) Laplacian detectors in the presence (dashed) and absence (solid) of unlearned interferents.

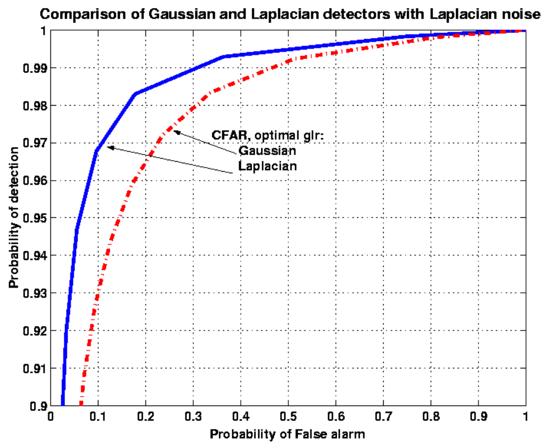

Figure 2: *ROC performance comparison of Laplacian and Gaussian detectors in the presence of Laplacian noise:* CFAR Optimal Laplacian or $\lambda_{1,ou}$ (solid) and Gaussian $\lambda_{2,ou}$ (dashed) detectors.

Figure 2 shows two plots that compare the probability of detection vs. false alarm performance of the Laplacian and Gaussian CFAR detectors, $\lambda_{1,ou}$ and $\lambda_{2,ou}$ respectively, in the presence of Laplacian noise ($\omega = .707$). The solid curve represents the Laplacian detector ($p = 1$), and the dash-dotted curve represents the Gaussian detector ($p = 2$). The results are a clear indication of *the deleterious effect on performance of the widely held practice of assuming Gaussian noise and using Gaussian based statistics.*

5 Conclusion

We design non-Gaussian matched subspace detectors that are robust to interferents whose subspace is unknown. To do so, we formulate a hypothesis test that is insensitive to unlearned interferents and simultaneously sensitive to the signal of interest. In particular, generalized likelihood ratio (GLR) detectors are derived for the family of Generalized Gaussian density functions, which includes the Gaussian and Laplacian pdf's. Simulation based results are shown here, and these detectors are successfully applied to functional magnetic resonance images (fMRI) in [1].

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