SPECTRAL ESTIMATION USING DYNAMICAL SYSTEM THEORY

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ABSTRACT

This paper addresses the problem of spectral estimation by using dynamical system theory. A new parametric model motivated by the Takens theorem is defined. This model referred to as dynamical AR model is showed to outperform the usual AR model to estimate the frequencies of sinusoidal signals.

1. INTRODUCTION

A fundamental problem in signal processing consists of modeling observations x_n, $n = 1, \ldots, N$ by an appropriate parametric model. Most parametric models can be expressed as follows:

$$x_n = F(x_{n-1}, \ldots, x_{n-d}) + e_n, \tag{1}$$

where d is referred to as model order, F is a function from $\mathbb{R}^d$ to $\mathbb{R}$, and e_n is an additive white Gaussian noise. Typical examples include AR models [1] (with $F(x_1, \ldots, x_d) = \sum_{i=1}^{d} a_i x_i$) or Volterra series [2] (with $F(x_1, \ldots, x_d) = \sum_i a_i x_i + \sum_{i,j} a_{ij} x_i x_j + \ldots$). This paper studies a specific class of models which has received much attention in dynamical system theory. These models can be justified by the Takens Theorem [3]. The Takens theorem states that any dynamical system can be described accurately from successive values of an observed scalar time series. More precisely, any dynamical system can be defined by the following model, referred to as *dynamical model*:

$$x_n = F(x_{n-\tau}, \ldots, x_{n-d\tau}) + e_n, \tag{2}$$

where τ and d are the embedding delay and the embedding dimension, respectively.

This paper shows that model (2) has interesting properties with respect to model (1) for the spectral estimation problem. In particular, model (2) provides a parsimonious representation of sinusoidal signals which makes it attractive for spectral estimation.

Section 2 addresses the challenging problem of frequency estimation by using the dynamical model (2). Simulation results are presented in section 3. Conclusions are reported to section 4.

2. FREQUENCY ESTIMATION USING DYNAMICAL AR MODELS

2.1. Problem formulation

The observed signal is the sum of p noisy sinusoidal signals:

$$x_n = \sum_{i=1}^{p} A_i \sin(2\pi f_i n + \phi_i) + e_n, \tag{3}$$

where $n = 1, \ldots, N$, and e_n is an additive white Gaussian noise. The problem of estimating the frequencies f_i from the observed samples x_n, $n = 1, \ldots, N$ has received considerable attention in the signal processing literature (see for instance [1] and references therein). As a consequence, many algorithms have been studied to solve this spectral estimation problem. These algorithms include nonlinear least squares (LS), High-order Yule-Walker, Pisarenko and MUSIC methods [1]. Algorithms based on the singular value decomposition of the autocorrelation matrix have became very popular because of their high resolution properties and their insensitivity to model order overestimation. In this paper, we focus on the high-order Yule-Walker (HOYW) frequency estimation method ([1], p. 151) which is summarized below:

1. estimate the $M \times L$ ($L \geq 2p$) autocorrelation matrix of x_n defined by

   $$R_x = [c_x(1), \ldots, c_x(L)] \tag{4}$$

   $$c_x(k) = [r_x(M_0 - k), \ldots, r_x(M_0 + M - 1 - k)]^T$$

   where $r_x(i) = E[x_j x_{j-i}]$, $M = \frac{N}{2}$ and $M_0 = \frac{L+1-M}{2}$,
2. compute the SVD of the estimated autocorrelation matrix denoted $\hat{R}_x$,
3. solve the rank-truncated HOYW system of equations in the LS sense, which yields the estimated AR parameter vector denoted $\hat{a}$,

4. compute the roots of the estimated AR polynomial $A(z)$ (Z transform of the AR parameter estimates), and keep only the roots $\hat{z}_k = \hat{r}_k \exp(j2\pi \hat{f}_k)$ ensuring $\hat{f}_k \geq 0$,
5. keep the p positive frequencies $\hat{f}_k$ that correspond to the greatest peaks in the pseudospectrum

$$S(e^{j2\pi f}) = \frac{1}{|A(e^{j2\pi f})|^2}.$$

2.2. Amplitude estimation using the Kalman filter

An alternative to step 5 consists of estimating the amplitudes A_k associated to each frequency $\hat{f}_k$ by the Kalman filter as if the signal were a linear combination of sinusoidal signals.

A full description of the Kalman filter theory is described in ([4], Chapter 7). The algorithm used in this part is based on the following decomposition (see [4], p. 321):

$$s_k(n+1) = s_k(n)\cos(2\pi \hat{f}_k) + c_k(n)\sin(2\pi \hat{f}_k), \quad (5)$$

with

$$s_k(n) = A_k \sin(2\pi \hat{f}_k n + \phi_k),$$
$$c_k(n) = A_k \cos(2\pi \hat{f}_k n + \phi_k).$$

According to the model (3), the state model of the Kalman filter is defined by:

$$\begin{cases} Y(n+1) = AY(n) + B(n) \\ x_n = CY(n) + e_n \end{cases} \quad (6)$$

where:

- $Y(n) = [A_0, s_1(n), c_1(n), \ldots, s_q(n), c_q(n)]^T$ is the state vector (q is the number of positive frequencies $\hat{f}_k$),
- A is the state transition matrix (a block diagonal matrix composed of the terms $\cos(2\pi \hat{f}_k)$ and $\sin(2\pi \hat{f}_k)$ of (5)),
- $C = [1, 1, 0, 1, 0 \ldots]$ is the measurement vector,
- B and e are white Gaussian noises.

The Kalman filter classically estimates the state vector $Y(n)$ from the observed samples x_n, $n = 1, \ldots, N$. The amplitudes A_k can then be estimated as follows:

$$\hat{A}_k = \sqrt{s_k^2(N) + c_k^2(N)} \quad (7)$$

In this case, step 5 is defined as follows:

5. keep the p positive frequencies $\hat{f}_k$ corresponding to the p largest amplitudes.

2.3. Dynamical AR Models

The frequency estimation strategy detailed in section 2 is implicitly based on the equivalent ARMA model for a sum of sinusoidal signals embedded in additive noise ([1], p. 144):

$$x_n = \sum_{k=1}^{2p} a_k x_{n-k} + \epsilon_n, \quad (8)$$

where $\epsilon_n = \sum_{k=0}^{2p} a_k e_{n-k}$, and the parameters a_k depend on the amplitudes A_i, frequencies f_i and phases ϕ_i defined in (3). However, there are many other representations which could be used for frequency estimation. This paper proposes to model the signal x_n by a so-called *dynamical AR model* defined as follows:

$$x_n = \sum_{k=1}^{2p} a_{k\tau} x_{n-k\tau} + \epsilon_n, \quad (9)$$

where $\tau \in \mathbb{N}$ is an appropriate delay and $2p$ is the embedding dimension. The new HOYW frequency estimation algorithm based on (9) is similar to the algorithm of section 2.1 except the autocorrelation matrix is replaced by:

$$\hat{R}_x(\tau) = [\hat{c}_x(\tau), \hat{c}_x(2\tau), \ldots, \hat{c}_x(2p\tau)] \quad (10)$$

This strategy requires to estimate the embedding delay τ.

2.4. Embedding delay estimation

This section addresses the problem of estimating the embedding delay τ appearing in (9) and (10). This estimation has received much attention in the dynamical system literature, without any a priori information regarding the function F. The embedding delay τ is usually estimated by minimizing an appropriate correlation measure between x_n and $x_{n-\tau}$. Standard correlation measures include the autocorrelation function [5] or the mutual information [5], [6]. This paper estimates the embedding delay by minimizing the mutual information defined as [7]:

$$I(\tau) = \sum_{ij} p_{ij}(\tau) \ln p_{ij}(\tau) - 2\sum_i p_i \ln p_i \quad (11)$$

where p_i is the probability that x_n is in the ith bin of its histogram, and $p_{ij}(\tau)$ is the probability that x_n is in bin i and $x_{n+\tau}$ in bin j. A full description of the mutual information theory is described in [8].

Once τ has been estimated, the determination of the embedding dimension d can be effected by means of the so-called correlation dimension [9], the false nearest neighbors [5], [10], [6], or recurrent neural networks [11]. However, the embedding dimension d is assumed to be known in this paper.

3. SIMULATION RESULTS

This section shows that dynamical models can be useful tools for spectral estimation. In all examples, the number of sinusoidal signals p is assumed to be known, which corresponds to an embedding dimension $d = 2p$. Moreover, the embedding delay is estimated from the mutual information (11). The simulation results have been obtained by averaging the results of 100 Monte Carlo runs.

3.1. Example 1

Consider a single sinusoidal signal embedded in white Gaussian noise:

$$x_n = A_1 \sin(2\pi f_1 n + \phi_1) + e_n, \tag{12}$$

where $n = 1, ..., N$, $N = 1000$, $A_1 = \sqrt{2}$, ϕ_1 is uniformly distributed on $[0, 2\pi[$ and e_n is a white Gaussian noise. The simulation results presented in this section have been obtained for a second order dynamical AR model (9). Two cases are considered:

- τ is fixed a priori to $\tau = \text{round}\{\frac{1}{4f_1}\}$, where (round($x$) is the nearest integer of x),
- τ is estimated by the mutual information (11).

Figure 1 displays the absolute relative error between the estimated and actual frequencies averaged over 100 Monte Carlo runs, i.e.

$$\frac{\Delta f_1}{f_1} = \frac{1}{100} \sum_{i=1}^{100} \frac{|\hat{f_1}^{(i)} - f_1|}{f_1}, \tag{13}$$

where $\hat{f}_1^{(i)}$ denotes the ith frequency estimate.

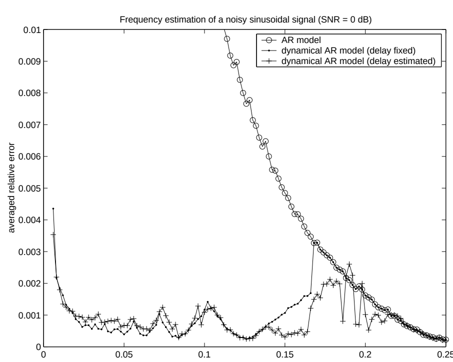

Figure 1: Relative error $\frac{\Delta f_1}{f_1}$ versus f_1 ($SNR = \mathbf{0}\ dB$)

Figure 1 shows that the second order dynamical AR model (9) provides better frequency estimates than the conventional second order AR model (8), for each frequency f_1. Note that the results have been presented for $f_1 \leq 0.25$. Indeed, the proposed methodology determines the relevant time delays associated to the observed signal. This strategy fails for small sampling rates. Figure 1 also shows that the relative error $\frac{\Delta f_1}{f_1}$ is not affected when the embedding delay is estimated.

Figure 2 shows the relative error $\frac{\Delta f_1}{f_1}$, for different values of the AR model order. This figure indicates that large AR model orders are necessary to obtain a performance (in term of relative error) which is comparable to that obtained using the second order dynamical AR model. In other words, the dynamical AR model provides a parsimonious representation of the sinusoidal signal compared to the usual AR model.

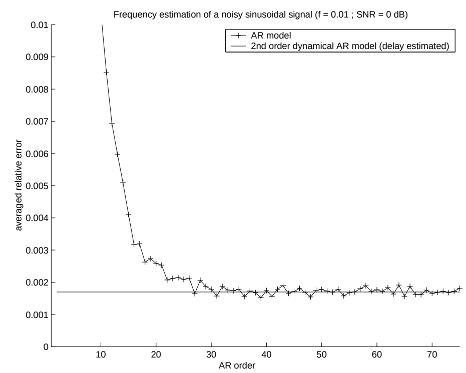

Figure 2: Relative error $\frac{\Delta f_1}{f_1}$ versus AR order ($f = \mathbf{0.01}$ and $SNR = \mathbf{0}\ dB$)

The relative error $\frac{\Delta f_1}{f_1}$ is represented in figure 3 as a function of the embedding delay τ. For SNR $= 0$ dB, the performance of the proposed frequency estimation strategy is not very sensitive to the value of τ, when τ is in a neighborhood of $\tau = \text{round}\{\frac{1}{4f_1}\}$. The results are more sensitive to τ for SNR $= -10$ dB.

3.2. Example 2

For illustration purposes, the second example considers two sinusoidal signals embedded in white Gaussian noise. However, the analysis could be generalized to more realistic situations involving more harmonics. The observed signal is defined by :

$$x_n = \sum_{k=1}^{2} A_k \sin(2\pi f_k n + \phi_k) + e_n, \tag{14}$$

where $n = 1, ..., N$, $N = 1000$, $A_1 = A_2 = 1$. The phases ϕ_1 and ϕ_2 are uniformly distributed on $[0, 2\pi[$ and e_n is a white Gaussian noise. The simulation results presented in this section have been obtained for

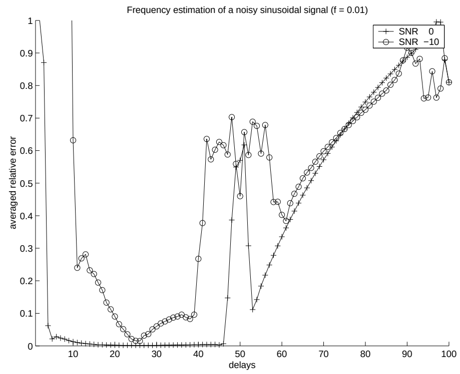

Figure 3: Relative error $\frac{\Delta f_1}{f_1}$ versus embedding delay τ ($f = \mathbf{0.01}$)

a fourth order dynamical AR model (9). The embedding delay τ is estimated by minimization of the mutual information $I(\tau)$. The frequencies f_1 and f_2 are estimated following the algorithm of section 2.3. Figure 4 displays the averaged relative error

$$\Delta f = \frac{1}{2}\left(\frac{\Delta f_1}{f_1} + \frac{\Delta f_2}{f_2}\right), \tag{15}$$

as a function of f_1, for $f_2 = 0.15$ and SNR $= 0$ dB. The fourth order dynamical model (9) provides better frequency estimates than the conventional fourth order AR model (8), for each frequency f_1.

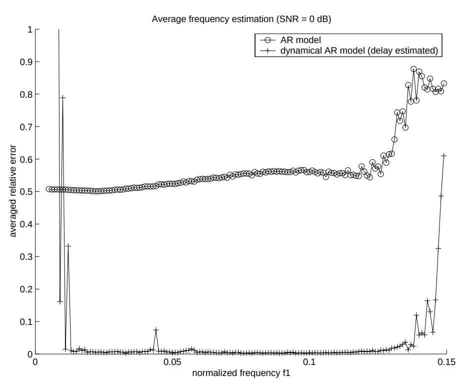

Figure 4: Relative error Δf versus f_1 ($f_2 = \mathbf{0.15}$, $SNR = \mathbf{0}\ dB$)

4. CONCLUSION

This paper addressed the problem of spectral estimation by using dynamical system theory. It is well known that the dynamics of many systems can be studied by the evolution of a state vector called state space. Takens has shown that the state vector can be characterized by measurements separated by a same delay time called embedding delay. This paper studied a so-called dynamical AR model motivated by this theory. The dynamical AR model was shown to outperform the usual AR model for the frequency estimation problem. However, the dynamical model order (or embedding dimension) was assumed to be known in this contribution. The estimation of this parameter (usually referred to as model selection problem) is currently under investigation.

5. REFERENCES

[1] P. Stoica and R. Moses, *Introduction to Spectral Analysis.* Englewood Cliffs NJ: Prentice-Hall, 1997.

[2] M. Schetzen, *The Volterra and Wiener Theories of Nonlinear Systems.* New York: Wiley, 1980.

[3] F. Takens, "Detecting strange attractors in turbulence," *Lecture Notes in Mathematics*, pp. 366–381, 1981.

[4] S. Haykin, *Adaptive Filter Theory.* Englewood Cliffs NJ: Prentice-Hall, 3nd ed., 1996.

[5] R. Hegger, H. Kantz, and T. Schreiber, "Practical implementation of nonlinear time series methods : The tisean package." available to http://www.mpipks-dresden.mpg.de/~tisean, 1998.

[6] M. G. Signorini, F. Lombardi, and S. Cerutti, "Nonlinearity parameters for the classification of high risk myocardial infarction subjects," *Computers in Cardiology*, vol. 25, pp. 545–548, 1998.

[7] H. Kantz and T. Schreiber, *Nonlinear Time Series Analysis.* Cambridge university, 1997.

[8] T. M. Cover and J. A. Thomas, *Elements of Information Theory.* New York: John Wiley & Sons, 1991.

[9] R. M. Kil, S. H. Park, and S. Kim, "Optimum window size for time series prediction," in *Proc. IEEE Engineering in Medicine and Biology Society*, vol. 4, pp. 1421–1424, 1997.

[10] J.-G. Choi, J.-K. Park, K.-H. Kim, and J.-C. Kim, "A daily peak load forecasting system using a chaotic time series," in *Proc. IEEE Intelligent Systems Applications to Power Systems*, pp. 283–287, 1996.

[11] J.-N. Hwang and E. Little, "Real time recurrent neural networks for time series prediction and confidence estimation," *IEEE Neural Networks*, vol. 4, pp. 1889–1894, 1996.