Capacity of MIMO Channels: Asymptotic Results for Correlated Fading

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ABSTRACT

This paper deals with the characterization of the asymptotic capacity of multiple input multiple output (MIMO) systems when both the number of inputs and outputs increase without bound at the same rate. The effect of the correlation at the input/output of the system is investigated and an expression for the asymptotic capacity is given in closed form as a function of the eigenvalue spread of the correlation matrix.

1 Introduction

We focus on a general MIMO channel model with M inputs and N outputs, where the output signal can be modelled as a column vector

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n},$$

with $\mathbf{H}$ an $N \times M$ channel matrix with complex entries, $\mathbf{s}$ a column vector containing the signals transmitted by the M inputs, and $\mathbf{n}$ the noise component at the input of the receiver, which is modeled as a circularly-symmetric complex Gaussian distributed random vector with zero mean and covariance $E\left[\mathbf{n}\mathbf{n}^H\right] = \sigma^2\mathbf{I}$. This signal model accommodates a variety of situations, such as a communications scenario where different users transmit to a single basestation with multiples antennas, or another where both mobile and basestation are equipped with antenna arrays. In any case, we will assume that perfect channel state information is only available at the receiver, i.e. the channel realization is perfectly known at the receiving stage. Note that this is not very far from the reality, taking into account the fact that training signals are usually sent to the receiver for that purpose[1].

It is often interesting to evaluate the actual capacity in terms of bits/sec/Hz that a MIMO system can support. Several authors have considered this problem: for instance, in [1][2] the channel matrix entries are modeled as independent circularly symmetric random variables with zero mean and unit variance and the capacity under an average transmitter power constraint is shown to be the expected value (in terms of the channel statistics) of

$$C(\beta) = \log_2 \det\left[\mathbf{I} + \frac{\beta}{M}\mathbf{H}\mathbf{H}^H\right], \tag{1}$$

where β is the average signal to noise ratio. If we denote by $\{\lambda_i\}_{i=1\ldots N}$ the eigenvalues of $N^{-1}\mathbf{H}\mathbf{H}^H$ (counting multiplicities), we can express

$$C(\beta) = \sum_{i=1}^{N} \log_2\left[1 + \frac{N}{M}\beta\lambda_i\right] \tag{2}$$

and the capacity would be obtained after averaging with respect to the eigenvalues statistics. Unfortunately, as shown in [2, Theorem 2] the general expression for such a capacity does not seem to have a closed analytical form, and the actual formula should be evaluated using numerical integration. For this reason, several authors have considered the *asymptotic* expression of such capacity, i.e. the capacity when the two dimensions of the MIMO system increase without bound at the same rate ($M, N \to \infty$, $M/N = c$ with $0 < c < \infty$). The analysis is based on random matrix theory and the study of the empirical distribution function (i.e., the eigenvalue counting function $f^N(x) = \#\{\lambda_i^N \leq x\}/N$, with λ_i^N the eigenvalues of the $N \times N$ matrix) when the matrix dimensions increase without bound. The basic concept behind this is that, as the dimensions of the random matrix grow, the empirical distribution function of the eigenvectors of some random matrix models tends to a non-random density. For the case considered above, where the entries of $\mathbf{H}$ are i.i.d. circularly symmetric Gaussian-distributed with zero mean and unit variance, it has been shown (c.f. [3]) that the empirical distribution function of the eigenvalues of $\mathbf{H}\mathbf{H}^H$ tends to the so-called Marčenko-Pastur distribution:

$$f(x) = \max\{0, 1-c\}\,\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x}1_{[a,b]}(x),$$

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[1] Of course, the presence of these training signals should be taken into account in the evaluation of the capacity. We obviate this for simplicity.

where $a = (\sqrt{c}-1)^2$ and $b = (\sqrt{c}+1)^2$. From the weak convergence of the measures $f^N(x)$ towards $f(x)$ follows the convergence in probability of $C(\beta)$ in (2) to $N \cdot C^*(\beta)$, where

$$C^*(\beta) = \int_{-\infty}^{\infty} \log_2\left(1+\frac{\beta}{c}x\right) f(x)dx, \qquad (3)$$

which is a non-random quantity that will referred to as asymptotic capacity per antenna. This had been pointed out in several papers, but it was not until [4] that a closed expression for this asymptotic capacity was given. In this paper, we try to generalize that expression including the effect of correlation between the entries of $\mathbf{H}$ at either the transmitter or the receiver.

2 Capacity under correlated fading model

The evaluation of the MIMO capacity under a correlated fading model has been considered by several authors. The usual approach, e.g. [5][6] is to propose a scattering model and evaluate the capacity according to that physical model. More recently, in [7], the asymptotic capacity under correlated model at both the transmit and receive ends was studied. The capacity was shown to be the solution of a functional equation depending on the general inter-element fading correlation functions. Here, we try to give a closed expression for that asymptotic capacity instead of a functional equation, at the expense of some loss of generality of the final expressions.

We will model the channel matrix $\mathbf{H}$ in (1) as

$$\mathbf{H} = \mathbf{C}^{1/2}\mathbf{U},$$

where $\mathbf{U}$ is an $N \times M$ matrix with i.i.d. circularly symmetric complex Gaussian entries (zero mean, unit variance) and $\mathbf{C}$ is an $N \times N$ Hermitian Toeplitz matrix that contains, at each of its entries, the fading correlation between two receive elements[2] [Here $\mathbf{C}^{1/2}$ stands for the Hermitian Square Root of $\mathbf{C}$]. Note that the Toeplitz assumption restricts the actual configuration of the system to a linear structure (for instance, a uniform linear array if we consider the reception with multiple antennas). On the other hand, the expression of the capacity in (1)-(3) is symmetric from the transmit/receive point of view, and that replacing c by c^{-1} in these expressions one would obtain the asymptotic capacity of the reciprocal channel.

From now on, and to avoid an unnecessary notation complication, we consider the capacity normalized by $\log 2$, which will be denoted $\bar{C}(\beta) = C^*(\beta)\log 2$. Consider thus the expression of the asymptotic capacity (3) properly normalized and take derivatives with respect to the signal to noise ratio:

$$\frac{\delta \bar{C}(\beta)}{\delta \beta} = \frac{1}{\beta}\int_{-\infty}^{\infty} f(x)dx - \frac{c}{\beta^2}\int_{-\infty}^{\infty} \frac{1}{\frac{c}{\beta}+x} f(x)dx$$
$$= \frac{1}{\beta} + \frac{c}{\beta^2} m\left(-\frac{c}{\beta}\right) \qquad (4)$$

with $m(z)$ the Stieltjes transform of the density $f(x)$,

$$m(z) = \int_{-\infty}^{\infty} \frac{1}{z-x} f(x)dx. \qquad (5)$$

Thus, obtaining the Stieltjes transform of the asymptotic density of eigenvalues of $\mathbf{HH}^H$, one can obtain the actual expression of the asymptotic capacity per antenna integrating (4) and forcing $\bar{C}(0) = 0$. In order to obtain $m(z)$, we will make use of results from Free Probability Theory [8] and, in particular, the theory of multiplicative free convolution of measures. That is a useful tool that describes the eigenvalue distribution function of a product of infinite-dimensional matrices as a function of the eigenvalue distribution of each matrix (see [8] for more details). The key observation in our problem is that, as shown in [9, Corollary 4.3.8], when the dimensions of the problem increase without bound, the two matrices $\{\mathbf{C}\}$, $\{\mathbf{UU}^H\}$ become asymptotically free (a concept similar to independence but applied to non-commutative random variables). Under these circumstances, one can obtain the asymptotic eigenvalue distribution function of the product of the two matrices from the asymptotic eigenvalue distribution of each one. This is achieved by means of the so-called S-transform, which can be obtained from the Stieltjes transform defined above as

$$S(z) = \frac{1+z}{z}\chi(z),$$

with $\chi(z)$ the formal inverse of $\psi(z)$ (i.e. $\psi(\chi(z)) = z$),

$$\psi(z) = z^{-1}m(z^{-1}) - 1$$

and $m(z)$ as in (5). The importance of the S-transform stems from the fact that the S-transform of the density of eigenvalues of a product of two freely independent random matrices is equal to the product of S-transforms of each matrix, in our case

$$S(z) = S_U(z)S_C(z),$$

where $S_U(z)$ and $S_C(z)$ are the S-transforms of the densities of eigenvalues of $\{\mathbf{UU}^H\}$ and $\{\mathbf{C}\}$ respectively. Now, it can easily be shown that the S-transform of the Marčenko-Pastur law is given by

$$S_U(z) = \frac{1}{z+c},$$

so that the S-transform of the final density takes the form

$$S(z) = \frac{1}{z+c}S_C(z) \qquad (6)$$

[2] Note that, if the signal model corresponds to a multiple antenna system, the Toeplitz structure of $\mathbf{C}$ is imposes a linear interelement separation constraint

with $S_C(z)$ the S-transform of the density of eigenvalues of $\mathbf{C}$. At this point, one should choose a model for the fading correlation, obtain the S-transform of the corresponding asymptotic eigenvalue distribution $S_C(z)$, insert it into (6) and undo the transformation to get to the corresponding Stieltjes transform $m(z)$. From that point, one would obtain the asymptotic normalized capacity expression by integration in (4). The only obstacle here is the fact that in order to get to the Stieltjes transform from the S-transform, one must find the formal inverse of $\chi(z) = \frac{z}{1+z}S(z)$, and that can only be done with relative ease of manipulation when $\chi(z)$ is a quotient of second-order polynomials in z. Of course, a closed expression can be given for that inverse even when $\chi(z)$ is a quotient of third and fourth order polynomials, but the analytical manipulation of the capacity expressions turns out to be very difficult. Thus, it seems advisable to model the asymptotic density of eigenvalues of $\mathbf{C}$ with a density that yields a first order polynomial as S-transform. Such density can be described as a tilted semi-circular law and takes the form[3]

$$f_C(x) = \frac{2\sqrt{\sigma_1\sigma_2}}{\left(\sqrt{\sigma_2} - \sqrt{\sigma_1}\right)^2} \frac{\sqrt{(x-\sigma_1)(\sigma_2 - x)}}{\pi x^2} \quad (7)$$

with support $[\sigma_1, \sigma_2]$. Note that we have two degrees of freedom to fix the support: one will be used to normalize $\mathbf{C}$ (setting for instance its diagonal entries to 1), while the other will be used as a correlation parameter related to the eigenvalue spread of $\mathbf{C}$ (namely, a zero eigenvalue spread will be related to a zero-correlation model $\mathbf{C} = \mathbf{I}$ while higher spreads will be associated with higher correlation modes). In a real situation, the observed eigenvalue spread should be used to fix these two parameters. For example, assume that we want to model an exponentially decaying correlation model, i.e. $\{\mathbf{C}^{\exp}\}_{i,j} = \rho^{|i-j|}$, with $0 \le \rho < 1$ the correlation parameter. It is well known that the asymptotic density of eigenvalues of a Toeplitz matrix can be extracted from the Fourier transform of the sequence of its skew-diagonal entries (see [10] for details). In our case, it can be shown that the density of eigenvalues of $\mathbf{C}^{\exp}$ has as support $\left[\frac{1-\rho}{1+\rho}, \frac{1+\rho}{1-\rho}\right]$. Thus, if we want to model an exponentially decaying fading model, we can fix

$$\frac{\sigma_2}{\sigma_1} = \left(\frac{1+\rho}{1-\rho}\right)^2$$

so that the distribution in (7) generates the same eigenvalue spread as the exponential model, and also $\sigma_1\sigma_2 = 1$ to guarantee[4] that the diagonal elements of $\mathbf{C}$ are equal to 1. In Figure 1 we represent the two asymptotic eigenvalue distributions for different values of ρ. The conclusion that should be drawn is that even though we are imposing a certain shape of the eigenvalue distribution, we can still accomodate a more realistic scenario.

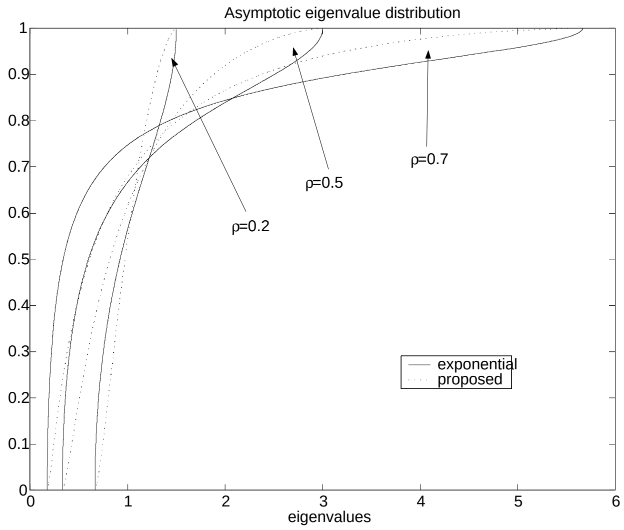

Figure 1: Asymptotic eigenvalue distribution for the exponential and proposed models.

Now, getting back to the derivation of the asymptotic capacity, the S-transform of the distribution $f_C(x)$ takes the form (see [11] for the derivation)

$$S_C(z) = \frac{1}{\sqrt{\sigma_1\sigma_2}} - \frac{\left(\sqrt{\sigma_2} - \sqrt{\sigma_1}\right)^2}{4\sigma_1\sigma_2} z \overset{\text{def}}{=} 1 - \mu z,$$

where we have assumed that $\sigma_1\sigma_2 = 1$ so that the diagonal elements of $\mathbf{C}$ are equal to one. Note that here μ is a parameter that controls the degree of fading correlation: $\mu = 0$ would correspond to an uncorrelated fading model whereas $\mu > 0$ introduces some correlation (the higher μ the more correlated the fading is). Inserting this expression in (6) and using the formulas above, one easily gets to the Stieltjes transform of the final global matrix

$$m(z) = \frac{z + 2z\mu + 1 - c + \sqrt{[z - (1+c)]^2 - 4c(1+\mu z)}}{2z(1+\mu z)}.$$

From this point, one can obtain a closed expression for the asymptotic capacity per antenna element integrating (4) with respect to the signal to noise ratio and forcing $C^*(0) = 0$. Due to the obvious space limitations, we avoid the derivation of the final expression here. The interested reader is referred to [11] for the complete proof. The capacity per antenna takes the form

$$C^* = \log_2\left[\frac{w(\beta)}{\beta^{-1}}\right] + \frac{1}{\mu}\log_2|1 - \mu c \cdot v(\beta)| + (c-1)\log_2\left|\frac{1}{u(\beta)}\right|,$$

[3] It is important that the density chosen corresponds to a probability measure, as it is the case. Otherwise, one can not use the theory of free convolution of measures

[4] This can be imposed forcing the first moment of the eigenvalue distribution in (7) to 1.

where

$$u(\beta) = \frac{c + (1-c)\,\beta - \sqrt{R} + 2\mu c^2}{2c\,(\beta - \mu c)}$$

$$v(\beta) = \frac{c + \beta\,(1+c) - \sqrt{R} + 2\beta c\mu}{2c\beta\,[1 + (1+c)\,\mu + c\mu^2]}$$

$$w(\beta) = \frac{\phi\left[c + \beta\,(1+c) + \sqrt{R}\right] - 2c\mu\psi}{2c\beta\,[1 + (1+c)\,\mu + c\mu^2]}$$

and

$$R = [c + (1+c)\,\beta]^2 - 4c\beta\,(\beta - c\mu)$$
$$\phi = 1 + (1+c)\,\mu$$
$$\psi = \beta - \mu c.$$

Note that the expression obtained is quite similar to the one presented in [4], which can be recovered setting $\mu = 0$. In Figure 2 we represent the asymptotic capacity per antenna as a function of the signal to noise plus interference ratio for different values of the correlation parameter μ. We also show the corresponding ρ that gives the same eigenvalue spread of the correlation matrix in the exponential model. Note that the degradation of the asymptotic capacity does not seem very serious for typical values of the correlation parameter ρ.

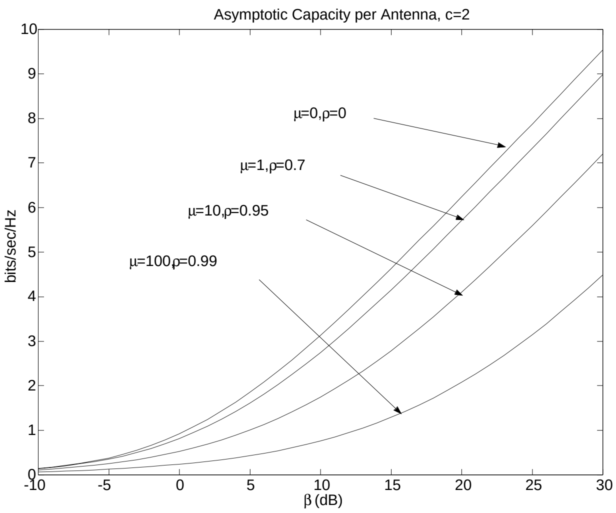

Figure 2: Asymptotic capacity as a function of the Signal to Noise ratio for different values of the parameter μ.

3 Conclusions

We have presented a closed form expression of the asymptotic capacity of a MIMO system assuming correlated fading at one side of the communications link. The expression is obtained modeling the asymptotic eigenvalue distribution of the fading correlation matrix as a tilted semi-circular law. The length of the support of this distribution law is a direct measure of the eigenvalue spread of the correlation matrix, which is in turn related to the degree of fading correlation. This way, we have a simple way of relating the channel capacity with the fading correlation through a single parameter.

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