AN HOS BASED STATISTICAL TEST FOR MULTIPLICATIVE NOISE DETECTION

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ABSTRACT

This paper addresses the problem of detecting the presence of multiplicative noise, when the information process can be modelled by a parametric AR process. A suboptimal detector based on higher-order cumulants (HOC) is studied. This detector consists of filtering the data by the fitted AR filter. HOC of the residual data are shown to be efficient for the detection problem.

1 INTRODUCTION AND PROBLEM FORMULATION

Many signal processing applications deal with multiplicative noise models. For instance, harmonic signals corrupted by multiplicative and/or additive noise have been intensively studied for random communication models (fading channels), Sonar, Doppler systems. Harmonic signals, as well as many other processes (including processes with continuous spectra), can be modelled accurately by AR processes. The signal parameter estimation problem then reduces to the estimation of AR parameters. Appropriate techniques for AR estimation in additive noise environment can fail dramatically in multiplicative noise environment, and vice-versa. This paper studies a statistical test for detecting the presence of multiplicative noise, when the information process can be modelled by an AR process. A similar problem was studied in [7]. However, the proposed algorithm was restricted to discerning random from constant amplitude harmonics. The detection problem is the following composite hypothesis testing problem:

$$\begin{aligned} H_0: \ y(n) = y_0(n) &\triangleq x(n) + b_0(n) \\ H_1: \ y(n) = y_1(n) &\triangleq e(n)x(n) + b_1(n) \end{aligned} \qquad (1)$$

$x(n)$ is an AR(p) process driven by an iid sequence $g(n)$:

$$x(n) = -\sum_{j=1}^{p} a_j x(n-j) + g(n)$$

$b_0(n)$ and $b_1(n)$ are the additive (possibly colored) Gaussian noises. $e(n)$ is an iid multiplicative noise, whose distribution is unknown. The sequences $g(n)$, $b_0(n)$, $b_1(n)$, and $e(n)$ are mutually independent. Optimal detectors based on the Neyman-Pearson criterion can be derived, when statistical properties concerning signal and noises are avalaible. Such a detection problem was studied in [6], for the detection and the classification of signals corrupted by additive and multiplicative noise. However, the Neyman-Pearson detector (NPD) can lead to intractable computation and of course, the signal and noises pdf's have to be known. Consequently, the NPD cannot be derived for the multiplicative noise detection problem, since $x(n)$ and $e(n)$ have unknown pdf's. This paper studies a suboptimal detector based on Higher-Order Cumulants (HOC).

When the AR process $x(n)$ is Gaussian (i.e. $g(n)$ Gaussian), the observed signal $y(n)$ is Gaussian under hypothesis H_0 and non-Gaussian under hypothesis H_1. Any Gaussianity test such as Hinich test [4] can then be used for the multiplicative noise detection problem . Unfortunately, when the AR process $x(n)$ is non-Gaussian, Gaussianity tests are not relevant for our problem, since the observed signal $y(n)$ is non-Gaussian under both hyopthesis. This paper studies a suboptimal multiplicative noise detector based on HOC. This detector does not require a known distribution for the AR process $x(n)$. However, it is restricted to non-Gaussian process $x(n)$. The proposed detection algorithm is a four-step procedure, which can be summarized as follows:

1. Estimate the AR parameters using HOC computed at appropriate lags.
2. Filter the data $y(n)$ by the estimated AR polynomial to form the residual process $z(n)$.
3. Compute the HOC of $z(n)$ at non-zero lags.
4. Compare these HOC with zero, and take a decision.

2 AR PARAMETER ESTIMATION

Denote $\mu_u = E[u(n)]$ the mean of a random process $u(n)$, and $C_k^u(\rho)$ its kth-order cumulant computed at lag $\rho = (\rho_1, \rho_2 \ldots, \rho_{k-1})$. It is well known that higher-order cumulants ($k > 2$) are blind to Gaussian processes. Consequently, the multiplicative noise detection problem (1) yields:

$$\begin{array}{l} H_0: C_k^y(\rho) = C_k^{y_0}(\rho) = C_k^x(\rho), \\ H_1: C_k^y(\rho) = C_k^{y_1}(\rho) = C_k^{ex}(\rho), \end{array} \quad (2)$$

where ex denote the process $e(n)x(n)$. HOC of multiplicative processes were studied in [5]. In particular, the following property was obtained:

$$C_k^{ex}(\rho) = \mu_e^k C_k^x(\rho) \; \forall \rho \in S_k \quad (3)$$

where

$$S_k = \{\rho / \rho_i \neq 0 \; \forall i \in \{1, \ldots, k-1\}, \rho_i \neq \rho_j \text{ for } i \neq j\}$$

This section assumes $\mu_e \neq 0$. The proposed algorithm can be modified for zero-mean multiplicative noise (see section 5). Eq. (3) shows that $kth-$order cumulants of $y(n)$ computed at specific lags ($\rho \in S_k$) are proportional under both hypotheses H_0 and H_1. Consequently, AR parameters can be determined under both hypotheses from the well-known $kth-$order Yule-Walker equations:

$$\sum_{j=1}^{p} a_j C_k^y(\rho_1 - j, \rho_2, \ldots, \rho_{k-1}) = -C_k^y(\rho), \forall \rho_1 > 0 \quad (4)$$

However, lags ρ have to be chosen carefully such that:

A) eq. (3) is satisfied for any $\rho = (\rho_1 - j, \rho_2, \cdots, \rho_{k-1})$, $\forall j = 0, \ldots, p$, i.e. $\rho \in S_k$, which implies $\rho_1 \neq j$, $\forall j \in \{0, \ldots, p\}$ and $\rho_1 \neq \rho_l + j$, $\forall j \in \{0, \ldots, p\}$, $\forall l \in \{2, \ldots, k-1\}$. For example, we could choose

$$\begin{array}{rcl} \rho_1 & = & p+1, \ldots, p+1+m \\ \rho_l & = & p+m+l, \; l \in \{2, \ldots, k-1\} \end{array}$$

for some $m \geq p$.

B) the Toeplitz matrix obtained by concatenation of (4) is full-rank.

Condition A) can be satisfied by removing the lines involving $C_k^y(\rho)$, $\rho \notin S_k$ in eq. (4). Condition B) can be fulfilled by collecting $m = p(p+1)$ cumulant slices (see [2] for a similar problem).

Define $\underline{C}_k(p)$ as the Toeplitz matrix whose first row is

$$(C_k^x(p, \rho_2, \ldots, \rho_{k-1}), \ldots, C_k^x(1, \rho_2, \ldots, \rho_{k-1}))$$

and first column

$$(C_k^x(p, \rho_2, \ldots, \rho_{k-1}), \ldots, C_k^x(p+m, \rho_2, \ldots, \rho_{k-1}))^T$$

Denote

$$\begin{array}{rcl} \underline{c}_k(p) & = & (C_k^x(p+1, \rho_2, \ldots, \rho_{k-1}), \ldots, \\ & & \ldots, C_k^x(p+1+m, \rho_2, \ldots, \rho_{k-1}))^T \end{array}$$

AR parameters are obtained from the following matrix equation:

$$\underline{C}_k(p).(a_1, \ldots, a_p)^T = -\underline{c}_k(p) \quad (5)$$

Several remarks are now appropriate:

1. In practice, sample cumulants replace the theoretical cumulants for AR parameter estimation.
2. eq. (5) was derived assuming that p is known. However, eq. (5) can be easily modified to yield unique AR parameters when an upper bound $\overline{p}$ of the AR order is avalaible.

3 MA DETECTOR

Denote $z_i(n)$ the output of the FIR filter with Z-transform $A(z) = \sum_{k=0}^{p} a_k z^{-k}$ driven by $y_i(n)$. The detection problem can be rewritten as:

$$\begin{array}{l} H_0: z(n) = z_0(n) = g(n) + \sum_{j=0}^{p} a_j b_0(n-j) \\ H_1: z(n) = z_1(n) = \sum_{j=0}^{p} a_j e(n-j) x(n-j) \\ \qquad + \sum_{j=0}^{p} a_j b_1(n-j) \end{array} \quad (6)$$

Eq. (6) shows that 1) $z_0(n)$ is the sum of a Gaussian MA(p) sequence and an iid non-Gaussian sequence $g(n)$. Consequently, the kth-order cumulants ($k > 2$) of $z_0(n)$ are zero except at lag $\rho = 0$; 2) $z_1(n)$ is the sum of a Gaussian MA(p) sequence and a non-Gaussian and non-linear sequence, whose kth-order cumulants are non-zero for a specific set of lags.

It is well-known that the parameters $(\theta_j)_{j=0,\ldots,p}$ of a non-Gaussian $MA(p)$ process $u(n)$ can be uniquely identified from the kth-order cumulant vector

$$\mathbf{C}_k^u \triangleq [C_k^u(p, j, 0, \ldots, 0) \; / \; j = 0, \ldots, p]^T$$

(since $\theta_j = \frac{C_k^u(p,j,0,\ldots,0)}{C_k^u(p,0,0,\ldots,0)}$, $1 \leq j \leq p$, $\theta_0 = 1$). Using basic properties of kth-order cumulants, the following binary hypothese testing problem can be considered:

$$\begin{array}{l} H_0: \mathbf{C}_k^z = \mathbf{C}_k^{z_0} = 0 \\ H_1: \mathbf{C}_k^z = \mathbf{C}_k^{z_1} \neq 0 \end{array} \quad (7)$$

Denote $\widehat{\mathbf{C}}_k^z$ the vector obtained by replacing the true cumulants in $\mathbf{C}_k^z$ by their usual estimates computed with N samples. The asymptotic statistical behavior of the HOS vector estimate $\widehat{\mathbf{C}}_k^z$ is:

$$\begin{array}{c} \sqrt{N}\widehat{\mathbf{C}}_k^z \sim \mathcal{N}(0, \Sigma_0) \\ \sqrt{N}\left(\widehat{\mathbf{C}}_k^z - \mathbf{C}_k^{z_1}\right) \sim \mathcal{N}(0, \Sigma_1) \end{array} \quad (8)$$

where Σ_0 and Σ_1 are two matrices independent of N. The asymptotic statistics of $\widehat{\mathbf{C}}_k^z$ can be used to derive kth-order cumulant based likelihood ratio detectors, when parameters Σ_0, Σ_1, and $\mathbf{C}_k^{z_1}$ are known. When these parameters are unknown, the multiplicative noise detection problem defined in (7) is a composite hypothesis test. Assume that M independent realizations of $\widehat{\mathbf{C}}_k^z$ (denoted $\left(\widehat{\mathbf{C}}_{k,j}^z\right)_{j=1,\ldots,M}$) are available. These M

measurements can be obtained from one single signal by segmentation. This segmentation procedure consists of considering a N-sample signal as M segments of K samples (with $N = MK$). Define $\overline{\mathbf{C}}$ and $\widehat{S}$ the sampled mean and covariance matrix of the sequence $\left(\widehat{\mathbf{C}}_{k,j}^{z}\right)_{j=1,\ldots,M}$:

$$\overline{\mathbf{C}} = \frac{1}{M}\sum_{j=1}^{M}\widehat{\mathbf{C}}_{k,j}^{z}, \quad (9)$$

$$\widehat{S} = \frac{1}{M-1}\sum_{j=1}^{M}\left(\widehat{\mathbf{C}}_{k,j}^{z}-\overline{\mathbf{C}}\right)\left(\widehat{\mathbf{C}}_{k,j}^{z}-\overline{\mathbf{C}}\right)^{T} \quad (10)$$

Using the asymptotic normality of vector $\left(\widehat{\mathbf{C}}_{k,1}^{z},\ldots,\widehat{\mathbf{C}}_{k,M}^{z}\right)^{T}$, the generalized likelihood ratio detector for the detection problem (7) is defined by:

$$H_0 \text{ rejected if } T^2 = M\overline{\mathbf{C}}^{T}\widehat{S}^{-1}\overline{\mathbf{C}} > \lambda \quad (11)$$

λ is a threshold which can be determined from the distribution of T^2 under the null hypothesis and a fixed probability of false alarm (PFA). Giri [3] showed that the statistic $\frac{M-\tau}{(M-1)\tau}T^2$ has an F-distribution with $(\tau, M-\tau)$ degrees of freedom, under the null hypothesis, where τ denotes the size of vector $\overline{\mathbf{C}}$ ($\tau = p+1$). Moreover, the distribution of $\frac{M-\tau}{(M-1)\tau}T^2$ is a non-central F-distribution with $(\tau, M-\tau)$ degrees of freedom and non-central parameter $\nu = MK\left(\mathbf{C}_k^{z_1 T}\Sigma_1^{-1}\mathbf{C}_k^{z_1}\right)$, under hypothesis H_1. The probability of detection (PD) can then be obtained from the PFA as follows:

$$\lambda = \frac{(M-1)\tau}{M-\tau}f^{-1}\left[\tau, M\text{-}\tau, 0\right](1-PFA)$$

$$PD = 1 - f\left[\tau, M\text{-}\tau, \nu\right]\left(\frac{M-\tau}{(M-1)\tau}\lambda\right) \quad (12)$$

where $f\left[d_1, d_2, \mu\right](.)$ denotes the cumulative distribution function of a non-central F-distribution with d_1 and d_2 degrees of freedom and non-centrality parameter μ, and $f^{-1}\left[d_1, d_2, \mu\right](.)$ its inverse.

It is interesting to note that eq. (6) was derived assuming that the AR parameter vector $\underline{a}$ is known. In practical applications, this vector is unknown and has to be estimated using the procedure described in section 2. The kth-order cumulant vector $\mathbf{C}_k^z$ is then estimated by cumulants of the output of the FIR filter with Z-transform $\widehat{A}(z) = \sum_{k=0}^{p}\widehat{a}_k z^{-k}$ driven by $y_i(n)$. Consistency of sample cumulant estimators guarantees the convergence with probability 1 of $\widehat{a}_k$ to a_k [2].

4 CASE OF ZERO-MEAN MULTIPLICATIVE NOISE

If the multiplicative noise $e(n)$ is zero-mean, the hypothesis testing problem (2) can be expressed as:

$$\begin{array}{ll} H_0 : C_k^y(\rho) = C_k^{y_0}(\rho) = C_k^x(\rho), & \forall\rho \in S_k \\ H_1 : C_k^y(\rho) = C_k^{y_1}(\rho) = 0\ , & \forall\rho \in S_k \end{array} \quad (13)$$

Since $C_k^y(\rho)$ is asymptotically Gaussian, and $C_k^x(\rho) \neq 0$, the problem (13) is similar to (7), with interchanged hypotheses. Thus, the AR parameter estimation and the filtering are not necessary, since a statistical test can be developed directly on the cumulants of the received data $y(n)$ using a signal segmentation, as for problem (7).

5 SIMULATION RESULTS

Many simulations have been performed to validate the theoretical results. Fig.'s 1, 2 and 3 show the first detector Receiver Operational Characteristics (ROC's) for three $AR(4)$ processes with poles $\rho_1 = [0.1e^{\pm j\pi/4}; 0.3e^{\pm j\pi/3}]$, $\rho_2 = [0.15e^{\pm j\pi/4}; 0.35e^{\pm j\pi/3}]$, $\rho_3 = [0.2e^{\pm j\pi/4}; 0.4e^{\pm j\pi/3}]$. The number of samples is $N = 10000$, and the number of slices is $M = 7$. The input sequence is a zero-mean iid exponentially distributed process $g(n)$, with variance $\sigma_g^2 = 1$. $b_0(n)$ and $b_1(n)$ are zero-mean Gaussian processes with unit variance. The multiplicative noise $e(n)$ is an exponentially distributed sequence such that $\mu_e = 1$.The signal to noise ratio between the signal $x(n)$ and the multiplicative noise $e(n)$ is $SNR_{x,e} = 0dB$ (fig. 1), $SNR_{x,e} = 10dB$ (fig. 2) and $SNR_{x,e} = 20dB$ (fig. 3). It can be noted that the detector performance is not very sensitive to $SNR_{x,e}$. Fig. 4 shows the second detector ROC's (with zero-mean multiplicative noise), for the same $AR(4)$ processes, except $\mu_x = 1$, $\mu_e = 0$, and $N = 1000$. The performance is similar to those obtained in fig's 1, 2, and 3. However, it is interesting to emphasize that the last results have been conducted with $N = 1000$. Moreover, it should be noted that the second detector ROC's are unsensitive to the multiplicative noise parameters. Indeed, eq. (12) show that the theoretical performance only depends on parameter ν (which does not depend on the multiplicative noise parameters, since hypotheses H_0 and H_1 have to be interchanged).

6 CONCLUSION

This paper studied two multiplicative noise detectors. The information signal was modelled by a non Gaussian AR process. **In the case of a non-zero-mean multiplicative noise**, the AR parameters were first estimated using appropriate higher-order cumulants. The data were then filtered by the fitted AR filter. An HOS-based detector on the residual data was finally developed. **In the case of a zero-mean multiplicative noise**, the detection was achieved using the cumulants of the received signal (the AR estimation procedure was not necessary). The paper was restricted to iid multiplicative noise. However, the detectors can be generalized to colored multiplicative noise [1]. Moreover, it can be noted that the proposed algorithms can be adapted to ARMA processes. Indeed, the AR estimation can be conducted using cumulants computed at different lags. After filtering, the residual process under hypothesis H_0

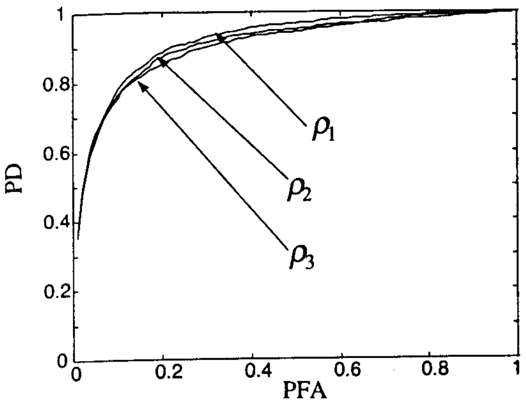

Figure 1: ROC's for an AR(4) process with poles ρ_1, ρ_2, ρ_3 - $SNR_{x,e} = 0$.

is the sum of two MA processes, whose cumulants are identically zero except on a finite support. This property can then be used for detection and ARMA parameter estimation (problem similar to (7)).

7 References

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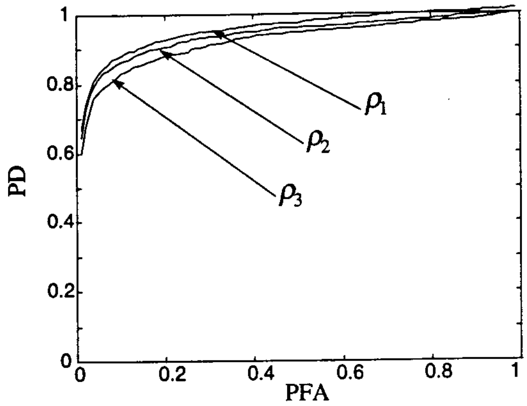

Figure 2: ROC's for an AR(4) process with poles ρ_1, ρ_2, ρ_3 - $SNR_{x,e} = 10$.

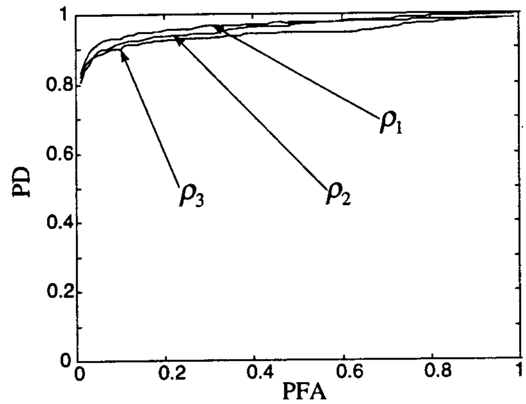

Figure 3: ROC's for an AR(4) process with poles ρ_1, ρ_2, ρ_3 - $SNR_{x,e} = 20$.

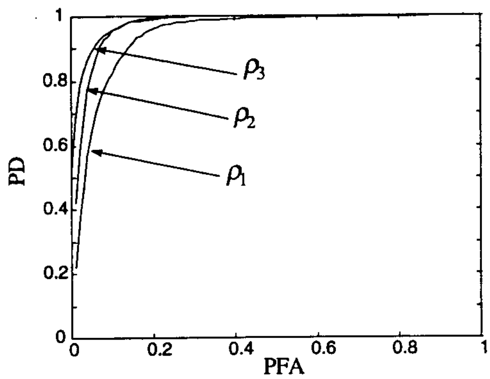

Figure 4: ROC's for the zero-mean multiplicative noise detector for an AR(4) process with poles ρ_1, ρ_2, ρ_3 - $N = 1000$ - $SNR_{x,e} = 10$.