LIMITATIONS OF THE ARMA w-SLICE METHOD USING A LINEAR COMBINATION OF HIGHER-ORDER CUMULANTS

Damjan Zazula
Faculty of EE and Computer Science
Smetanova 17
2000 MARIBOR
S L O V E N I A
Phone: +386 62 220 74 80
E-mail: zazula@uni-mb.si

Jean-Luc Vuattoux
Institut de Recherche en Cybernétique de Nantes
1, rue de la Noë
44321 NANTES cedex 03
F R A N C E

ABSTRACT

The paper analyses autoregressive moving-average (ARMA) system identification method. This method belongs to higher-order statistical methods of a linear algebra type, showing a unique feature that the method works for any kind of model, i.e. MA, AR, or ARMA, and that the model's order (p,q) need not be known in advance. Our analyses of the ARMA approach proved that there is a class of systems not being identifiable. All these systems having poles $s_i;\ i=1,\ldots,p$, and at least one zero of type of $(s_{i_1}s_{i_2}\cdots s_{i_{k-1}})^{-1};\ i_1,i_2,\ldots,i_{k-1}\in(1,\ldots,p)$ cannot be identified by ARMA w-slices using k^{th}-order cumulants, no matter whether with single cumulants, linear combination of cumulants, 1-D slices, or multidimensional slices. The analytical result is backed by simulations. Finally, we propose a procedure of verification of ARMA identifiability and an extension of ARMA w-slice in order to assure the identifiability.

1 INTRODUCTION

During the last few years, the system identification methods based on higher-order statistics (HOS) have gained a lot of attention in many applications. A variety of approaches has been proposed, basically divided into closed-form, linear algebra, and optimisation solutions [1]. Nevertheless, none of them exhibit thoroughly acceptable behaviour, especially multiple constraints diminish their value.

Recently, a new linear algebra solution has been introduced under the name of w-slice methods [2]. It has been found robust and applicable to any type of the model, i.e. MA [3], AR [4], and ARMA [5], having an order of (p,q). In case of MA and AR models, identifiablility was proved [2, 3, 4], while the same property was supposed in case of the ARMA models [5]. Nevertheless, a class of ARMA models exists that cannot be identified by the w-slice approach, as we will show here.

Recapitulate briefly the basic formulation of the w-slice method [5]:

$$c_w(\tau) = w_2C_{2,y}(\tau)+\sum_{\tau_2=-M}^{N}w_3(\tau_2)C_{3,y}(\tau,\tau_2)+\sum_{\tau_2=-M}^{N}\sum_{\tau_3=-M}^{\tau_2}w_4(\tau_2,\tau_3)C_{4,y}(\tau,\tau_2,\tau_3)+\cdots \quad (1)$$

where $C_{k,y}$ stands for the k^{th}-order cumulant of signal y (the observed system output) and w_k for the corresponding weights.

The identification algorithm is based on two steps: first, the weights are calculated according to the chosen set of cumulants, and afterwards, the weights are used in estimation of a causal system response, $h(n)$. Practical implementation of this approach is preferably, because of lower computational complexity, used with 1-D slices of either a single cumulant or a combination of several higher-order cumulants. In the sequel, we show that when the w-slice method fails with a single cumulant, there is no linear combination of this cumulant with any other higher-order cumulants that would guarantee identifiability (Section 2). In Section 3, we develop an identifiability verification procedure and an extension of ARMA w-slices in order to assure the identifiability.

2 IDENTIFIABILITY USING w-SLICES WITH A SUM OF SEVERAL HIGHER-ORDER CUMULANTS

We have investigated the identifiability of ARMA w-slice method using 1-D slices of single higher-order cumulants and proved the class of ARMA(p,q) models that cannot be identified through this approach [8]. We modified Eq. (1) to:

$$c_w(\tau) = \sum_{\tau_2=-M}^{N}w_k(\tau_2)C_{k,y}(\tau,\tau_2,0,\ldots,0)$$

$$= \sum_{n=0}^{\infty} h(n+\tau) g_k(n) \quad (2)$$

where

$$\begin{aligned} g_k(n) &= \gamma_{k,x} h^{k-2}(n) \sum_{\tau_2=-M}^{N} w_k(\tau_2) h(n+\tau_2) \\ &= \gamma_{k,x} h^{k-2}(n) \sum_{j=0}^{M+N} u_k(j) h(n+N-j) \quad (3) \end{aligned}$$

and $\gamma_{k,x}$ denotes k^{th}-order cumulant of the system's input non-Gaussian i.i.d. noise and the weights, $u_k(j) = w_k(N-j)$. These weights are calculated according to the chosen limits, M and N, out of a linear system with K equations [7]. We have proved that a certain form of the weights, $u_k(j)$, is mandatory if the proper solution for the system response, $h(n)$, is to be obtained via the w-slice method. This form yields:

$$u_k(n) = \sum_{i=0}^{p} a(i) r(n-i); \; n = 0, \ldots, N+M \quad (4)$$

where $a(i)$ designates the system AR coefficients and

$$r(n) = \begin{cases} \frac{1}{\gamma_{k,x} b(q)}, & n = N-q \\ \text{arbitrary value}, & 0 \le n \le N-q-1 \\ 0, & \text{otherwise} \end{cases} .$$

It has been believed that the weights of the necessary form appear if a solution of $c_w(\tau) = 0$ for $\tau < 0$ is forced. The authors of [5] and [6] based their proof on an erroneous assumption; namely, they claim that if $c_w(\tau) = 0$ for $\tau < 0$ then the same is valid for $g_k(n)$. They derived the corresponding conditions for M, N and K believing that these conditions are sufficient. However, they are only necessary for the mandatory form of the weights $u_k(j)$, as we showed in [7].

The main question in conjunction with the w-slice method is whether the solution for the weights obtained via the anticausal cumulant slices and the condition $c_w(\tau) = 0, \; \tau \le 0$ fulfil the necessary form of (4). In order to find out the answers, we observed the anticausal part of Eq. (3) in the z-domain:

$$1 = \left[H(z) G_k(z^{-1}) \right]_a . \quad (5)$$

We focused on $G_k(z^{-1})$. An obvious transform of the convolution sum over j was straight forward. But the time-domain multiplication of this sum by $h^{k-2}(n)$ needed a proper solution. We introduced expansion of all the system poles into the infinite geometric series [8]. As we talk about the causal, LTI stable system, $H(z)$ can have only inner poles. Therefore, for a solution of (5), it is enough that $G_k(z^{-1})$ possesses no outer poles. Interesting is then only the part of $G_k(z^{-1})$ that is beyond the area of independent samples caused by the MA part. Denote this part of $G_k(z^{-1})$ by index 2, writing $G_{k,2}(z^{-1})$. After multiple computational steps and conclusions [8], we obtained the following form:

$$\begin{aligned} G_{k,2}(z) &= z^{-(M+q)} \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{M+N} u_k(j) \right. \\ & \left. \left[\sum_{i_1=1}^{p} e_{i_1} s_{i_1}^{n} \right]^{k-2} \sum_{i_2=1}^{p} f_{i_2} s_{i_2}^{n-j} \right\} z^{-n} \quad (6) \end{aligned}$$

where s_i stand for the poles of the system model $H(z)$, $e_i = d_i s_i^M$ and $f_i = e_i s_i^N$, while d_i equals $d_i = \lim_{z \to s_i} \frac{1 - s_i z^{-1}}{A(z)} \sum_{m=0}^{q} b(m) s_i^{-m+q}$ with $A(z)$ representing the AR part of the model, $b(m)$ MA coefficients, and (p, q) the system's order. An additional modification leads to:

$$\begin{aligned} G_{k,2}(z) &= z^{-(M+q)} \sum_{i_1=1}^{p} \cdots \sum_{i_{k-1}=1}^{p} U_k(s_{i_1}^{-1} \cdots s_{i_{k-2}}^{-1} z) \\ & \frac{e_{i_1} \cdots e_{i_{k-2}} f_{i_{k-1}}}{1 - s_{i_1} \cdots s_{i_{k-1}} z^{-1}}, \quad (7) \end{aligned}$$

which is the basis for recognition of the class of non-identifiable systems using ARMA w-slice 1-D single cumulants. The class comprises all the systems having at least one outer zero with the value of one of the inverted poles in Eq. (7). In other words: for the k^{th}-order cumulant, the poles generated in $G_{k,2}(z^{-1})$ are of the form $(s_{i_1} \cdot s_{i_2} \cdots s_{i_{k-1}})^{-1}; \, i_1, i_2, \ldots, i_{k-1} \in (1, \ldots, p)$. If there is a zero in $H(z)$ with the value of any of these poles, the two cancel in Eq. (5). This leads to identification problems of $H(z)$ using 1-D slices of the k^{th}-order cumulant.

Now, extend Eq. (3) to a sum of multiple higher-order cumulants:

$$g(n) = \sum_{k \in I} \gamma_{k,x} h^{k-2}(n) \sum_{j=0}^{M+N} u_k(j) h(n+N-j), \quad (8)$$

where $I = \{k_1, k_2, k_3, \ldots\}$ represents a set of orders of combined higher-order cumulants.

Applying this extension to the z-domain part of $g_k(n)$, from (7) we have:

$$\begin{aligned} G_2(z) &= z^{-(M+q)} \sum_{k \in I} \sum_{i_1=1}^{p} \cdots \sum_{i_{k-1}=1}^{p} U_k(s_{i_1}^{-1} \cdots s_{i_{k-2}}^{-1} z) \\ & \frac{e_{i_1} \cdots e_{i_{k-2}} f_{i_{k-1}}}{1 - s_{i_1} \cdots s_{i_{k-1}} z^{-1}}. \quad (9) \end{aligned}$$

This final form enables conclusions on identifiability of the ARMA w-slice method using multiple 1-D higher-order cumulants.

Expression (9) was derived with a supposition that the number of weights was the same for all the cumulant orders respected. So, all the polynomials

$U_k(s_{i_1}^{-1}\cdots s_{i_{k-2}}^{-1}z)$ have the same length. If we take into account different numbers of weights, it changes only the lengths of U_k. The poles of $G_2(z)$ remain the same. That only is what matters. As we saw, the poles of $G_2(z)$ may prevent identification if one of them coincides with an outer zero of $H(z)$. Therefore, no linear combination of cumulants of which one caused identification problems can improve the identification either.

An illustrative example

This conclusion is supported also by simulations. We tried to identified in 100 Monte-Carlo runs the following system, where the system input was a zero-mean i.i.d. exponential noise with variance 1:

$$H(z) = \frac{1 - 1.5z^{-1} - z^{-2}}{1 + 0.5z^{-2}};\ p = 2,\ q = 2.$$

The system has two zeros at 2 and -0.5, and two poles at $\pm\frac{\sqrt{2}}{2}\jmath$ ($\jmath$ being imaginary unit). The zero at 2 equals an inverse of the product of two poles, which prevents identification by the 3^{rd}-order cumulants (Table 1). In Table 2, the results were obtained for the same system by the 4^{th}-order cumulants. As we expected, the identification was sucessful. However, the results in Table 3, gathered from the identification with a sum of the 3^{rd}- and the 4^{th}-order cumulants, show that the identification failed. This was also expected according to the derivation in previous sections. Namely, if a cumulant of a certain order prevents identification, no combination with any other cumulants can provide the identifiability.

Table 1: Simulation results for the 3^{rd}-order cumulants, length of the input noise of 8192, $N = q,\ M = p,\ K = M + q$.

Parameters	$a(1)$	$a(2)$	$b(1)$	$b(2)$
True value	0	0.5	-1.5	-1
Mean Std	-0.0072 ± 0.0751	0.4996 ± 0.0498	1.8473 ± 8.7957	0.5834 ± 3.8928

Table 2: Simulation results for the 4^{th}-order cumulants, length of the input noise of 8192, $N = q,\ M = p,\ K = M + q$.

Parameters	$a(1)$	$a(2)$	$b(1)$	$b(2)$
True value	0	0.5	-1.5	-1
Mean Std	-0.0037 ± 0.0903	0.4803 ± 0.0742	-1.2772 ± 2.8434	-0.8752 ± 1.1082

3 CHECKING THE IDENTIFIABILITY

Another interpretation of the non-identifiability pointed out here may be as follows: such systems give in an

Table 3: Simulation results for a combination of the 3^{rd}- and the 4^{th}-order cumulants, length of the input noise of 8192, $N = q,\ M = p,\ K = 2(M + q) + 1$.

Parameters	$a(1)$	$a(2)$	$b(1)$	$b(2)$
True value	0	0.5	-1.5	-1
Mean Std	-0.0137 ± 0.0941	0.5061 ± 0.0814	0.2178 ± 4.3038	-0.0686 ± 2.4363

ARMA w-slice identification the correct AR part, but the MA part is incorrect because of the cancelled zero(s). We realised this fact also observing our simulation results.

Hence, if $\hat{H}(z)$ is the identified estimate of the real system $H(z)$ with $X(z)$ and $Y(z)$ being its input and output sequences, then the following derivation applies:

$$Y(z) = H(z)X(z) \Longrightarrow$$
$$\hat{X}(z) = \frac{Y(z)}{\hat{H}(z)} = \frac{H(z)}{\hat{H}(z)}X(z) = F(z)X(z). \quad (10)$$

Suppose $\hat{H}(z)$ lacks some outer zeros owing to erroneous identification by ARMA w-slices. Otherwise, all the poles and other zeros are considered the same as within $H(z)$. This means that $F(z)$ must be a finite, i.e. MA, sequence. Moreover, if the identification is correct, $F(z)$ must be a constant, theoretically. Thus, $F(z)$ actually gives a measure of a system identifiability using ARMA w-slice approach.

From Eq. (10), $\hat{X}(z)$ can be obtained as a deconvolution outcome of the system's output signal and the system estimate. Another MA w-slice, or similar, procedure may be applied to (10) in order to extract $F(z)$. Once $F(z)$ is known while non-identifiability was suspected, the very same $F(z)$ represents the missing zeros of $\hat{H}(z)$:

$$H(z) = \hat{H}(z)F(z). \quad (11)$$

Eqs. (10) and (11) fix the ARMA w-slice identifiability problem and mean an extension of the original method.

4 CONCLUSIONS

The w-slice identification method works satisfactory in non-parametric identification without a priori knowledge of the system order and it may be applied to all kinds of models, i.e., MA, AR, and ARMA. In spite of these very nice features, identifiability problems are encountered in the ARMA cases. In this paper we define a class of such non-identifiable systems. In opposition to the general conviction that a sum of multiple higher-order cumulants circumvents the identification problems, we have proved the following fact: if an ARMA system cannot be identified by 1-D w-slices using a single cumulant of a certain order k, there is no linear combination of any higher-order cumulants, the

k^{th}-order included, that would lead to successful identification. The reason is that such a combination includes again the poles that prevented identification with a single cumulant – see Eqs. (7) and (9).

Finally, return to the most general definition of the w-slice identification method, i.e. to Eq. (1). We revealed here analytically the reasons for non-identifiability in the case of 1-D cumulant slices used for the calculation. We also saw that a combination of different cumulants cannot resolve the identifiability problems. Thus, the same conclusion may be extended to the multidimensional slices. Also these cases merely mean a combination of different 1-D slices. If only one of them couldn't be identified, its "fatal" poles would appear in any combination with other slices. Hence, the multidimensional slices bring no solution to the ARMA w-slice identification problems either. However, a possible extension of the method proposed in the previous section can help, firstly, detecting the non-identifiability and, secondly, avoiding the errors introduced by the original ARMA w-slice algorithm.

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