Nth-Band Filter Design

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ABSTRACT

Digital Nth-band linear-phase nonrecursive and Nth-band recursive filters are special digital filter classes playing an important role in various applications. Both these filter classes are named according to their frequency-domain characteristics. This paper reviews the properties of these filters as well as their usefulness in several digital signal processing applications. Also their optimization for various applications is considered.

1 Introduction

Digital Nth-band FIR and IIR filters [1]–[11] (see also references in [11]) have somewhat different frequency domain and time domain properties, but they posses also many common characteristics. In the lowpass case, these filters have a (3 dB or 6 dB) bandwidth of π/N and the transition band is approximatively symmetric around this frequency. Both FIR and IIR Nth-band filters are quite efficient to implement. Especially, downsampling and upsampling operations can be combined very efficiently with these filters. This property makes Nth-band filters very interesting for all multirate signal processing applications. Also bandpass and highpass version can be obtained, e.g., by complex or cosine-modulation. In this way, also efficient Hilbert transfomers can be derived from lowpass Nth-band filters [12]. Also critically-sampled perfect reconstruction analysis-synthesis filter banks and transmultiplexers have a close relationship to Nth-band filters [13]–[18]. However, in this paper we concentrate on the lowpass case.

In the time domain, a characteristic property of Nth-band filters is that its impulse response has zero crossings at a regular distance, at all multiples of N samples away from the central sample. This is the so-called zero intersymbol interference property of the Nyquist pulse-shaping filters, a concept which is one of the corner-stones of bandwidth-efficient digital transmission systems [19]. Traditionally, raised cosine filters or a pair of square-root raised-cosine filters have has been utilized in digital transmission systems. A digital implementation of a raised cosine filter is actually a special type of Nth-band filter. In practise, raised-cosine filters suffer from the nonideal stopband response due to truncation of the ideal infinite-length impulse response, and better solutions can be found by filter optimization techniques [1], [3], [6], [9].

This paper reviews some basic properties of Nth-band FIR and IIR filters and illustrates their usefulness in various digital signal processing applications in terms of examples.

2 Nth-Band Linear-Phase FIR Filters

This section considers some basic properties of Nth-band linear-phase FIR filters. Their transfer function is of the

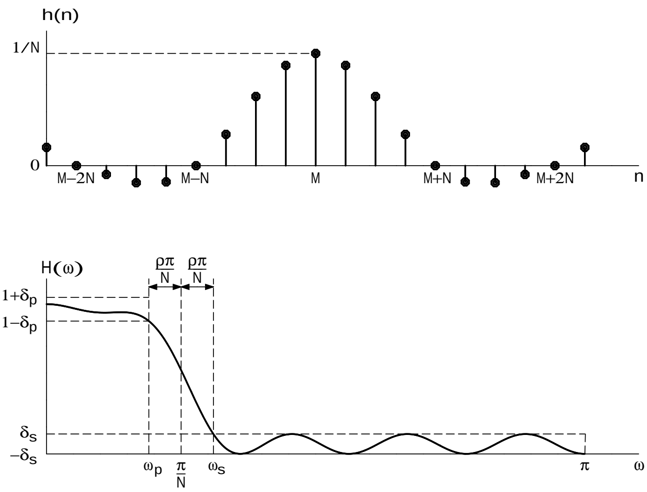

Figure 1: *Typical impulse response and zero-phase frequency response for an FIR Nth-band FIR filter.*

form

$$H(z) = \sum_{n=0}^{2M} h(n) z^{-n}, \tag{1}$$

where the impulse-response coefficients are symmetric, that is, $h(2M-n) = h(n)$ for $n = 0, 1, \cdots, 2M$. This filter is said to be an Nth-band filter if (see Fig. 1)

$$h(M) = 1/N, \quad h(M \pm rN) = 0 \ \text{ for } \ r = 1, 2, \cdots, \lfloor M/N \rfloor, \tag{2}$$

where $\lfloor x \rfloor$ stands for integer part of x.

The frequency response of the above filter is expressible as

$$H(e^{j\omega}) = e^{-jM\omega} H(\omega), \tag{3a}$$

where the zero-phase frequency response $H(\omega)$ is given by

$$H(\omega) = 1/N + 2 \sum_{n=1}^{M} h(M-n) \cos(n\omega). \tag{3b}$$

It can be shown [2] that the time-domain conditions of Eq. (2) are equivalent to the following frequency-domain condition:

$$\sum_{r=0}^{N-1} H(\omega + 2\pi r/N) = 1. \tag{4}$$

Based on this condition, the passband (stopband) in the lowpass case is restricted to be smaller (larger) than π/N.

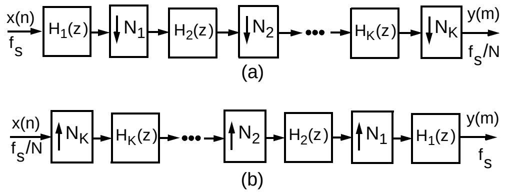

Figure 2: *Implementations of a multistage FIR Nth-band filter for sampling rate conversion. (a) Decimator. (b) Interpolator.*

Usually, the passband and stopband edge angles, denoted by ω_p and ω_s, are specified as (see Fig. 1)

$$\omega_p = (1-\rho)\pi/N, \quad \omega_s = (1+\rho)\pi/N, \tag{5}$$

where $\rho > 0$ is called the rolloff of the filter. For designing Nth-band FIR filters for other cases, see [10].

It follows also that if the maximum deviation of $H(\omega)$ from zero in the stopband region $[\omega_s,\ \pi]$ is δ_s, then δ_p, the maximum deviation of $H(\omega)$ from unity in the passband region $[0,\ \omega_p]$ is satisfies $\delta_p \leq (L-1)\delta_s$. This implies that for a small value of δ_s, δ_p is automatically small. Therefore, when designing FIR Nth-band filters the synthesis can concentrate on shaping the stopband response.

3 Various Classes of Nth-Band FIR filters

This section considers different classes of Nth-band FIR filters.

3.1 Multistage Filters

If N is factorizable into the product $N = N_1 \cdot N_2 \cdots N_K$, then the overall Nth-band filter can be contsructed in terms of K N_kth-band FIR filters with transfer functions of the form

$$H_k(z) = \sum_{n=0}^{2M_k} h_k(n) z^{-n}, \quad \text{for} \quad k = 1, 2, \cdots, K, \tag{6}$$

where each impulse response $h_k(n)$ is symmetric and satisfies the conditions of Eq. (2) with $N = N_k$ and $M = M_k$. The desired overall filter is then expressible as [6]

$$H(z) = \prod_{k=1}^{K} H_k(z^{L_k}), \tag{7a}$$

where

$$L_K = 1, \quad L_k = \prod_{l=k+1}^{K} N_k \quad k = 1, 2, \cdots K-1. \tag{7b}$$

In the above equation, instead of a unit delay z^{-1}, there is a block delay z^{-L_k} for all the terms except for $H_K(z)$. The order of this filter is $2M = 2(L_1M_1 + L_2M_2 + \cdots + L_KM_K)$.

The main advantage of the above decomposition is that the number of multipliers is significantly reduced compared to the direct-form implementation. Furthermore, if the overall filter is used for decimation or interpolation by a factor of N, then it can be implemented as shown in Fig. 2. Note that in these implementations unit delays are used.

The zero-phase frequency response of the above multistage filter is given by

$$H(\omega) = \prod_{k=1}^{K} H_k(L_k\omega). \tag{8}$$

3.2 Separable Filters

In pulse shaping in telecommunication applications, it is desired that the overall Nth-band (Nyquist) filter is factorizable as [9], [19]

$$H(z) = T(z)R(z) \tag{9a}$$

where

$$T(z) = \prod_{k=1}^{K} T_k(z^{L_k}), \quad R(z) = \prod_{k=1}^{K} R_k(z^{L_k}). \tag{9b}$$

Here, the half-Nyquist filters $T_k(z)$ and $R_k(z)$ are obtained by factorizing $H_k(z)$ as

$$H_k(z) = T_k(z)R_k(z) \tag{10}$$

where $T_k(z)$ and $R_k(z)$ have the same magnitude responses and their impulse responses are time-reversed versions of each other, that is, $R_k(z) = z^{-M_k}T_k(z^{-1})$, where M_k is half the order of $H_k(z)$.

In this case, it is required that the zero-phase frequency responses $H_k(\omega)$ for $k = 1, 2, \cdots, K$ are non-negative on $[0,\ \pi]$ in order to make $H_k(z)$ factorizable in the desired manner.

In communication theory, $T(z)$ and $R(z)$ are referred to as a matched filter pair and they are used as transmitter and reciever filters, respectively. $T(z)$ and $R(z)$ and can be effectively implemented in a manner similar to Figs. 2(b) and 2(a), respectively.

4 Optimization of Nth-Band FIR Filters and Design Examples

This section illustrates the filter optimization in terms of examples.

4.1 Example 1: Design of Nonseparable Filters

It is desired to design an Nth-band FIR filter to meet in the minimax sense the criteria: $N = 8$, $\rho = 0.2$, and the minimum stopband attenuation is at least 40 dB. Given K, the number of stages, the problem is to find N_k's and the minimum overall orders $2M_k$ for $k = 1, 2, \cdots, K$ to meet the given criteria and then to optimize the filter parameters to minimize

$$E_\infty = \max_{\omega \in [(1+\rho)\pi/N,\ \pi]} |W(\omega)H(\omega)|, \tag{11}$$

where $H(\omega)$ is given by Eq.(8) and $W(\omega)$ is a positive weight function on $[(1+\rho)\pi/N,\ \pi]$.

For $K = 1$, these criteria are met by $2M = 74$. When exploiting the coefficient symmetry and the facts that $h(37) = 2^{-3}$ and $h(37 \pm 8r) = 0$ for $r = 1, 2, 3, 4$, only 32 multipliers are required to implement this filter. The implementation of the central coefficient $h(37) = 2^{-3}$ is trivial. For $K = 3$, the given criteria are met by $N_1 = N_2 = N_3 = 2$ and $2M_1 = 18$, $2M_2 = 2M_3 = 6$. In this case, only $4 + 2 + 2 = 8$ multipliers are required. The price paid for the reduction in the number of multipliers from 32 to 8 compared to the direct design is a slight increase in the overall filter order (from 74 to 90). Figures 3(a) and 3(b) show the responses for $H_1(z^4)$, $H_2(z^2)$, and $H_3(z)$ as well as that of the overall filter. The subfilters have been iteratively designed using the technique proposed in [6]. As seen from these two figures, $H_1(z^4)$ provides for the overall filter an equiripple stopband behavior on $[(1+\rho)/8,\ \pi/4]$, whereas $H_2(z^2)$ and $H_3(z)$ and attenuate in the minimax sense the extra passbands and transition bands of $H_4(z^4)$ located around $\omega = \pi/2$ and $\omega = \pi$, respectively.

The impulse response of the overall filter is depicted in Fig. 3(c), whereas the overall filter optimized in the least-mean-square sense with the same subfilter orders is shown

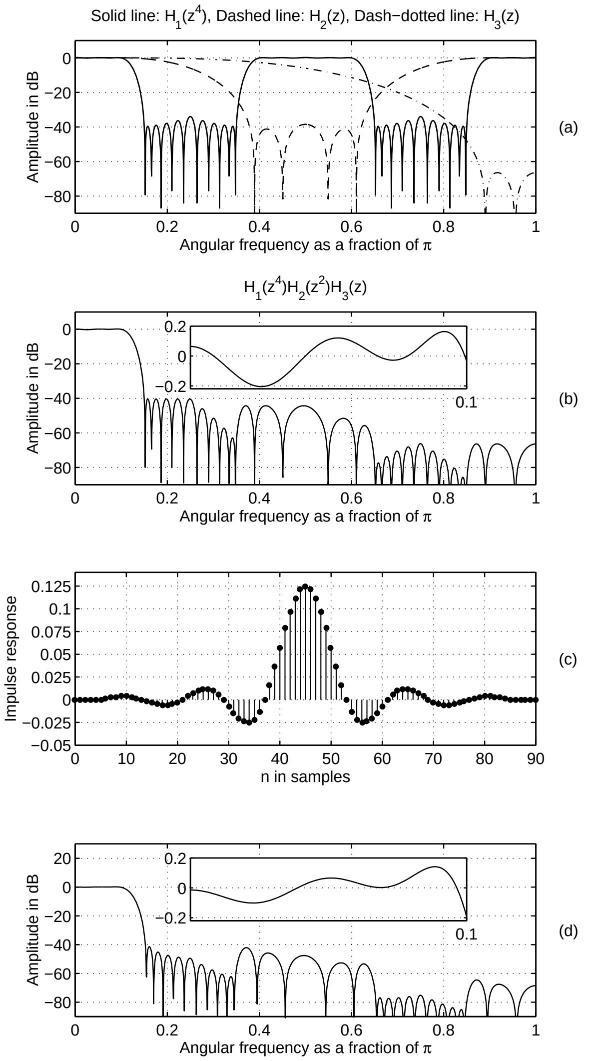

Figure 3: *Responses for three-stage filters of Example 1. (a) Amplitude responses for the subfilters. (b) and (c) Amplitude and impulse responses for the overall minimax Nth-band FIR filter. (d) Amplitude response for the least squared filter design.*

in Fig. 3(d). In this case, the impulse-response coefficients are determined to minimize

$$E_2 = \int_{(1+\rho)\pi/N}^{\pi} [W(\omega)H(\omega)]^2 d\omega. \tag{12}$$

The frequency-response-shaping responsibilities are shared like for the corresponding minimax filter design. It should be point out that linear programming [7], [8] can be also used for designing subfilters in the minimax sense, whereas the synthesis method proposed in [5] can be used for designing these filters in the least-mean square sense.

4.2 Example 2: Design of Separable Filters

It is desired to design a separable Nth-band FIR filter to meet in the minimax sense the criteria: $N = 8$, $\rho = 0.2$, and the minimum stopband attenuation is at least 40 dB for $R(z)$ and $T(z)$. For the overall separable filter, the minimum attenuation is thus 80 dB. In this case, the problem is to find the filter parameters to mininize E_∞ as given by Eq. (11) subject to the condition that $H(\omega)$ is nonnegative.

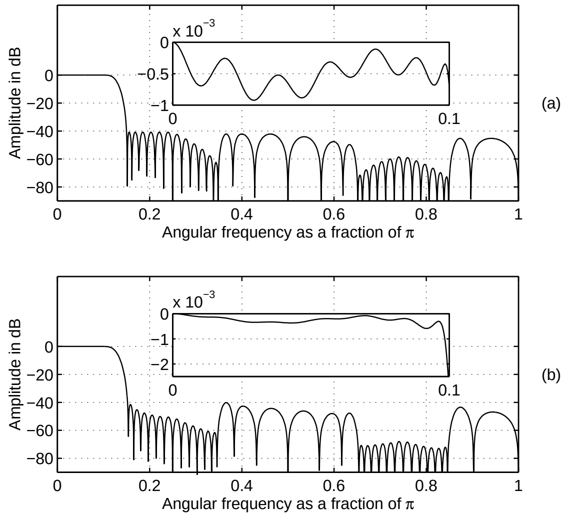

Figure 4: *Responses for three-stage filters of Example 2. (a) Common amplitude response for $T(z)$ and $R(z)$ designed in the minimax sense. (b) Common amplitude response for $T(z)$ and $R(z)$ designed in the least-mean-square sense.*

The subfilters can be effectively optimized using the synthesis scheme proposed in [20]. To meet these criteria with $K = 1$, $2M = 202$ is required For $K = 3$, the criteria are met by $N_1 = N_2 = N_3 = 2$, $2M_1 = 50$, $2M_2 = 18$, and $2M_3 = 10$. When the overall filter is split into the minimim-phase part $T(z)$ and the maximum phase part $R(z)$, both $T(z)$ and $R(z)$ require 102 multipliers in the $K = 1$ case and 42 multipliers in the $K = 3$ case. The price paid for this reduction is a slight increase in the overall filter order (from 202 to 246). Figure 4(a) shows in the three-stage case the common amplitude response $T(z)$ and $R(z)$, whereas Fig. 4(b) shows the corresponding response for a filter designed in the least-mean-square sense. In this case, the filter parameters are desired to be determined to minimize

$$\widehat{E}_2 = \int_{(1+\rho)\pi/N}^{\pi} [W(\omega)H(\omega)] d\omega \tag{13}$$

subject to the condition that $H(\omega)$ is nonnegative on $[0,\ \pi]$. For this purpose, the authors have generated a MATLAB routine. Note that in this case, $H(\omega) = |H_T(e^{j\omega})|^2 = |H_R(e^{j\omega})|^2$.

5 Nth-Band IIR filters

This section considers some basic properties of N-band IIR filters. These filters are a special class of filters having the following polyphase decomposition

$$H(z) = \frac{1}{N} \sum_{n=0}^{N-1} z^{-n} A_n(z^N). \tag{14}$$

For these filters, the $A_n(z)$'s are stable allpass filters of the form

$$A_n(z) = z^{-k_n} \frac{\sum_{l=0}^{K_n} a^{(n)}(l) z^{-(K_n - l)}}{\sum_{l=0}^{K_n} a^{(n)}(l) z^{-l}}. \tag{15}$$

The order of $A_n(z)$ is $k_n + K_n$ and it contains K_n adjustable parameters so that the overall number of parameters is

$$K = \sum_{n=0}^{N-1} K_n \tag{16}$$