RANDOMIZED REGRESSION IN DIFFERENTIATORS

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ABSTRACT

Differentiation of a signal is required in many applications in the field of signal processing. Linear differentiators fail to give good results for signals corrupted by both Gaussian and impulsive type of noise. For such cases nonlinear methods can be used in order to obtain better results. In this paper we propose a method, which we call randomized regression differentiator, based on random sampling and giving very good results in the presence of Gaussian and impulsive types of noise. We also present a modification of this general method which is designed for piecewise linear signals and utilizes random samplings of one window position in the next one. The output of this differentiator has sharp transitions when the slope value changes and the constant slope areas are smooth having only few small deviations around the correct slope value.

1 INTRODUCTION

Differentiation is a method to approximate instantaneous rate of change or slope of a signal. There are many important applications in the field of signal processing in which differentiators can be utilized. In biomedical engineering the slope of a signal can be used, e.g., to measure the rate of saturation or second derivative can give important information about the beginning or the end of a particular phenomenon. For example differentiation has been used for obtaining the time derivative of left ventricular pressure in [1] and in [2] differentiation has been applied to ECG signal processing. Differentiation can also be used in time domain interpolation [3]. This is done by expressing the interpolated value in terms of the Taylor series expansion around the nearest sample value. By using suitable approximations the evaluation of the interpolated value does not require higher than second order derivatives.

Because the signals are noisy in most of the real life signal processing applications, there is a need for such a differentiator which can estimate the signal slope also under noisy conditions. If noise is Gaussian then the optimal attenuation can be obtained by using linear differentiators, e.g., an FIR differentiator proposed in [4]. In Figure 1 is a test signal corrupted by Gaussian and impulsive noise and in Figure 2 is the result obtained by the FIR differentiator in [4] when applied to this test signal. However, if the type of noise is such that the deviations from the correct values can also be large then the linear methods fail as can be observed from Figure 2. In such conditions some nonlinear method needs to be used. In [5] we presented two nonlinear methods giving good results when signal is corrupted both by Gaussian and impulsive type of noise and in this paper we continue further by developing a new method called randomized regression.

2 LINEAR REGRESSION

Assuming a sliding window of length $2N+1$, we wish to approximate the signal values in the window by a straight line $y = ax + b$. Signal values in the window are $y_1, y_2, \ldots, y_{2N+1}$ and occur at the time instants $x_1, x_2, \ldots, x_{2N+1}$. The slope a of the line $y = ax + b$ is interpreted as the derivative of the signal at the middle point of the window. So we want to study the dependence of a random variable Y on variable X. For this purpose we can use the method of linear regression. In this approach the unknowns a and b are solved by using the principle of least squares, i.e., by minimizing $L = E\{Y - aX - b\}^2$. This square is minimized when the two partial derivatives $\frac{\partial L}{\partial a}$ and $\frac{\partial L}{\partial b}$ are set equal to zero and a and b can then be solved from the two obtained equations (cf. e.g. [6]). This way we obtain

$$a = \frac{E\{XY\} - E\{X\}E\{Y\}}{E\{X^2\} - E\{X\}^2} = \frac{\mathrm{cov}(X,Y)}{\mathrm{var}(X)} \quad (1)$$

and

$$b = E\{Y\} - aE\{X\}. \quad (2)$$

The line of regression of Y on X now obtains the form

$$y = \frac{\mathrm{cov}(X,Y)}{\mathrm{var}(X)}(x - E\{X\}) + E\{Y\}.$$

Because we have now only $2N+1$ samples for calculations, we have to approximate expected values, for example, $E\{XY\} = \frac{1}{2N+1}\sum_{i=1}^{2N+1} x_i y_i$.

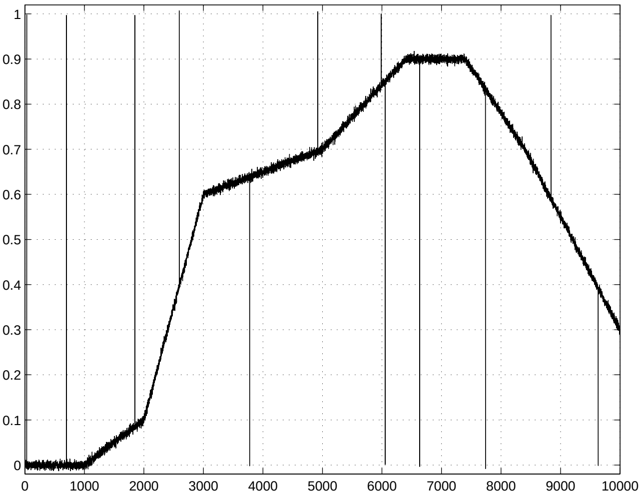

Figure 1: Noisy test signal.

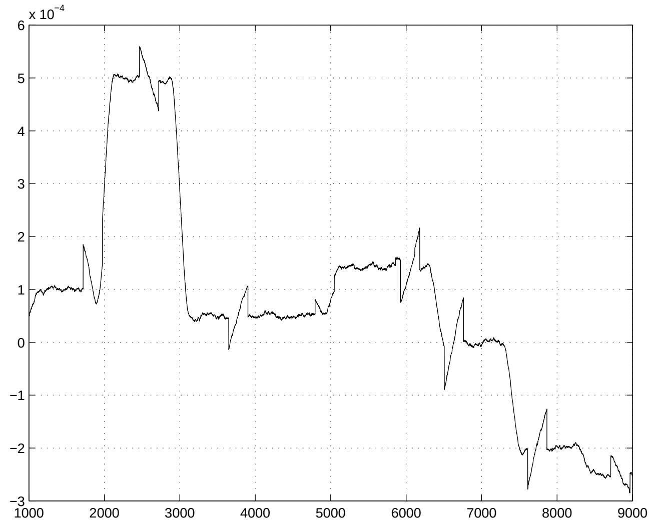

Figure 2: Output of linear FIR differentiator with window length 255.

As a measure of deviation of the signal values y_i from the predicted counterparts $\hat{y}_i = ax_i + b$ we can use the residual $\hat{e}_i = y_i - \hat{y}_i$ or squared residual $\hat{e}_i^2$. The residual describes how well at each sample point y_i in the window the calculated straight line approximates this sample point. When the samples y_i are noisy the squared residuals $\hat{e}_i^2$ obtain larger values and gross outliers can have very disturbing influence on the estimate. Since we are considering cases where the outliers are in y-direction they can have quite large influence on the regression line. However in this case the outliers often also possess large positive or negative residuals measured from the regression line. So residuals can quite reliably be used to indicate the outlying samples.

3 ROBUST REGRESSION

There are various robust estimators which could be used to remove the influence of outliers on the estimate (see e.g. [7]). In [5] we proposed a nonlinear robust regression differentiation method for reducing the susceptibility to noise in the signal that linear regression described in the previous section has. In this method we estimate the expected value in (1) and (2) by using the idea behind the nonlinear WMMR filter [8]. In WMMR filter we select m of the windowed values with the smallest range and weight these samples. The range of a set of values $\{y_1, y_2, \ldots, y_{2N+1}\}$ is defined to be $\max\{|y_i - y_j|, i \neq j, 1 \leq i, j \leq 2N+1\}$. So the selection of the minimum range finds the most condensed concentration of the values and rejects the outliers.

Our method is a combination of linear regression and WMMR filtering. First we calculate an initial estimate with linear regression as described in Section 2. After that we find the smallest range of the squared residuals $\hat{e}_1^2, \hat{e}_2^2, \ldots, \hat{e}_{2N+1}^2$ to obtain the m subindexes identifying the samples y_i and x_i taken into consideration in the calculations. The final estimate is then calculated by linear regression of only these m samples. So we have modified the WMMR filter in such a way that we filter samples corresponding to the subindexes obtained from the range calculations. The weights are all set equal to $\frac{1}{m}$, i.e., we calculate the mean of the m selected samples. In Figure 3 is the test signal of Figure 1 differentiated by robust regression. As can be seen impulses do not corrupt the output of this differentiator.

4 RANDOMIZED REGRESSION

The robust regression given in Section 3 however has some drawbacks. The number of outliers has to be estimated correctly in order to reject all the outliers and windowing also deteriorates the result both in robust regression and in linear methods. Especially when the signal to be differentiated is very noisy, the window length has to be set quite long. As a consequence also the transition area of the slope, when its value changes from one level to another, gets longer. So a method giving sharper transitions and better attenuation of noise is needed. For this purpose we propose in this paper randomized regression, a method based on randomized Hough Transform [9].

4.1 Description of the Method

In randomized regression we randomly sample the signal in the window in order to choose a pair of signal points (x_1, y_1) and (x_2, y_2). After this the slope a of a straight line between these two random points is calculated and the cell corresponding to the slope value a is accumulated in the accumulator space. The random sampling and accumulation is repeated until the value of some cell in the accumulator space exceeds threshold value which can be either fixed or variable. The slope corresponding to this cell is taken as the differentiator output. When a change in the slope value is approach-

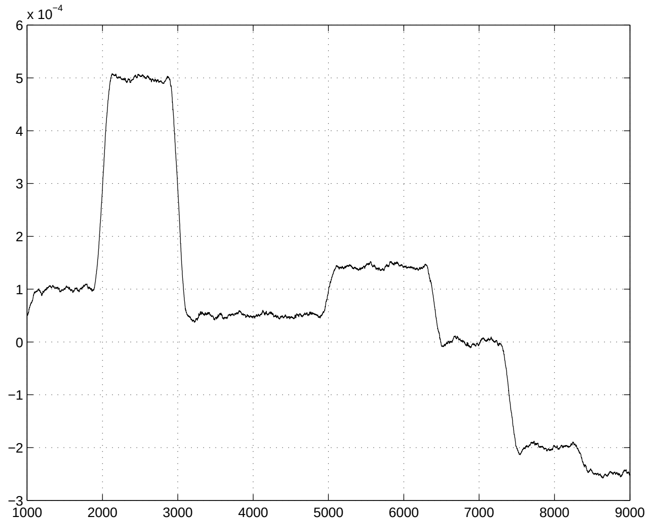

Figure 3: Output of the robust regression differentiator with window length 255 and $m = 253$.

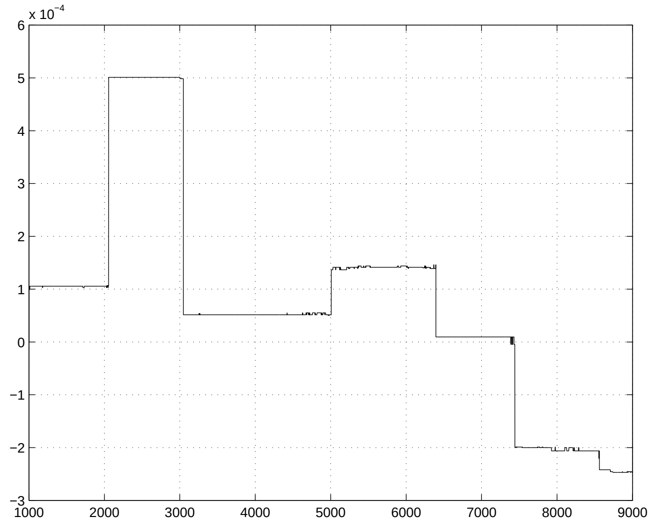

Figure 4: Output of the randomized regression differentiator with window length 2001 and threshold 100.

ing there are two peaks in the accumulator space. One of these peaks corresponds to the current slope value and the other one to the new slope value. The peak corresponding to the new slope value grows and finally exceeds the other peak when the window has moved to the position where more than half of its values are from the area of the new slope value. So the randomized regression differentiator output can have sharp transitions from one slope value to another and the window length is not similarly connected to the transition length as in robust or linear regression.

We consider the accumulator space as one-dimensional, and use only the value a of the slope and not the value b of the intercept. Of course intercept could also be utilized but the obtained benefit would be negligible compared to the increased complexity. Accumulator array can either be a fixed size array consisting of equal length intervals or a dynamic array structure. The fixed size array has the problem that it takes a lot of space when resolution is high; the dynamic array on the other hand has the problem that it can take a very long time to achieve the threshold at any cell, since very slightly differing slope values a occupy different cells. Fuzzyness can be added to these implementations in a similar manner as was done for Hough Transform in [10]. Not only the correct cell but also the the cells near the correct cell can be accumulated. For example, if the correct cell is accumulated by one, then the cells on both sides of it can be accumulated by 0.5 and the next cells by 0.1. Fuzzyness makes the method more robust since the accumulations obtained from a noisy signal differing only slightly from the correct value will accumulate also the correct cell.

A real-time implementation of the method could be done by utilizing parallel computation. In that case all the random sampling operations would be performed simultaneously.

4.2 Saving of the Valid Accumulations

A significantly faster implementation can be obtained for randomized regression by saving all the accumulations which still are valid from the previous window position to the next one. So only those accumulations are removed where one of the random sample points is no longer inside the window when the window is moved one step ahead. This also makes the obtained differentiator output smoother since it eliminates small deviations present when accumulator space is completely emptied as the window is moved ahead.

Critical points are those where the slope should change from one value to another. The algorithm which saves valid accumulations from one window position to the next one delays the transition to the correct slope value. This delay can however be removed by monitoring the accumulator space where an upcoming change in the slope for a piecewise linear signal can be detected as two high peaks instead of only one peak. In order to make the new peak at the position of the new slope value more easily recognizable the method is modified so that at each window position first a set of slope values is obtained by random sampling of samples in the right half of the window. This sampling is done even if the threshold would be reached without any new accumulations. The aim is to emphasize the influence of future samples while saving of valid accumulations emphasizes the influence of past samples. If these accumulations are not enough to reach the threshold value then two more accumulations are performed, one by random sampling of the right half of the window and the other one by random sampling of the left half. This sampling is then continued until the threshold is reached.

The above procedure does not remove the delay in the transition totally but it makes the peak corresponding to the correct output higher. So when the slope value should change there are two distinctive peaks in the accumulator space. Now the correct position for transition can be found by checking whether there are two peaks in the accumulator space and whether the height of the lower peak exceeds a given percentage of the height of the higher peak. If the situation is this then the transition is forced to happen by taking the slope value corresponding to the second highest peak and not the highest as the output. This procedure removes the delay and forces the transitions to happen at correct indexes.

4.3 Experimental Results

In Figure 4 is the test signal of Figure 1 differentiated by randomized regression where all the valid accumulations are saved when window moves ahead. In this experiment the length of the window is 2001 and the threshold of accumulator space is 100. So 100 accumulations to one cell are needed to make the slope corresponding to this cell as the output. Resolution of the accumulator space is such that each cell has width of $1.20 \cdot 10^{-6}$ which is small enough compared to the slope values in the test signal of Figure 1. When slope value is changing the two peaks in the accumulator space are searched by finding all the cells that have larger values than their 30 left and 30 right neighboring cells and by choosing two highest of these. Elimination of delay is done by taking the output to be the slope corresponding to the second highest peak when its height is more than 70% of the height of the highest peak.

When compared to the Figure 3, Figure 4 shows much improvement. The transitions are now extremely sharp and the constant parts are much smoother than in Figure 3. There are only small deviations from the correct value which result from the fact that the cells nearby the correct value in the accumulator space have also high values and sometimes their value will temporarily exceed the value of accumulations in the correct cell. Window length in Figure 4 is longer than in Figure 3, but the window has different role in these two approaches. In randomized regression longer window length does not imply essentially longer computation time, since window only gives the area from where random sampling will be performed. So these window lengths are not comparable to each other.

Experiments on other signals gave similar results as those obtained in the above test. Inclusion of fuzzyness gave also good results but slightly worse concerning the deviations around the correct value.

5 CONCLUSIONS

In this paper we have presented a randomized regression differentiator working well also in conditions where the signal to be differentiated is corrupted by both Gaussian and impulsive type of noise. Good properties of this differentiator are its capability to make sharp transitions when the value of the slope changes and to attenuate noise well in the areas where the slope remains unchanged.

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