DESIGN OF POLYNOMIAL INTERPOLATION FILTERS BASED ON TAYLOR SERIES

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ABSTRACT

Interpolation filters are used to interpolate new sample values at arbitrary time instants between the existing discrete-time samples. Interpolation filters utilizing polynomial-based interpolation can be efficiently implemented using the Farrow structure. This paper introduces a new method for designing Farrow-structured interpolation filters. The proposed synthesis method is based on the relationship between the Farrow structure and the Taylor series of the interpolating continuous-time signal formed based on the existing sample values.

1. INTRODUCTION

Interpolation filters (or interpolators) are used to interpolate new sample values at arbitrary points between the existing discrete-time samples. The applications of these filters include; echo cancellation in modems [1], symbol synchronization in digital receivers [2]–[4], and arbitrary sampling rate conversion [5]–[6].

One alternative for implementing an interpolation filter is to use an FIR filter to delay the input samples by $D = D_{\text{int}} + \mu$, where D_{int} is an integer delay determined by the length of the filter and $\mu \in [0,1)$ is the fractional delay (or fractional interval). In order to be able to generate new samples at arbitrary time instants, the fractional delay of the filter should be adjustable. In this case, the filter coefficients can be precalculated for each value of the fractional delay and can be stored in the coefficient memory. Because the size of the memory becomes usually very large, this technique is not very useful.

A very efficient way to get around this problem is to use polynomial-based interpolation filters where a polynomial approximation is calculated for each sample interval. The sample values at desired time instants are then obtained by evaluating the corresponding values of these polynomials. The main advantage of using polynomial-based interpolation is that it can be efficiently implemented using the so-called Farrow structure [1]. This discrete-time filter structure consists of FIR branch filters having fixed coefficient values. The interpolated samples are obtained by weighting the output samples of these FIR filters by the fractional interval μ.

In many applications of digital signal processing (DSP), it is desired to know the frequency-domain behavior of the interpolator, in addition to its time domain behavior. This can be done conveniently by analyzing the Farrow structure by using an equivalent hybrid analog/digital model. In this model, the digital input sequence is first converted to weighted analog impulses by using a digital-to-analog (D/A) converter. This sequence of weighted impulses is then filtered by a continuous-time anti-imaging filter. Finally, the interpolated sample values are obtained by sampling this reconstructed analog signal at any desired time instants.

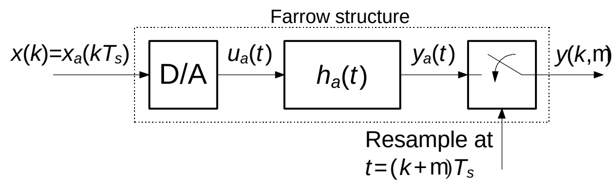

Fig. 1. The hybrid analog/digital model for the interpolation filter.

In this paper, a new method for designing polynomial-based interpolation filters is introduced. This method is based on the relationship between the Taylor series of the approximating continuous-time signal and the Farrow structure as introduced in [7]. It enables us to design the FIR filters in the Farrow structure separately. Because these FIR filter are linear-phase filters, they can be easily designed by using, e.g., the Remez algorithm.

2. IMPLEMENTATION OF POLYNOMIAL-BASED INTERPOLATION FILTERS

The polynomial-based interpolation methods are traditionally considered as time-domain procedures, where some approximating polynomial is fitted to the data and then this polynomial is evaluated at the desired time instants to get the interpolated sample values.

However, in DSP applications, our signal is not a predefined deterministic function and, therefore, a time-domain analysis is not very informative. What is known about the signal is its frequency band of interest. Consequently, the interpolation should be considered as a filtering problem in the frequency domain.

This can be done by using the hybrid analog/digital model for the interpolation filter as shown in Fig. 1 [5]–[6]. In this model, the digital input sequence $x(k)$ is first converted to a sequence of weighted analog impulses $u_a(t)$ which are filtered by a lowpass anti-imaging filter with an impulse response $h_a(t)$. After that, the reconstructed analog signal $y_a(t)$ is resampled at the desired time instant determined by $t = (k+\mu)T_s$ to obtain the output sample $y(k,\mu)$. Here, $\mu \in [0,1)$ is the fractional interval and k is any integer. Assuming that the impulse response of the anti-imaging filter $h_a(t)$ is non-zero only in the interval $0 \le t \le NT_s$, the desired output sample is given by

$$y(k,\mu) = y_a\big((k+\mu)T_s\big) = \sum_{n=0}^{N-1} x(k-n)h_a\big((n+\mu)T_s\big), \quad (1)$$

where T_s is the sampling interval of the input signal $x(k)$. It is desired that the delay of the anti-imaging filter, which is equal to $NT_s/2$, is a multiple of the sampling interval T_s. Therefore, N has to be even.

The use of the hybrid analog/digital structure for implementing interpolation filters is not very practical due to the need of analog anti-imaging filter, and D/A- and A/D-converters. Fortunately, there exist more efficient digital implementation structure, the so-called modified Farrow structure [8], which is equivalent to the hybrid model in Fig. 1. The connection between the original Farrow structure and the hybrid analog/digital model has been derived in [3]. The basic assumption for deriving the modified Farrow structure is that the anti-imaging filter $h_a(t) = h_a((n+\mu)T_s)$ for $n = 0, 1,..., N-1$ and for $\mu \in [0,1)$ is a piecewise polynomial in $(2\mu - 1)$ and is, therefore, expressible as

$$h_a(t) \equiv h_a\big((n+\mu)T_s\big) = \sum_{l=0}^{L} c_l(n)(2\mu-1)^l, \quad (2)$$

where the $c_l(n)$'s are the polynomial coefficients and L is the degree of polynomials. By substituting Eq. (2) into Eq. (1), the interpolated output sample can be given by

$$y(k,\mu) = y_a\big((k+\mu)T_s\big) = \sum_{l=0}^{L} v_l(k)(2\mu-1)^l, \quad (3)$$

where

$$v_l(k) = \sum_{n=0}^{N-1} x(k-n)c_l(n) \quad (4)$$

are the output samples of the FIR branch filters having the following transfer functions:

$$C_l(z) = \sum_{n=0}^{N-1} c_l(n)z^{-n} \quad \text{for } l = 0,\ 1, \ldots,\ L. \quad (5)$$

The corresponding filter structure is shown in Fig. 2.

In the original Farrow structure [1], the basis function is μ instead of $(2\mu - 1)$. The advantage of using $(2\mu - 1)$ is that the coefficients of the corresponding FIR filters $C_l(z)$ are always symmetrical, which is not generally true for the Farrow structure. The equation to translate the coefficients

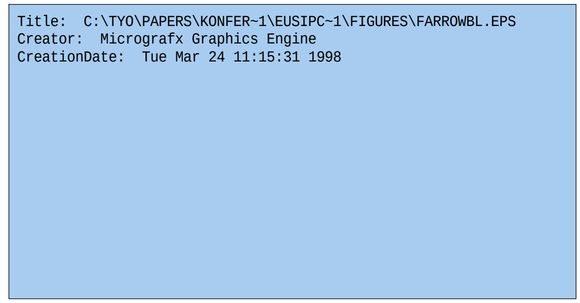

Fig. 2. The modified Farrow structure.

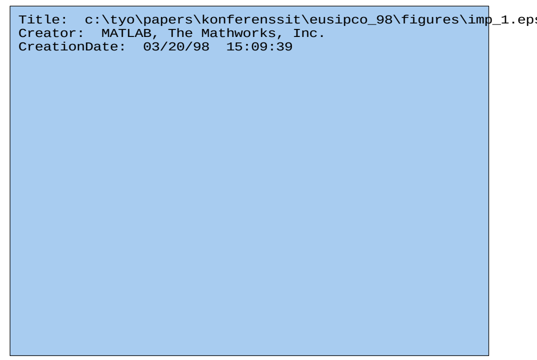

Fig. 3. Polynomial-based impulse response $h_a(t)$ for the minimax interpolation filter.

of the original Farrow structure to the coefficients of the modified Farrow structure has been given in [9].

The filter coefficients for the Farrow structure can be obtained, e.g., using the Lagrange interpolation [3] or the minimax design [8]. The Lagrange interpolation is a conventional time-domain approach, whereas the minimax design optimizes the coefficients of the Farrow structure directly in the frequency domain. A combined time-frequency-domain approach has been introduced in [7].

As an example, Fig. 3 shows a polynomial-based impulse response $h_a(t)$ for the interpolation filter being designed using the minimax synthesis method. The filter specifications are: the filter length is $N = 8$, the degree of the interpolation is $L = 3$, and the passband and stopband edges are at $0.35F_s$ and $0.65F_s$, respectively. Here, $F_s = 1/T_s$ is the sampling frequency.

The continuous-time frequency responses $H_a(j2\pi f)$ for this minimax and the cubic Lagrange interpolation filters are shown in Fig. 4. For the Lagrange filter, $L = 3$ and $N = 4$.

3. FILTER DESIGN BASED ON TAYLOR SERIES

The synthesis technique to be introduced in this paper enables us to design FIR branch filters in the modified

Farrow structure separately. Furthermore, because these FIR filters are linear-phase filters, they can be easily designed by using almost any existing algorithm proposed for synthesizing linear-phase FIR filters.

The derivation of the desired responses for the FIR filters with the transfer functions $C_l(z)$ is based on two facts:

1. The FIR filters $C_l(z)$ for $l = 0, 1, ..., L$ in the Farrow structure form an L^{th} order Taylor series approximation to the continuous-time interpolated signal $x_a(t)$ [7].
2. In the modified Farrow structure, the FIR filters $C_l(z)$ are linear-phase Type II filters when l is even and Type IV filters when l is odd [8].

We derive the desired responses for the FIR filters by first studying the amplitude responses $|C_l(e^{j\omega})|$ of the above-mentioned minimax interpolation filter. These responses $|C_l(e^{j\omega})|$ for $l = 0, 1, 2,$ and 3 are shown in Fig. 5. As can be seen, they approximately follow the curves $k_l\omega^l$, where the k_l's are some constant. This shape of the amplitude responses $|C_l(e^{j\omega})|$ seems to be a feature of other interpolation filters as well, like Lagrange interpolators. This means that the FIR filter with transfer function $C_l(z)$ in the Farrow structure is an l^{th} order differentiator [recall that the ideal frequency response of the l^{th} order differentiator is $(j\omega)^l$].

The fact that the FIR filters $C_l(z)$ for $l = 1, 2, ..., L$ are l^{th} order differentiators and $C_0(z)$ is an all-pass filter can be explained by using the Taylor's theorem. The relationship between the Taylor series and the original Farrow structure has been introduced in [7].

Taylor's theorem states that if the function $f(\mu)$ has L+1 continuous derivatives on the interval $\mu \in [0, 1]$, then it can be approximated by the following L^{th} degree polynomial:

$$f(\mu) \approx \sum_{l=0}^{L} \frac{1}{l!} f^{(l)}(0)\mu^l, \qquad (6)$$

where $f^{(l)}(0)$ is the l^{th} degree derivative of $f(\mu)$ at $\mu = 0$. The approximation error is defined by the remainder term $R_{L+1}(\mu)$ (see any numerical-analysis textbook).

In the modified Farrow structure, the polynomial approximation can be formed in the similar manner to the Taylor series approximation. This can be done by using the linear-phase FIR filter transfer function $C_l(z)$ to calculate the l^{th} order differential of the input signal. This differential, denoted by $v_l(k)$ (see Fig. 2), is then used to approximate the l^{th} order derivative of the original continuous-time signal $x_a(t)$ at $t = (k+\mu)T_s$. The derivative is determined at $\mu = 0.5$ because the filter lengths are even and, therefore, the differential is always calculated in the middle of two existing input samples.

Because the frequency response of the ideal l^{th} order differentiator is $(j\omega)^l$, the desired frequency response for the FIR filter with transfer function $C_l(z)$ takes, after some manipulations, the following form:

$$\hat{C}_l(e^{j\omega}) = e^{-j\omega(N-1)/2}\frac{(-j\omega)^l}{2^l\, l!}\,. \qquad (7)$$

Here, the term 2^l in the denominator and −1 inside the parenthesis are due to the fact that for the modified Farrow structure $2\mu - 1$, instead of μ, is used [compare Eqs. (3) and (6)].

Fig. 4. Continuous-time frequency responses for the minimax (solid line) and cubic Lagrange (dashed line) interpolation filters.

Fig. 5. Amplitude responses $|C_l(e^{j2\pi f/F_s})||$ of the FIR filters in the Farrow structure. The minimax design in the previous section (solid line) and the desired responses (dashed line).

These desired responses $|\hat{C}_l(e^{j\omega})|$ for $l = 0, 1, 2,$ and 3 are shown in Fig. 5

For the proposed design procedure, the input parameters are N, L, F_s , and $f_p = \alpha F_s/2$ with $\alpha < 1$, the passband edge of the interpolator filter. Since the FIR branch filters are linear phase filters, the design involves finding the coefficients of the L+1 transfer functions $C_l(z)$ in such a way that the following error function:

$$E_l(\omega) = \left| \sum_{n=0}^{(N-1)/2} c_l\big[(N-1)/2 - n\big]\gamma(l,n,\omega) - D(l,\omega) \right| \qquad (8)$$

is minimized in either the minimax or least-mean-square sense on $[0, \alpha\pi]$. Here, $D(l,\omega) = (-1)^{\lfloor 3l/2 \rfloor}\omega^l/(2^l l!)$, where $\lfloor x \rfloor$

stands for integer part of x, and $\gamma(l,n,\omega) = 2\cos[(n+1/2)\omega]$ for l even and $\gamma(l,n,\omega) = 2\sin[(n+1/2)\omega]$ for l odd. The region $[\alpha\pi, \pi]$ is considered as a don't care band. Note that for l even $c_l(N-1-n) = c_l(n)$ and for l odd $c_l(N-1-n) = -c_l(n)$.

The coefficients of the transfer functions $C_l(z)$ can be found directly in the minimax or least-mean-square sense using in MATLAB the routine *remez.m* or *firls.m*, respectively. However, the routine *remez.m* has to be modified in such a way that the desired function becomes $D(l,\omega) = (-1)^{\lfloor 3l/2 \rfloor}\omega^l/(2^l l!)$.

4. DESIGN EXAMPLES

To exemplify the use of the above-mentioned method, we consider the design of a polynomial-based interpolation filter with the following parameters: $N = 8$, $L = 5$, $f_p = 0.7F_s/2$ ($\alpha = 0.7$).

The $C_l(z)$'s for $l = 0, 1,\ldots, 5$ have been designed in the least-mean-square sense using the routine *firls.m* in MATLAB. The amplitude responses $|C_l(e^{j\omega})|$ for $l = 0, 1,\ldots, 5$ are shown in Fig. 6. The polynomial-based impulse response $h_a(t)$ and the continuous-time frequency response $H_a(j2\pi f)$ for the interpolation filter are shown in Figs. 7 and 8, respectively. Note that the use of the don't care band when designing the transfer functions $C_l(z)$ causes don't care bands also to the continuous-time frequency response of the interpolation filter. These bands are given by $[nF_s+f_p, (n+1)F_s-f_p]$ for $n = 1, 2, 3,\ldots$, where f_p is the passband edge.

If the highest frequency component of the signal under consideration is less than or equal to $0.35F_s$, then there is no aliasing from these bands.

5. CONCLUSIONS

A new synthesis technique for polynomial-based interpolation filters was presented. In this technique, the linear-phase Type II and Type IV FIR filters in the modified Farrow structure are designed separately. Consequently, the design of the interpolation filter reduces to the design of L+1 linear-phase FIR filters.

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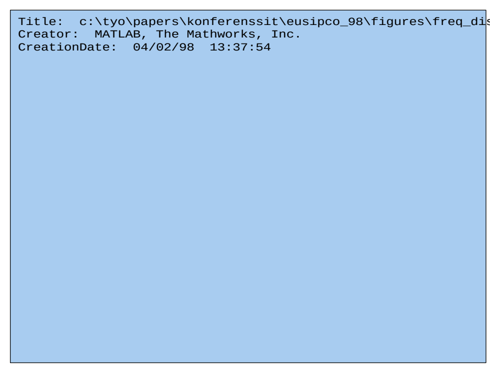

Fig. 6. Amplitude responses $|C_l(e^{j2\pi f/F_s})|$ for the FIR filters $C_l(z)$ for $l = 0, 1,\ldots, 5$ (solid line) and the ideal responses (dashed line).

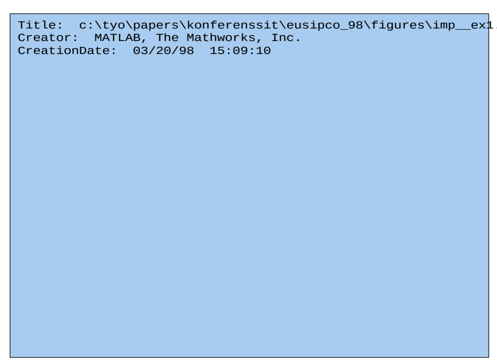

Fig. 7. Impulse response $h_a(t)$ for the interpolation filter.

Fig. 8. Frequency response $H_a(j2\pi f)$ for the interpolation filter.