Relationship Between The Wigner-Distribution And The Teager Energy

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ABSTRACT

The link between the Teager energy (TE) operator and the Wigner distribution (WD) is established in this paper. This link results in a simple way to calculate the second conditional moment in frequency of a Wigner distribution. Furthermore, this link helps explain the negative 'energy' via the Teager operator, and the negative conditional second moment via the Wigner distribution. We present the similarity between the WD and the time-variant periodogram. Also, we calculate the TE of a time-variant finite-time complex spectrum, and the TE of a time-variant periodogram.

1 Introduction

Teager energy (TE) operator has been used in a number of applications since its introduction in [1]-[2] including one-dimensional signal processing [3]-[4], image processing [5] and color image processing [6]-[7]-[8].

The relationship of the TE and the Ambiguity function (AF) has been established by the authors in [4] and [9]. The Ambiguity function is a time-frequency correlation function which is useful in many signal communication systems, especially radar signals where it answers questions about resolution, ambiguities, measurement precision, and clutter rejection. The AF is also the characteristic function of the time-frequency Wigner distribution (WD) [10].

In this paper, we establish the connection between the TE operator and the Wigner Distribution. This connection points to an interesting relationship and a simple way to calculate the second conditional moment in frequency of a Wigner distribution. Also, since other time-frequency distributions have different second moments, this points to generalizations of the Teager energy. Moreover, the established connection helps explain the curious results that can be obtained via the TE operator and the Wigner distribution. The former provides negative "energy" and the latter negative conditional second moment, leading to complex standard deviations.

Besides, we relate the TE of a time-variant finite-time complex spectrum with the continuous-time signal, and the TE of a time-variant periodogram with the time-variant correlogram.

The paper is organized as follows, Section 2 defines the Wigner distribution and the ambiguity function. The connection between the TE and the WD is established in Section 3. The similarity between the WD and the time-variant periodogram is presented in Section 4. The TE operator of the pseudo-spectrum, and the TE operator of the time-variant periodogram are calculated in Section 5. Conclusion are drawn in Section 6.

2 Wigner Distribution and Ambiguity Function: Definitions

The Wigner distribution of a signal $x(t)$ is defined by [10]

$$W(t,v) = \frac{1}{2\pi}\int x^*(t-\frac{\tau}{2})x(t+\frac{\tau}{2})e^{-jv\tau}d\tau. \quad (1)$$

Similarly, the WD in terms of the spectrum is defined by

$$W(t,v) = \frac{1}{2\pi}\int X^*(v+\frac{u}{2})X(v-\frac{u}{2})e^{-jtu}du \quad (2)$$

where $X(v)$ is the Fourier transform of $x(t)$, assumed to be normalized symmetrically;

$$X(v) = \frac{1}{\sqrt{2\pi}}\int x(t)e^{-jvt}dt$$

$$x(t) = \frac{1}{\sqrt{2\pi}}\int X(v)e^{jtv}dv.$$

The characteristic function of the WD is defined by [10];

$$M(u,\tau) = \int\int W(t,v)e^{jut+j\tau v}dtdv. \quad (3)$$

Substituting Eq. (1) for $W(t,v)$ in the above equation, we obtain

$$\begin{aligned} M(u,\tau) &= \int x^*(t-\frac{\tau}{2})x(t+\frac{\tau}{2})e^{jut}dt \quad (4) \\ &= \Xi(u,\tau). \end{aligned}$$

This specific characteristic function $\Xi(u,\tau)$, is known as the Ambiguity function (AF) and is expressed in terms of the spectrum by [10]

$$\Xi(u,\tau)=\int X^*(v+\frac{u}{2})X(v-\frac{u}{2})e^{j\tau v}dv. \quad (5)$$

Notice that the WD is just the inverse double Fourier transform of this symmetric time-frequency AF. Also, the relation $\Xi(u,\tau)=\Xi^*(-u,-\tau)$ implies the reality property of the WD;

$$W^*(t,v)=W(t,v). \quad (6)$$

In fact, characteristic functions are considered very often more revealing than the distributions mainly because they represent the main tool for obtaining a distribution and determining its properties besides their capabilities in calculating the mixed moments [10]-[11].

3 Teager Energy and Wigner Distribution

The Teager energy for a complex-valued signal $x(t)$ is defined by [9]

$$\Psi_C[x(t)]=\dot{x}(t)\dot{x}^*(t)-\frac{1}{2}[\ddot{x}(t)x^*(t)+x(t)\ddot{x}^*(t)]. \quad (7)$$

When $x(t)$ is real, Eq. (7) reduces to the Teager energy of a real-valued signal, defined in [1] as follows;

$$\Psi_R[x(t)]=\dot{x}^2(t)-x(t)\ddot{x}(t). \quad (8)$$

It was shown in [9] that the TE of a complex signal is equal to the sum of the Teager energies of its real and imaginary parts;

$$\Psi_C[x(t)]=\Psi_R[x_r(t)]+\Psi_R[x_i(t)]. \quad (9)$$

The connection between the TE operator and the AF is derived in [9] and [12]. It is shown that the TE operator is related to the Fourier transform of the AF as follows;

$$\Psi_C[x(t)]=\frac{1}{4\pi}\int\int v^2X(v+\frac{u}{2})X^*(v-\frac{u}{2})e^{jtu}dudv. \quad (10)$$

Under the assumption that the signal and its spectrum are symmetrically normalized as mentioned earlier, also that the signal has unity energy.

Since the WD is always real, conjugating Eq. (2) does not change anything;

$$W(t,v)=\frac{1}{2\pi}\int X(v+\frac{u}{2})X^*(v-\frac{u}{2})e^{jtu}du. \quad (11)$$

Evaluating the integral over du in Eq. (10), we obtain

$$\Psi_C[x(t)]=\frac{1}{2}\int v^2W(t,v)dv, \quad (12)$$

thus, we relate the TE operator to the WD.

The second conditional moment in frequency of the Wigner distribution is defined by ([10], chap. 8)

$$<v^2>_t=\frac{1}{|x(t)|^2}\int v^2W(t,v)dv. \quad (13)$$

Finally, expressions (12) and (13) give the explicit relation between the Teager energy operator and the second conditional moment in frequency of the Wigner distribution as follows:

$$\Psi_C[x(t)]=\frac{1}{2}|x(t)|^2<v^2>_t\ . \quad (14)$$

The above equation points to an interesting relationship and simple way to calculate the second conditional moment in frequency of a Wigner distribution, namely as,

$$<v^2>_t=2\frac{\Psi_C[x(t)]}{|x(t)|^2}. \quad (15)$$

4 Similarity of the WD and the Time-Variant Periodogram

In fact, the Wigner distribution (time-frequency energy density) bear some resemblance with the limit cyclic autocorrelation and the limit cyclic spectrum (spectrogram) [10]-[13]. However, this relationship is limited because the WD is a distribution and does not involve a limiting time average, analogous to the limit autocorrelation and spectral density [13].

Unlike the WD which sometimes provide negative results that can not be interpreted, the spectrogram is manifestly positive and its results can always be analyzed. Hence, some modified WD trying to achieve a positive distribution by convolving the WD with a smoothing window were developed. Although, the time-variant finite-time spectrum $S_{x_T}(t,v)$ (called also periodogram) represents a smoothed version of the energy density WD ([13], chap. 10)

$$S_{x_T}(t,v)=\int\int W(\tau,u)w(\tau-t,u-v)d\tau du, \quad (16)$$

for which the smoothing widow $w(t,v)$ is the WD of a unity height rectangle pulse.
The time-variant periodogram is also defined by

$$S_{x_T}(t,v)\triangleq\frac{1}{T}|\chi_T(t,v)|^2, \quad (17)$$

where $\chi_T(t,v)$ is the time-variant finite-time complex spectrum of $x(t)$ defined by

$$\chi_T(t,v)\triangleq\int_{t-T/2}^{t+T/2}x(\mu)e^{-jv\mu}d\mu. \quad (18)$$

Here, $\chi_T(t,v)$ is just the FT of a data segment length T centered at time t.

Similarly, it was shown in [13] that the time-variant periodogram $S_{x_T}(t,v)$ can also be expressed as the FT of the time-variant finite-time autocorrelation $R_{x_T}(t,\tau)$ (called also correlogram)

$$S_{x_T}(t,v) = FT\{R_{x_T}(t,\tau)\}, \tag{19}$$

for which the time-variant correlogram $R_{x_T}(t,\tau)$ is defined by

$$R_{x_T}(t,\tau) \triangleq \frac{1}{T}\int_{t-(T-|\tau|)/2}^{t+(T-|\tau|)/2} x(\mu+\frac{\tau}{2}) x(\mu-\frac{\tau}{2})d\mu[2Tw_{2T}(\tau)], \tag{20}$$

where $w_T(t)$ is a rectangular window function given by

$$w_T(t) = \begin{cases} 1/T & \text{if } |t| \le T/2 \\ 0 & \text{if } |t| > T/2. \end{cases}$$

5 TE of Pseudo-Spectrum and TE of Time-Variant Periodogram

In the following, we relate the TE of a time-variant finite-time complex spectrum with the continuous-time signal, and the TE of a time-variant periodogram with the time-variant correlogram.

For a normalized time-variant finite-time complex spectrum of tapered data ([13], chap. 5)

$$\bar{\chi}_T(t,v) \triangleq \frac{1}{\sqrt{T}}\chi(t,v), \tag{21}$$

where $\chi(t,v)$ is given by Eq. (18), the finite-autocorrelation in v of the time-variant finite-time complex spectrum of a rectangular data-tapered window (Eq. (21)) is derived as follows;

$$\int \bar{\chi}_T(t,v+\frac{u}{2})\bar{\chi}_T^*(t,v-\frac{u}{2})dv = \frac{1}{T}\int_{t-T/2}^{t+T/2} x^2(\mu)e^{-ju\mu}d\mu. \tag{22}$$

Differentiating Eq. (22) twice with respect to u, one gets

$$\int \frac{\partial^2[\bar{\chi}_T^*(t,v-\frac{u}{2})\bar{\chi}_T(t,v+\frac{u}{2})]}{\partial u^2}|_{u=0}dv = \frac{-1}{T}\int_{t-T/2}^{t+T/2} \mu^2 x^2(\mu)e^{-ju\mu}|_{u=0}d\mu. \tag{23}$$

Note that the first integrand evaluated at $u=0$ is just the TE of the time-variant finite-time complex spectrum with respect to v, resulting in:

$$\int \Psi_C[\bar{\chi}_T(t,v)]dv = \frac{1}{2T}\int_{t-T/2}^{t+T/2} \mu^2 x^2(\mu)d\mu. \tag{24}$$

Thus, we relate the TE of the time-variant finite-time complex spectrum with the continuous-time signal.

Similarly, for a time-variant finite-time complex spectrum of tapered data (Eq. (21)), the finite-autocorrelation in v of the time-variant periodogram $S_{x_T}(t,v)$ can also be expressed as follows;

$$\int S_{x_T}(t,v+\frac{u}{2})S_{x_T}(t,v-\frac{u}{2})dv = \int [R_{x_T}(t,\tau)]^2 e^{-jv\tau}d\tau. \tag{25}$$

Differentiating the above equation (25) twice with respect to u, we obtain

$$\int \frac{\partial^2[S_{x_T}(t,v+\frac{u}{2})S_{x_T}(t,v-\frac{u}{2})]}{\partial u^2}|_{u=0}dv = -\int \tau^2[R_{x_T}(t,\tau)]^2 e^{-ju\tau}|_{u=0}d\tau. \tag{26}$$

Note also that the first integrand evaluated at $u=0$ is just the TE of the time-variant periodogram with respect to v, resulting in:

$$\int \Psi_C[S_{x_T}(t,v)]dv = \frac{1}{2}\int \tau^2[R_{x_T}(t,\tau)]^2 d\tau. \tag{27}$$

Hence, we relate the TE of the time-variant periodogram with the time-variant correlogram.

6 Conclusion

The link between the Teager energy and the Wigner distribution is established in this paper. This link provides a simple way to calculate the second conditional moment in frequency of a Wigner distribution. Furthermore, this relation suggests possible generalization of the Teager operator by utilizing different characteristic functions corresponding to different time-frequency distributions. We presented the similarity between the WD and the time-variant periodogram. Also, we calculated the TE of a time-variant finite-time complex spectrum, and the TE of a time-variant periodogram.

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