Blind Multichannel Equalization with Controlled Delay

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ABSTRACT

In this contribution, we address a new second order approach for multichannel zero forcing equalization with controlled delay. The method basically exploits the second order whiteness input signal properties and condition of left invertibility of the multichannel. Channel identification is investigated in a second step. In comparison of existing methods, the proposed method has the interesting properties to involves some robustness with respect to channel order estimation, and similar complexity than subspace-like methods.

Keywords: Multichannel identification/equalization.

1. INTRODUCTION

During the last 5 years several blind second order multichannel identification/equalization approaches based on introducing channel diversity (due to oversampling the received analog signal and/or using a sensors array) have been proposed to suppress intersymbol interference in digital transmission systems ([2], [3], [6],...).

Unfortunately, these methods suffer from many drawbacks. The channel identification methods are known to be inconsistent when the channel order is not well estimated and/or when there is loss of channel disparity (*i.e.* when sub-channels $h_k(z)$ have close roots, see Figure 1) ([4]). The direct equalization methods (which consist in estimating a left inverse of the channel) are usually quite robust to the channel order estimation. However, in the existing methods (like linear prediction methods, for example [6]) the delay cannot be controlled at least in a one step procedure. Because this parameter is known to be important in practice in terms of the input/output Mean Square Error (MSE) performances ([8]), a method based on the estimation of equalizers corresponding to each possible delay has been developed [5]. Unfortunately this method increases considerably the computational cost and is very sensitive to the loss of channel disparity and channel order knowledge.

In this contribution, we propose an alternative second order approach, based on a specific parametrization of the 'left-null-space' of the desired global impulse response, leading to the minimization of a quadratic (convex) function in order to estimate a single equalizer with a controlled delay. The method is promising since the criterion involves some trade off robustness properties with respect over estimation of channel order. Bloc implementation with different constraints are considered. In a second step, from the zero forcing equalizer, channel identification based on input / output second order correlation methods is addressed.

2. MODEL

The multichannel equalization problem consists in choosing the $q \times 1$ Finite Impulse Response equalizer transfer function $g_\alpha(z) = (g_{\alpha,1}(z), ..., g_{\alpha,q}(z))^\top$, with $g_{\alpha,k}(z) = \sum_{p=0}^{N-1} g_{\alpha,k}(p)\, z^{-p}$ such that the output

$$v(n) = [g_\alpha(z)^\top]\; y(n) = \sum_{k=0}^{N-1} g_\alpha(k)^\top\; y(n-k) \qquad (1)$$

achieves a "good" estimate of the scalar uncorrelated. input sequence $s(n-\alpha)$ (with α an arbitrary integer delay), see Figure 1. Note that $g_\alpha(k)$ denotes the k^{th} sub-vector (of length q) of the N dimensional equalizer $g_\alpha = (g_\alpha(0)^\top, ..., g_\alpha(N-1)^\top)$. We suppose that the input signal is of variance $\sigma_s^2 = 1$. Herein, $y(n)$ is understood as the multichannel observations vector $y(n) = [h(z)]\, s(n) + w(n)$ where $h(z) = (h_1(z), ..., h_q(z))^\top$, with entries $h_k(z) = \sum_{p=0}^{Q} h_k(p)\, z^{-p}$ a polynomial function of degree Q. The additive Gaussian noise is described by the q-dimensional vector $w(n) = (w_1(n), ..., w_q(n))^\top$ independent of the input signal. We suppose that $w(n)$ is spatially and temporally white with variance σ_w^2. According to the previous model, the estimation of the input signal turns to the estimation of the vector g_α of dimension Nq (with entries the components of $g_\alpha(z)$) such that:

$$\begin{aligned} v(n) &= g_\alpha^\top Y_N(n) & (2)\\ &= \underbrace{g_\alpha^\top \mathcal{T}(\mathrm{h})}_{f^\top}\, S_{N+Q}(n) + g_\alpha^\top W_N(n) \simeq s(n-\alpha) & \end{aligned}$$

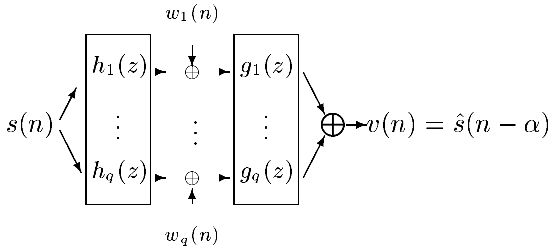

Figure 1: Multichannel Equalization Scheme

where $Y_N(n)$ (resp.$W_N(n)$) is a regressor vector of the N last q-dimensional observations $y(n)$ (resp.$w(n)$). $S_{N+Q}(n)$ contains the input sequence at $n, n-1, ..., n-N-Q+1$. $\mathcal{T}(\mathrm{h})$ is the $Nq \times (N+Q)$ Sylvester channel convolution matrix containing the taps of $h(z)$ (see [3]). Note that, $\mathcal{T}(\mathrm{h})$ is a full column-rank matrix, under the fundamental hypothesis of channel identifiability, i.e., when there is no common zeros between all components $h_k(z)$ $k = 1, ..., q$ and $N \geq Q$. For convenience we introduce the notation $M = N + Q$.

3. QUADRATIC CRITERION

In order to simplify the presentation, we give preliminary results in terms of the global impulse response f (*i.e.* channel + equalizer). First, note that (if we neglected the noise effects) perfect equalization is equivalent to estimate a vector g_α leading to $f = \lambda\,\delta_\alpha$,where $\delta_\alpha = (0...010...0)^\top$ is a canonical vector (with λ an arbitrary scalar factor) of dimension $N+Q$ such that $\delta_\alpha^\intercal\, S_{N+Q}(n) = s(n-\alpha)$. Under channel identifiability, it is *a priori* possible to introduce the following quadratic criterion (in terms of f),

$$Q_\alpha(f) \stackrel{def}{=} |\, g_\alpha^\top \mathcal{T}(\mathrm{h})\ P_\alpha\,|^2 = |\, f^\top P_\alpha\,|^2$$

where P_α is a matrix of dimension $M \times M$. The motivation comes from the consideration that if Kernell(P_α) is spanned by δ_α, then the following result holds,

$$f_\alpha = \arg\min_{f \in \Omega}\ Q_\alpha(f) \Leftrightarrow f_\alpha = \lambda\,\delta_\alpha \qquad (3)$$

where Ω is a constraint in order to avoid the trivial solution $f = 0$. Obviously, there is infinitely many solutions for P_α. We present here a straightforward solution deduced from a sum of Jordan matrices. Other variations may be developed. The simple proposed choice is,

$$P_\alpha = J(\alpha) + J(\alpha - M - 1) \qquad (4)$$

where J is a (Jordan) matrix of dimension $M \times M$, defined as $(J(m))_{ab} = 1$ if $a - b = m$ and 0 elsewhere. We may verify that dim span $(P_\alpha) = M - 1$ and that the null space (of dimension one) is spanned by δ_α. Note that each matrix P_α is associated to a specific delay $\alpha + 1$ where $\alpha \in \{0, ..., M\}$. The resulting criterion is therefore,

$$Q_\alpha(f) = f^\top P_\alpha P_\alpha^\top f = \sum_{j \neq \alpha}^{M} f_j^2 \geq 0 \qquad (5)$$

where $Q_\alpha(f) = 0$ if and only if all the components of f but the $(\alpha+1)^{th}$ are equal to zero.

A key point is to notice first that tanks to the whiteness assumption of input signal we have $S_\alpha = R_Y(\alpha) - \sigma_w^2 J(\alpha) + R_Y(\alpha - M - 1) - \sigma_w^2 J(\alpha - M - 1) = \mathcal{T}(\mathrm{h}) P_\alpha \mathcal{T}(\mathrm{h})^\intercal$ where $R_Y(k) = \mathbb{E}[Y_N(n) Y_N(n-k)^\top]$ denotes the covariance matrix of the observations. Note that this result stems form the whiteness input signal assumption. The corresponding criterion, in term of g, can be constructed therefore from the observations $Y_N(n)$. It results in a criterion of the form,

$$\widetilde{Q}_\alpha(f) \stackrel{def}{=} f^\top P_\alpha\ \left[\mathcal{T}(\mathrm{h})^\top W \mathcal{T}(\mathrm{h})\right]\ P_\alpha^\top\ f \qquad (6)$$

where the weighting matrix W is chosen such that $\mathcal{T}(\mathrm{h})^\intercal W \mathcal{T}(\mathrm{h})$ is definite positive. One may consider for example the following choices: $W = R_Y(0) - \sigma_w^2\, I$ leading to $\mathcal{T}(\mathrm{h})^\top W \mathcal{T}(\mathrm{h}) = (\mathcal{T}(\mathrm{h})^\intercal \mathcal{T}(\mathrm{h}))^2$, or $W = (R_Y(0) - \sigma_w^2\, I)^\#$ in order to get $\mathcal{T}(\mathrm{h})^\top W \mathcal{T}(\mathrm{h}) = I$ (note that in this case we obtain $\widetilde{Q}_\alpha(f) = Q_\alpha(f)$). We want to use such a criterion in order to force g_α to correspond to the desired impulse response $f_\alpha = \delta_\alpha$. An important point is that the previous criterion result in,

$$\widetilde{Q}_\alpha(g) \stackrel{def}{=} g^\top S_\alpha\ W\ S_\alpha^\top\ g \qquad (7)$$

Thus, the main result of the paper can be summarized by the relation below,

$$g_\alpha^\star = \arg\min_{g \in \Omega}\ \widetilde{Q}_\alpha(g) \Leftrightarrow \mathcal{T}(\mathrm{h})^\top g_\alpha^\star = \mu\,\delta_\alpha \qquad (8)$$

where Ω denotes some properly constraint avoiding the trivial solution $g_\alpha^\star = 0$. Note that since rank $(\mathcal{T}(\mathrm{h})) = Nq - M \geq 0$ the solution $g_\alpha^\star$ is not unique. More precisely it is subject to the specific choice of the constraint Ω. Note that other choices for W may be introduced in order to reduce the estimation variance of $g_\alpha^\star$ for example.

Proof. The proof is straightforward. Indeed, $\overline{Q}_\alpha(g) \geq 0$ with $g^\top S_\alpha\ W\ S_\alpha^\top\ g = 0$ if and only if $g^\top S_\alpha\ W = 0$ where $W > 0$. It implies that $f^\top P_\alpha = 0$. According to the definition of P_α (4) the only solutions $f$of the equation above are of the form $f = \mathcal{T}(\mathrm{h})^\top g_\alpha^\star = \mu\,\delta_\alpha = (0...0\mu 0...0)^\top$ □□□.

USEFUL CONSTRAINTS

Several constraints should be considered in the minimization of $\widetilde{Q}_\alpha(g)$. If we do not have information about the input signal distribution, we should consider, for example, the simple and classical following quadratic constraints: **a**) $\|g\|^2 = 1$, for which the solution (8) is a unit norm eigenvector associated to one zeros eigenvalue of $S_\alpha\ W\ S_\alpha^\intercal$, **b**) $\mathbb{E}\left[v^2\right] = \sigma_s^2 (= 1)$ or equivalently $\|f\|^2 = 1$, *i.e.* $g^\top (R_Y(0) - \sigma_w^2 I) g = 1$. The advantage of this constraint (at least theoretically) is to guarantee that $\mathcal{T}(\mathrm{h})^\top g_\alpha^\star = \delta_\alpha$, in other words the scale factor μ (see expression (8)) is equal to 1. This choice is more natural than the constraint (a), but it involves the minimization of a generalized Raleigh ratio increasing consequently the computational cost. A linear constraints without eigenvalue (and/or eigenvector) estimation may be also considered. In some applications partial informations on the input signal are available. We may use this knowledge to define more efficient constraints, for example, with a constant modulus input signal, we should consider : **c)** $T^{-1}\sum_{n=1}^{T}(g^\top Y_N(n)Y_N(n)^\top g-1)^2$. If a short input sequence $(s(n))_{1\le n\le T}$ is known by the receiver, a natural constraint is : **d)** $T^{-1}\sum_{n=1}^{T}(g^\top Y_N(n)-s(n))^2$.

EXAMPLE

We first illustrate the performances of estimation with 2-dimensional academics random channels $h(z)$ (see Table below for the zeros location of channel #1) driven by a BPSK sequence. The noise variance is $\sigma_w^2=0.01$.

Roots of channel #1

$h_1(z)$	1.40	-3.20
$h_2(z)$	0.50	1.90

The degree of the equalizers $g_\alpha(z)$ is chosen as $N=2$. Note that in this case $\mathcal{T}(\mathrm{h})$ is a square matrix. There is only one solution $g_\alpha^\star(z)$ corresponding to $g_\alpha^\star(z)=z^{-\alpha}h^\#(z)$. To perform the minimization of (7), we use 500 samples for the covariance matrices estimation and we minimized $\widetilde{Q}_\alpha(g)$ under the constraint $\|g_\alpha\|^2=1$. In Table I, we display the global impulse response $f_\alpha^\top=g_\alpha^\top\mathcal{T}(\mathrm{h})$ (corresponding to the average of 20 Monte Carlo realizations) for equalizers g_α leading to delay estimation $\alpha=1,3,5$. We can verify that the best result in terms of residual ISI is given by $\alpha=3$, see Table I.

Table I: Global Impulse Response

f_1	**0.9497**	0.1846	0.1922	0.1890	0.1655
f_3	0.0704	0.1059	**1.0000**	0.1490	0.0790
f_5	0.1195	0.2425	0.1217	0.2004	**1.0000**

ROBUSTNESS

One may notice that the criterion $\widetilde{Q}_\alpha(g)$ has some robustness properties in particular with respect to the channel degree estimation. Under the fundamental assumption of channel disparity (*i.e.* the sub-channels $h_k(z)$ for $k=1,...q$ do not have common roots, see [3]) the convolution matrix $\mathcal{T}(\mathrm{h})$ is full column rank when the sufficient condition given by $N\ge Q$ is met. In other words, we have only to overestimate the channel degree $h(z)$ in order to guarantee that the vector $g_\alpha^\star$ which minimize $\widetilde{Q}_\alpha(g)$ is a left inverse of the channel convolution matrix (up to a delay α), *i.e.*, $g_\alpha^{\star\top}\mathcal{T}(\mathrm{h})=\mu\,\delta$. Note that in practice all the delays α do not lead to the same MSE input/output performances because of the empirical estimation sensitivity with respect to α of the matrix $R_Y(-\alpha)+R_Y(M+1-\alpha)$, which is necessary to construct the criterion (6). In particular it is preferable to avoid the delays $\alpha=0$ and M (see the example above). A choice with a good trade off is $\alpha\sim(M+1)/2$.

Under loss of channel disparity i.e., when the multichannel is of the form $h(z)=h_0(z)\underline{h}(z)$ where $h_0(z)=\sum_{k=0}^{\gamma}h_0(k)\,z^{-k}$ is a scalar polynomial function containing the common roots of $h_k(z)$ and where $\underline{h}(z)$ is a $q\times1$ polynomial function of degree $(Q-\gamma)$. Equivalent expression in term of the convolution matrix is $\mathcal{T}(\mathrm{h})=\mathcal{T}(\underline{\mathrm{h}})\mathcal{T}(\mathrm{h}_0)$ where $\mathcal{T}(\underline{\mathrm{h}})$ is a Sylvester matrix of dimension $Nq\times(M-\gamma)$ of full-rank column $M-\gamma$ and $\mathcal{T}(\mathrm{h}_0)$ is a Sylvester matrix of dimension $(M-\gamma)\times M$ with full-rank row M. From to the relation $f=\mathcal{T}(\mathrm{h}_0)^\top\mathcal{T}(\underline{\mathrm{h}})^\top g_\alpha$ .the only achievable impulses responses f are belong to subspace spanned by the columns of $\mathcal{T}(\mathrm{h}_0)^\top$. This subspace is of dimension $M-\gamma$, *i.e.*, lower to dimension of f (equal to M). Thus, both previous remark imply that theoretically it is no possible to guarantee that the minimum $f^\star$ of the quadratic criterion (6) is δ_α even if in practice some quite good results can be obtained for a long enough degree $N-1$ of $g_\alpha(z)$ and a small γ. The performances depends in particular of the distances of the roots associated to $h_0(z)$ with respect to the unit circle, the degree N and the input/output delay α. More precisely, the criterion (6) can be written as $f^\top P_\alpha\;U\;P_\alpha^\top\;f$ with $U=\mathcal{T}(\mathrm{h}_0)^\top\mathcal{T}(\underline{\mathrm{h}})^\top W\mathcal{T}(\underline{\mathrm{h}})\mathcal{T}(\mathrm{h}_0)$, where dim span$(U)=M-\gamma$ (with W full-rank). In this case, the robustness to loss of disparity of the proposed criterion lies in solutions (in term of f) of equation $P_\alpha^\top f=\sum_{k=1}^{\gamma}\lambda_k u_k$ where $\{u_k\}$ denotes a basis of the kernel of $\mathcal{T}(\mathrm{h}_0)^\top$. This point is still under investigation.

4. CHANNEL IDENTIFICATION

From the zero forcing equalizer g_α, we are able to investigate channel identification. Let $h=(h(0)^\top,...,h(Q)^\top)^\top$the vector of dimension $q\times(Q+1)$ collecting the taps of $h(z)$. The identification of each sub-vector $h(p)$ (with $0\le p\le Q$) of length q, defined as $h(p)=(h_1(p),...,h_q(p))^\top$, is given by the following input / output correlation expression, $h(k+\alpha)=\mathbb{E}\left[y(n)\left(\left[\hat{g}_\alpha(z)^\top\right]y(n-k)\right)\right]-\mathbb{E}\left[w(n)\left(\left[\hat{g}_\alpha(z)^\top\right]w(n-k)\right)\right]$, where -$\alpha\le k\le Q-\alpha$. According the noise and signal whiteness assumptions, we get the consistent estimator:

$$h(k+\alpha)=\sum_{m=0}^{N-1}\;(R_y(k+m)-\sigma_w^2\;\delta_{0,k+m}\mathrm{I}_q)\;\hat{g}_\alpha(m)\quad(9)$$

where R_y is the covariance matrix associated to the q dimensional vector $y(n)$ defined as $R_y(p)=\mathbb{E}[y(n)y(n-p)^\top]$. A compactly expression of the channel estimation is:

$$\hat{h}=\mathcal{H}_\alpha(R_y)\hat{g}_\alpha\qquad(10)$$

where $\mathcal{H}_\alpha(R_y)$ is a block-Hankel triangular matrix defined in footnote (see next page).

Proof. We established the proof of (9) and (10). One can easily check that $\mathbb{E}[y(n)\;([\hat{g}_\alpha(z)^\top]\;y(n-k))]$ leads to $\sum_{m=0}^{Q}h(m)\;\mathbb{E}[s(n-m)s(n-k-\alpha)]+\mathbb{E}[w(n)s(n-k-\alpha)]$ $+\mathbb{E}[[h(z)]s(n)\;s(n-k-\alpha)]+\mathbb{E}[w(n)\;([\hat{g}_\alpha(z)^\top]\;w(n-k))]$.

From the independence hypotheses between signal and noise, we get (9). Note that $\mathbb{E}[y(n)\ ([\hat{g}_\alpha(z)^\top]\ y(n-k))]$ may also be written as $\mathbb{E}[y(n) \sum_{m=0}^{N-1} \widehat{g}_\alpha(m)^\top\ z^{-m} y(n-k)] = \sum_{m=0}^{N-1} \mathbb{E}[y(n)y(n-m-k)^\top]\ \widehat{g}_\alpha(m)$. For the noise contribution we get in the same way $\sum_{m=0}^{N-1} \mathbb{E}[w(n)w(n-(m+k))^\top]\ \widehat{g}_\alpha(m)$ where $\mathbb{E}[w(n)w(n-(m+k))^\top] = \sigma_w^2\ \mathrm{I}_q$ if $m+k=0$ and 0 elsewhere. By expanding the expression (9) over all $k \in \{-\alpha, ..., Q-\alpha\}$ we obtain the compact expression (10) □□□.

It is sometimes interesting to address different choices for the *delay* α. One may therefore estimate h as an average of the estimation (10) given by several equalizers $\hat{g}_\alpha$. The estimation procedure is therefore,

$$\hat{h} = \frac{1}{|K|} \sum_{\alpha \in K} \mathcal{H}_\alpha(R_y)\hat{g}_\alpha \tag{11}$$

where K denotes the subset of subscripts α of all zero forcing estimated equalizer where we recall that $\alpha \in \{0, ..., M\}$, $|K|$ is understood as the cardinal (*i.e.* the length) of K.[1]

5. SIMULATIONS

We consider the channel #2 describe from its roots in the table below. It correspond to a 2 dimensional urban radio-mobile channel simulated according to the (real value) model of Clarke applied to GSM (*COST 257). The channel taps are displayed in Table II.

Roots of channel #2

$h_1(z)$	7.2335	-1.0459	0.2364	0.0436
$h_2(z)$	6.8943	-0.5573	-0.1649	0.1693

The channel is driven by a BPSK source. We investigate at SNR=25dB a zero forcing equalizer $g(z)$ of degree $N-1=6$ leading to an input/output delay $\alpha=3$.We use the weighting matrix $W = (R_Y(0) - \sigma_w^2\, I)^\#$ and the minimization of (7) is subject to the constraint $\|f\|^2 = 1$. The covariance matrices $R_Y(k)$ are estimated with the empirical estimator $T^{-1}\sum_{n=1}^{T} Y_N(n)Y_N(n-k)^\top$, with $T = 500$. The estimated global impulse response $f_3^{\star\top} = \widehat{g}_3^{\star\top}\mathcal{T}(\mathrm{h})$ (averaged over 10 Monte Carlo runs) leads to $f_3^\star$ = (0.0252, 0.0614, 1.0000, 0.0944, 0.0643, 0.0152, 0.0267, 0.0636, 0.0086, 0.0433, 0.0071) with ISI($f_3^\star$)= $\sum_{j\neq 3} f_3(j)^2 = 0.0121$. One may verify that it is very close to the canonical vector δ_3.Note that the taps are normalized. Likewise the estimation of $f_6^\star$ leads to $f_6^\star$ = (0.0012, 0.1156, 0.0400, 0.1051, 0.6237, 1.0000, 0.0095, 0.0607, 0.1297, 0.0060, 0.0039) with an higher ISI (ISI($f_6^\star$)= 0.1982).

Table II: Taps of channel #2

$h_1(0)$	$h_1(1)$	$h_1(2)$	$h_1(3)$	$h_1(4)$
0.111	0.719	0.647	0.228	0.009
$h_2(0)$	$h_2(1)$	$h_2(2)$	$h_2(3)$	$h_2(4)$
0.134	0.847	0.513	0.026	0.014

The estimation of the channel $h(z)$ from the equalizer $g_3^\star$ is established by the procedure (10). The taps are displayed in the Table III. Note that although the channel is overestimated the estimation result is satisfactory. In particular the last taps $h(5)$ and $h(6)$ are almost equal to zeros ($\sim 10^{-4}$).

Table III: Channel estimation (with overestimation)

$\widehat{h}_1(0)$	$\widehat{h}_1(1)$	$\widehat{h}_1(2)$	$\widehat{h}_1(3)$	$\widehat{h}_1(4)$	$\widehat{h}_1(5)$	$\widehat{h}_1(6)$
0.219	0.650	0.699	0.203	0.021	0.006	0.000
$\widehat{h}_2(0)$	$\widehat{h}_2(1)$	$\widehat{h}_2(2)$	$\widehat{h}_2(3)$	$\widehat{h}_2(4)$	$\widehat{h}_2(5)$	$\widehat{h}_2(6)$
0.246	0.785	0.568	0.014	0.027	0.000	0.000

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$$\mathcal{H}_\alpha \stackrel{def}{=} \begin{bmatrix} R_Y(-\alpha) & R_Y(-\alpha+1) & \cdots & R_Y(0)-\hat{\sigma}^2 I & & \cdots & R_Y(Q-\alpha) \\ R_Y(-\alpha+1) & \cdots & R_Y(0)-\hat{\sigma}^2 I & \cdots & R_Y(Q-\alpha) & 0 & 0 \\ \vdots & & & & \vdots & & \\ R_Y(0)-\hat{\sigma}^2 I & \cdots & R_Y(Q-\alpha) & 0 & \cdots & & 0 \\ R_Y(Q-\alpha) & 0 & 0 & & \cdots & & 0 \end{bmatrix}$$