NONLINEAR FRAME-LIKE DECOMPOSITIONS

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ABSTRACT

In this paper, we revisit a number of concepts which have recently proven to be useful in multiscale signal analysis, specifically by replacing the now classical linear scale transition operators by nonlinear ones. Connections between nonlinear perfect reconstruction filter banks and PDE operators used in scale-space theory are established. An application of the proposed nonlinear tools is then given for extracting a signal embedded in noise. We also develop the important case of time invariant nonlinear representations.

1 Introduction

There has recently been an increasing interest in nonlinear multiscale analysis of signals and images [1], [2]. This is to a large extent due to the interest in preserving important sharp features such as transitions/edges present in a signal.

One possible approach consists of finding nonlinear extensions of linear subband decompositions [2]. These nonlinear structures are based on a nonlinear scale transition operator operating from fine to coarse resolution. The resulting coefficients are composed of the approximation coefficients of the signal at a given resolution level and of the details differentiating two consecutive resolution levels. This methodology is very general as it allows for a very wide class of linear or nonlinear operators. Thus far, one has been selecting "good" nonlinear filters on the basis of the given problem (lossy or lossless image compression, feature sieves, signal enhancement,...), somewhat heuristically. In this work, we will give some selection guidelines by establishing a connection between nonlinear filter banks and nonlinear Partial Differential Equations (PDE's) used in scale-space theory for which feature-driven operators have been well researched, albeit sometimes heuristic. We further extend these nonlinear decompositions by endowing them with a shift-invariance property. Finally, an example illustrating the importance of such nonlinear techniques in extracting a signal embedded in noise is provided.

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The paper is organized as follows: in Section 2 we briefly review the basic principles of discrete-time nonlinear multiresolution decompositions. Section 3 shows some connections existing between these operators and multiscale approaches based on PDE's. Then, in Section 4, we propose some extensions of these nonlinear decompositions allowing us to obtain a time-invariant denoising method. Finally, in Section 5, we give some illustrative simulation results.

2 Background and Notation

It is now well-known [3] that wavelet decompositions are implemented by cascading two-channel linear filter banks. If we now try to find nonlinear extensions of these linear scale transition operators, we can use the structure shown in Fig. 1 where $\mathcal{H}$ and $\mathcal{G}$ are arbitrary operators and $\mathcal{A}_1$ and $\mathcal{A}_2$ are one-to-one mappings. The decomposition process is initialized by taking $c_0(k) = x(k)$, where $x(k)$ is the signal to be analyzed. By an appropriate choice of $\mathcal{H}$ and $\mathcal{G}$, the coefficients $c_j(k)$ may be interpreted as the approximation coefficients of the signal at resolution level j and $d_j(k)$ corresponds to the details lost when passing from resolution level j to the next coarser one $(j+1)$. If the decomposition is iterated up to resolution level $j_{\rm m}$, it provides (under some weak conditions [2]) a periodically time-invariant, critically subsampled decomposition with perfect reconstruction. The analysis filter bank in Fig. 1 is associated to a dual synthesis filter bank in a straightforward way. When $\mathcal{H}$, $\mathcal{G}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ are linear operators, it can be shown that this structure allows us to generate all possible orthogonal/biorthogonal discrete-time wavelet decompositions. If we now wish to design nonlinear decompositions, the only constraint to be satisfied is the injection of the operators $\mathcal{A}_1$ and $\mathcal{A}_2$. Therefore, a great variety of filters may be envisaged such as Volterra filters, morphological or order statistics filters. Up to now, there was no guideline of what a "good" nonlinear filter should be for a given problem, as for a given application a choice was made with no particular feature-driven objective. In the next section, we will lift this limitation by providing some guidelines which naturally fall out of the established connection between this nonlinear analysis framework and existing PDE-based techniques.

3 Connections with PDE's

In scale-space theory, a representation $c(t,s)$ at scale s of the analyzed continuous-time signal $x(t)$ is defined as the solution of the following PDE:

$$\frac{\partial c(t,s)}{\partial s} = \mathcal{F}[c](t,s) \qquad (1)$$

with the initial condition:

$$c(t,0) = x(t) \ . \qquad (2)$$

Increasing the scale parameter s leads to "simpler" signal representations. The operator $\mathcal{F}$ is a function of $c(t,s)$ and some of its partial derivatives with respect to t. If, for instance,

$$\mathcal{F}[c](t,s) = \frac{\partial^2 c(t,s)}{\partial t^2} \qquad (3)$$

the linear heat diffusion equation is obtained. Its solution can be shown to amount to convolving $c(t,s)$ with increasingly broad Gaussian functions (with "variance" proportional to s) [5]. It is interesting to note that such an analysis is actually similar to the first pyramidal decompositions which have been proposed [4] in a discrete-time framework. To have a non uniform smoothing of the signal which can take into account its local regularity properties, more sophisticated forms of $\mathcal{F}$ must be chosen. In particular, introducing nonlinearities can help us in slowing down the diffusion process in parts of the signal where discontinuities arise. As suggested in [6], an interesting class of PDE's is defined by

$$\mathcal{F}[c](t,s) = \frac{\partial[F(\partial c(t,s)/\partial t)]}{\partial t} \qquad (4)$$

where $F : \mathbb{R} \longrightarrow \mathbb{R}$ is an appropriate "force" function. The case where F reduces to the identity function corresponds to the linear diffusion in Eq. (3) whereas, if

$$F(u) \propto u \exp(-\frac{u^2}{K}) \ , \qquad (5)$$

with a positive thickness parameter K, the nonlinear equation of Perona and Malik [7] is found.
Discretizing Eqs. (1) and (4) both in time and scale and defining

$$\widetilde{c}(k,j) = c(k\Delta t, j\Delta s) \ , \quad \bar{F}(u) = \frac{\Delta s}{\Delta t} F(\frac{u}{\Delta t}) \ ,$$

lead to

$$\widetilde{c}(k,j+1) = \widetilde{c}(k,j) + \bar{F}(\widetilde{c}(k+1,j) - \widetilde{c}(k,j)) - \bar{F}(\widetilde{c}(k,j) - \widetilde{c}(k-1,j)). \qquad (6)$$

Assume now that

$$\widetilde{c}(k,j) - \widetilde{c}(k-1,j) \simeq \widetilde{c}(k-1,j) - \widetilde{c}(k-2,j).$$

(This approximation can be justified by some arguments of equality of the left and right derivatives of $c(t,s)$ at a given time.) By decimating $\widetilde{c}(k,j+1)$ by a factor 2, Eq. (6) yields

$$\widetilde{c}(2k,j+1) = \widetilde{c}(2k,j) + \bar{F}(\widetilde{d}(2k,j+1)) - \bar{F}(\widetilde{d}(2k-2,j+1)) \ ,$$

where

$$\widetilde{d}(k,j+1) = \widetilde{c}(k+1,j) - \widetilde{c}(k,j) \ . \qquad (7)$$

This means that the relations between $(\widetilde{c}(2k,j), \widetilde{c}(2k+1,j))$ and $(\widetilde{c}(2k,j+1), \widetilde{d}(2k,j+1))$ are identical to those existing between $(c_j(2k), c_j(2k+1))$ and $(c_{j+1}(k), d_{j+1}(k))$ in Fig. 1. In the present case, $\mathcal{G}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ reduce to the identity operator and $\mathcal{H}$ is obtained by cascading the memoryless nonlinear transform $\bar{F}$ with the finite difference filter with z-transfer function $(1 - z^{-1})$.

This result does not mean that Fig. 1 corresponds to an efficient numerical method for solving a PDE since we used a relatively rough discretization scheme. The main point of this result is to rather provide an insight into the possible choices for the nonlinear transform $\mathcal{H}$. By reflecting on the classical choices made in scale-space theory, e.g. (5), we can simply select a nonlinear transform with a variety of well defined goals.
In addition to its denoising potential and similarly to the scale-space analysis framework, this nonlinear multiresolution framework also affords one the ability to retrieve the details of the analysis if so desired.

4 Translation Invariance

The reconstruction property which has been imposed on the nonlinear filter bank in Fig. 1 is useful to guarantee that the decomposition gives a complete representation of the analyzed signal. Given that the considered filter bank is critically subsampled, it cannot however be time-invariant. This implies that, if the decomposition is used in a denoising algorithm, the estimation error will be particularly sensitive to the positions of the discontinuities in the signal. A similar problem is encountered with wavelet decompositions for which time-invariant versions have also recently been proposed [8], [9]. These shift-invariant decompositions are based on an underlying frame which is the union of different orthonormal bases. This characteristic simplify the implementation of the decomposition and leads to simple reconstruction formula. We next show how these methods extend to the nonlinear case.

An undecimated decomposition can be deduced from the non-redundant decomposition in Fig. 1 (with $\mathcal{A}_1$ and $\mathcal{A}_2$ equal to identity) by the following recursive equations:

$$\begin{aligned} \bar{d}_{j+1}(k) &= \bar{c}_j(k+2^j) - \mathcal{G}[(\bar{c}_j(k+2^{j+1}\ell))_{\ell\in\mathbb{Z}}] , \\ \bar{c}_{j+1}(k) &= \bar{c}_j(k) + \mathcal{H}[(\bar{d}_{j+1}(k+2^{j+1}\ell))_{\ell\in\mathbb{Z}}] , \end{aligned} \quad (8)$$

where $\bar{c}_0(k) = c_0(k)$. The coefficients $\bar{c}_j(k)$ and $\bar{d}_j(k)$ may be interpreted as components of the signal at resolution j and time-localization k and, we clearly have $c_j(k) = \bar{c}_j(2^j k)$ and $d_j(k) = \bar{d}_j(2^j k)$. To compute the coefficients $\bar{c}_j(k)$ and $\bar{d}_j(k)$, we can note that if, for all $p \in \{0, \ldots, 2^j - 1\}$,

$$c_{j,p}(k) = \bar{c}_j(2^j k + p) , \quad d_{j,p}(k) = \bar{d}_j(2^j k + p) ,$$

then,

$$\begin{aligned} d_{j+1,p}(k) &= c_{j,p}(2k+1) - \mathcal{G}[(c_{j,p}(2k+2\ell))_{\ell\in\mathbb{Z}}] , \\ c_{j+1,p}(k) &= c_{j,p}(2k) + \mathcal{H}[(d_{j+1,p}(k+\ell))_{\ell\in\mathbb{Z}}] , \end{aligned}$$

and

$$\begin{aligned} d_{j+1,p+2^j}(k) &= c_{j,p}(2k+2) - \mathcal{G}[(c_{j,p}(2k+2\ell+1))_{\ell\in\mathbb{Z}}] , \\ c_{j+1,p+2^j}(k) &= c_{j,p}(2k+1) + \mathcal{H}[(d_{j+1,p+2^j}(k+\ell))_{\ell\in\mathbb{Z}}] . \end{aligned}$$

Thus, $(c_{j+1,p}(k), d_{j+1,p}(k))$ can be computed from $(c_{j,p}(2k), c_{j,p}(2k+1))$ by using the analysis filter bank in Fig. 1 and $(c_{j+1,p+2^j}(k), d_{j+1,p+2^j}(k))$ is obtained from $(c_{j,p}(2k+1), c_{j,p}(2k+2))$, in the same way. This means that, when $(c_{j,p}(2k), c_{j,p}(2k+1))$ have been computed, we have just to shift by one sample the input $c_{j,p}(k)$ of the filter bank to generate $(c_{j+1,p+2^j}(k), d_{j+1,p+2^j}(k))$. At each scale, we have two possibilities (shift or not), which leads to 2^{j_m} different critically subsampled representations of the analyzed signal.

Different fusion strategies may be thought of to reconstruct the signal from these different representations. We can for instance search for a "best" representation or average the different reconstructions. In this case, a linear or nonlinear "mean" may be used. The median filter is often preferred because of its attractive property of preserving edges.

5 Denoising with nonlinear decompositions

An underlying signal in noise problem is a frequently encountered scenario in applications. Let,

$$x(k) = x^*(k) + b(k)$$

where $x^*(k)$ is the signal of interest and $b(k)$ is an additive noise.

Such an example is provided in Fig. 2. The signal of interest is embedded in a noise whose probability distribution is a Gaussian mixture. The standard deviation of the noise is here equal to 92.97. The observed signal was decomposed according to a nonlinear filter bank decomposition using a Perona-Malik force function. In this simulation, the detail coefficients were simply truncated and a median based shift-invariant method was applied to reconstruct the signal. For comparison, a similar denoising approach was performed using a Daubechies 4 orthonormal wavelet decomposition. It is clear that the nonlinear method allows for a much improved preservation of the features in the original signal. This improvement is confirmed by the standard deviation of the reconstruction error which is equal to 39.19 for the wavelet based approach whereas it is equal to 23.09 for the nonlinear decomposition method.

6 Conclusions

In this paper, we have shown some connections between PDE's and nonlinear decompositions. These connections allowed us to build nonlinear filter banks using the tools of nonlinear diffusion. We have seen that such tools provide promising results for signal denoising. By the way, we have extended shift-invariant wavelet representations to shift-invariant nonlinear decompositions. Future work will be concerned with extensions of these results to anisotropic multiscale processing of images.

7 References

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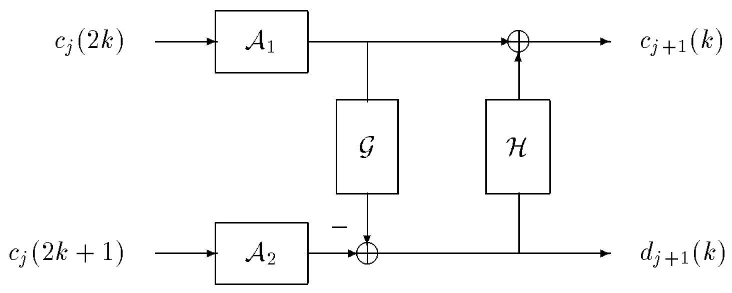

Figure 1: Nonlinear scale transition operator.

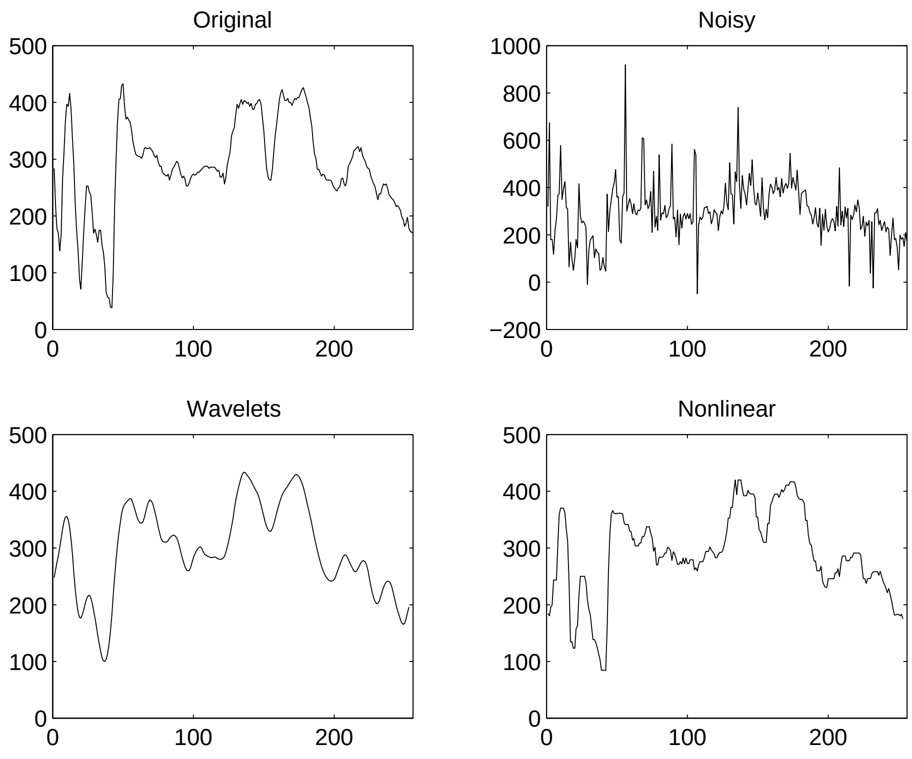

Figure 2: Denoising example.