TRANSFER FUNCTION MODELS FOR CONTINUOUS AND DISCRETE MULTIDIMENSIONAL SYSTEMS

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ABSTRACT

Continuous multidimensional systems, which are described by partial differential equations are usually discretized by standard methods from numerical mathematics. Here, a general approach for the transfer function description of multidimensional systems is presented. It allows a correct representation of initial and boundary values also for problems with spatially varying coefficients, general boundary conditions, three spatial dimensions and general differentiation operators. In spite of this generality, the resulting discrete systems can be realized with standard signal processing elements and are free of implicit loops.

1 INTRODUCTION

There are two principal methods for the design of discrete systems: Either the discrete system is modelled after an appropriate continuous system or it is designed according to requirements posed directly in the discrete domain. Methods for both cases are well established for onedimensional systems. Well known examples for the first case are analog-to-discrete transformations of the Laplace transfer function of a continuous system into the z-transfer function of a discrete system or alternatively, the synthesis of a discrete filter structure from a continuous network description, e. g. by wave digital filtering principles. An example for the second case is the design of digital filters according to a tolerance scheme.

The situation is different for multidimensional systems. Discrete systems, e. g. for image or seismic processing, are almost exclusively constructed from requirements in the discrete domain. One reason is that for many multidimensional signal processing tasks a continuous reference system does not exist. But even if there is a multidimensional continuous system model, often the proper methods for a transformation into an effective discrete system are lacking. An important example are multidimensional continuous systems, which are described by partial differential equations. Their discretization by standard methods of numerical mathematics leads to large systems of equations, which do not lend themselves to effective signal processing.

Recent advances from the signal processing community include the extension of the wave digital filter approach to the multidimensional case [2] with current applications to the simulation of wave propagation on transmission lines [3]. The general idea is to describe the multidimensional continuous system (e.g. the transmission line) by a multidimensional network, from which a discrete system is derived by wave digital filter principles.

A different approach is to set up a transfer function description of a multidimensional continuous system which takes the the initial and boundary conditions explicitely into account and to derive a discrete system by standard analog-to-discrete transformations. This approach relies on the proper selection of the functional transformation for the spatial variable according to the partial differential equation, the shape of the spatial domain and the type of boundary conditions for the given problem. A description for typical cases of parabolic and hyperbolic problems (heat flow, wave propagation) has been given in [6, 7].

It is the purpose of this contribution to describe the derivation of a proper multidimensional transfer function in a general framework. Section 2 presents a general MD problem with second order time and space derivation operators. Section 3 describes the corresponding transfer function model, explained by an example in section 4. The extension to more general problems is shown in section 5. Finally, section 6 discusses the discrete version of these transfer function models.

2 PROBLEM DESCRIPTION

A description of the general problem is given in Fig. 1. We consider continuous systems described by a partial differential equation (PDE) with an operator D for derivation with respect to time t and an operator L for the spatial derivatives. The space coordinate $\mathbf{x}$ may be one, two, or three-dimensional. The excitation function, the initial and the boundary values are denoted by $v(\mathbf{x}, t)$, $\mathbf{y}_i(\mathbf{x})$ and $\phi(\mathbf{x}, t)$, respectively. The spatial differentiation operator L is defined on the whole spatial domain V, while the boundary value operator f_b is defined on the surface S of V. For the moment, we assume that L is a linear self-adjoint operator of order two (for instance $L\{y\} = \Delta y$). Then f_b is of the general form

$$f_b\{y(\mathbf{x},t)\} = p(\mathbf{x})y(\mathbf{x},t) + q(\mathbf{x})\partial_n y(\mathbf{x},t) \quad \mathbf{x} \in S, \quad (1)$$

where $\partial_n y$ denotes the spatial derivative normal to S.

The time derivation operator D is also a second order operator with constant coefficients ($\dot{y}$ denotes time derivativation)

$$D\{y(\mathbf{x},t)\} = a_2\ddot{y}(\mathbf{x},t) + a_1\dot{y}(\mathbf{x},t) + a_0 y(\mathbf{x},t) \ . \quad (2)$$

The corresponding initial conditions are

$$\mathbf{f}_i\{y(\mathbf{x},0)\} = \begin{bmatrix} y(\mathbf{x},0) \\ \dot{y}(\mathbf{x},0) \end{bmatrix} = \begin{bmatrix} y_{i0}(\mathbf{x}) \\ y_{i1}(\mathbf{x}) \end{bmatrix} = \mathbf{y}_i(\mathbf{x}) \quad \mathbf{x} \in V \ . \quad (3)$$

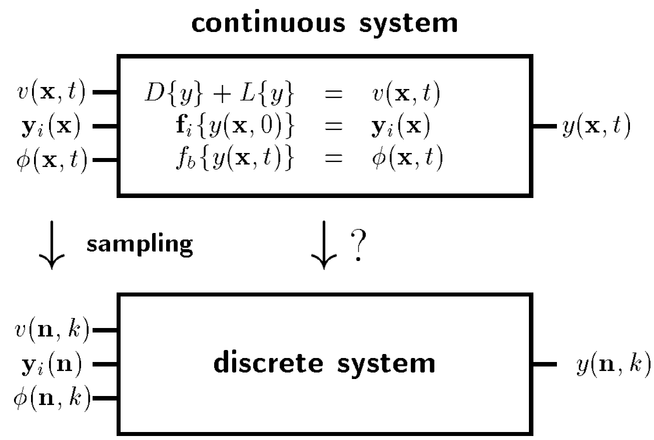

Figure 1: Problem description

We do not assume the point of view of numerical mathematics, which considers this initial-boundary value problem for given excitation, initial and boundary value functions and tries to determine some approximation of the solution $y(\mathbf{x},t)$. Instead we consider the given PDE as the description of a continuous multidimensional system with excitation, initial and boundary values as input signals and $y(\mathbf{x},t)$ as output signal. Our aim is to derive the description of a discrete system, which behaves similar as the continuous system. This means that we expect some discrete space ($\mathbf{n}$) and discrete time (k) approximation $y(\mathbf{n},k)$ at the output of the discrete system, when the input is fed with samples of the continuous input signals.

The problem is at first to chose a complete description of the continuous system, which encompasses not only the PDE itself but also the initial and boundary conditions. The second part is the derivation of a realizable discrete system from this continuous system description.

3 TRANSFER FUNCTION MODEL

An appropriate description of multidimensional (MD) continuous systems are transfer function models. Since a detailed account is given in [9], only the main results are compiled here. To shorten the presentation, we assume no excitation, i.e. $v(\mathbf{x},t) = 0$.

The derivation of a transfer function model from the PDE description of a MD system goes along the same lines as the derivation of the transfer function of a one-dimensional system from an ordinary differential equation. However, for time and space dependend systems according to Fig. 1, two functional transformations are required.

With respect to time, we apply the Laplace transformation

$$\mathcal{L}\{y(\mathbf{x},t)\} = Y(\mathbf{x},s) = \int_0^\infty y(\mathbf{x},t)\, e^{-st}\, dt \ . \quad (4)$$

The transform of the operator of time derivatives ($\mathcal{L}\{D\{y\}\}$) is obtained from (2) with the standard differentiation theorem of the Laplace transformation

$$\mathcal{L}\{D\{y(\mathbf{x},t)\}\} = \gamma^2(s)Y(\mathbf{x},s) - \mathbf{B}_i^T(s)\mathbf{y}_i(\mathbf{x}) \quad (5)$$

with

$$\gamma^2(s) = a_2 s^2 + a_1 s + a_0\,, \quad \mathbf{B}_i(s) = \begin{bmatrix} a_2 s + a_1 \\ a_2 \end{bmatrix} . \quad (6)$$

Laplace transformation with respect to time turns the initial-boundary value problem from Fig. 1 into a pure boundary value problem for $Y(\mathbf{x},s)$

$$\begin{aligned} \gamma^2(s)Y(\mathbf{x},s) + L\{Y(\mathbf{x},s)\} &= \mathbf{B}_i^T(s)\mathbf{y}_i(\mathbf{x}), \ \mathbf{x} \in V \quad (7) \\ f_b\{Y(\mathbf{x},s)\} &= \Phi(\mathbf{x},s), \qquad \mathbf{x} \in S. \quad (8) \end{aligned}$$

With respect to space, we construct a similar functional transformation. The transformation kernel $K(\mathbf{x},\beta)$ and the spatial frequency variable β are yet unspecified.

$$\mathcal{T}\{Y(\mathbf{x},s)\} = \bar{Y}(\beta,s) = \int_V Y(\mathbf{x},s)K(\mathbf{x},\beta)\, dV \ . \quad (9)$$

To apply this transformation to the operator of spatial derivatives, we need a differentiation theorem of the generic form

$$\mathcal{T}\{L\{Y(\mathbf{x},s)\}\} = \beta^2\bar{Y}(\beta,s) - \bar{\Phi}_b(\beta,s) \ . \quad (10)$$

The term $\bar{\Phi}_b(\beta,s)$ is obtained from the boundary values $\Phi(\mathbf{x},s)$ and plays the same role as the term $\mathbf{B}_i^T(s)\mathbf{y}_i(\mathbf{x})$ in (5). For the moment, we assume that a spatial transformation $\mathcal{T}$ with the differentiation property (10) exists. Then application of $\mathcal{T}$ turns the boundary value problem (8) into an algebraic equation. Solving this equation for $\bar{Y}(\beta,s)$ gives finally the desired transfer function description

$$\bar{Y}(\beta,s) = \bar{\mathbf{G}}_i^T(\beta,s)\bar{\mathbf{y}}_i(\beta) + \bar{G}_b(\beta,s)\bar{\Phi}_b(\beta,s) \quad (11)$$

with the transfer functions for the initial and boundary values

$$\begin{aligned} \bar{\mathbf{G}}_i(\beta,s) &= \frac{1}{\gamma^2(s)+\beta^2}\,\mathbf{B}_i(s) \quad (12) \\ \bar{G}_b(\beta,s) &= \frac{1}{\gamma^2(s)+\beta^2} \ . \quad (13) \end{aligned}$$

These transfer functions serve as starting point for deriving corresponding discrete systems by standard analog-to-discrete transformations.

The only problem left is the determination of the transformation kernel $K(\mathbf{x}, \beta)$ in (9) such that the relation (10) holds. The key element for the solution is Green's formula (or Green's identity), which is fulfilled by self-adjoint linear differential operators [1, 4, 5]

$$\int_V L\{Y\} K\, dV - \int_V Y L\{K\}\, dV = \tag{14}$$
$$= \int_S g_b\{Y\}\, f_b\{K\}\, dS - \int_S g_b\{K\}\, f_b\{Y\}\, dS\,.$$

Now we can set up conditions for the determination of the transformation kernel. If $K(\mathbf{x}, \beta)$ satisfies a so called Sturm-Liouville problem of the form

$$L\{K(\mathbf{x}, \beta)\} = \beta^2 K(\mathbf{x}, \beta) \tag{15}$$
$$f_b\{K(\mathbf{x}, \beta)\} = 0\,, \tag{16}$$

then Green's formula (14) turns into the differentiation theorem (10) with

$$\bar{\Phi}_b(\beta, s) = \int_S g_b\{K(\mathbf{x}, \beta)\}\, \Phi(\mathbf{x}, s)\, dS\,. \tag{17}$$

From the theory of Sturm-Liouville problems follows that the spatial frequency β is a discrete variable (eigenvalue of (15,16)) and that the transformation kernels $K(\mathbf{x}, \beta)$ are orthogonal functions. Thus the inverse transformation $\mathcal{T}^{-1}$ is given as an orthogonal expansion.

4 EXAMPLE

As an example, we consider a problem with a one-dimensional space coordinate x with $V = \{x | 0 < x < d\}$ and $S = \{0, d\}$. The differential operators represent a diffusion or a heat flow problem

$$D\{y(x,t)\} = c\dot{y}(x,t), \quad L\{y(x,t)\} = -[\lambda(x) y'(x,t)]'\,. \tag{18}$$

y' denotes spatial derivation. c and $\lambda(x)$ are material parameters, where $\lambda(x)$ may be spatially varying. The validity of Green's formula for this problem follows from integration by parts

$$\int_0^d L\{Y\} K\, dx - \int_0^d Y L\{K\}\, dx = \tag{19}$$
$$= \lambda[YK' - Y'K]\big|_0^d = [g_b\{Y\} f_b\{K\} - g_b\{K\} f_b\{Y\}]\big|_0^d\,.$$

The assignment of the boundary operators f_b and g_b is not unique. It can be adapted to the boundary conditions at hand. The operators for boundary conditions of the first, second and third kind are compiled in table 1.

For this problem, the transfer function for the initial condition is scalar and equal to the transfer function of the boundary conditions

$$\bar{G}_i(\beta, s) = \bar{G}_b(\beta, s) = \frac{1}{cs + \beta^2}\,. \tag{20}$$

boundary condition	$f_b\{Y\}$	$g_b\{Y\}$
1. kind (Dirichlet)	Y	$-\lambda Y'$
2. kind (Neumann)	$\pm\lambda Y'$	$\pm Y$
3. kind (Robin)	$pY \pm q\lambda Y'$	$\pm\frac{1}{q} Y$

Table 1: boundary operators f_b and g_b for different kinds of boundary conditions

Each spatial frequency is modelled by a first order system in time.

5 ADVANCED PROBLEMS

The transfer function models discussed in section 3 do not cover the most general case. Some generalizations to more advanced problems are discussed briefly here.

Space Dependent Coefficients. The 1D example in section 4 included a space dependent parameter $\lambda(x)$, but the parameter c was constant (see (18)). To set up the transfer function of a system with 3D space dependent parameter $c(\mathbf{x})$, the spatial transformation $\mathcal{T}$ has to be defined with $c(\mathbf{x})$ as a weighting factor

$$\mathcal{T}\{Y(\mathbf{x}, s)\} = \bar{Y}(\beta, s) = \int_V c(\mathbf{x}) Y(\mathbf{x}, s) K(\mathbf{x}, \beta)\, dV. \tag{21}$$

Higher order differential operators. Many technical problems are modelled by PDEs with up to second order differential operators. In elasiticity theory also fourth order operators occur. A higher order spatial operator L results in a more complicated Green's formula and a higher order Sturm-Liouville problem. However, once this is solved, the order of the transfer functions is only determined by the differential operator D. So most problems of practical importance result in transfer functions with a denominator polynomial in s of order one to four.

Non Self-Adjoint Differential Operators. The concept presented so far can also be applied to systems with non self-adjoint spatial operators L. In this case, Green's formula has to be formulated for L and the corresponding adjoint operator $\tilde{L}$

$$\int_V L\{Y\} K\, dV - \int_V Y \tilde{L}\{K\}\, dV = \tag{22}$$
$$= \int_S g_b\{Y\}\, \tilde{f}_b\{K\}\, dS - \int_S \tilde{g}_b\{K\}\, f_b\{Y\}\, dS\,.$$

While L and f_b are given by the PDE of the MD system, $\tilde{L}$, $\tilde{f}_b$, g_b, and $\tilde{g}_b$ follow from the derivation of Green's formula (22). To each of the operators L and $\tilde{L}$ belongs a different set of eigenfunctions. Neither of them is orthogonal, but together they form a set of biorthogonal functions. This means, that also for non self-adjoint operators L, the inverse transformation $\mathcal{T}^{-1}$ is given by a series expansion.

6 DISCRETE SYSTEMS

The extension to increasingly advanced problems presented in the last section is reflected by a corresponding increase in the complexity of the associated eigenvalue problems for the determination of the spatial transformation. However, due to the general formulation, this complexity only affects the operators L and f_b and neither the general structure of the transformation $\mathcal{T}$ nor the form of the differentiation theorem $\mathcal{T}\{L\{Y\}\}$. As a consequence, the simple form of the transfer functions is the same also for more advanced problems. This means that the structure of the discrete systems derived for simpler classes of problems in [6, 7, 8] remains also valid for the advanced problems considered here.

As an example, we consider a problem with a boundary value of the form

$$\phi(\mathbf{x}, t) = \sum_{\kappa=1}^{K} \phi_\kappa(\mathbf{x}) \psi_\kappa(t). \tag{23}$$

It describes K sources at the boundary which vary with time, but do not move. Then $\bar{\Phi}_b(\beta, s)$ has the form

$$\bar{\Phi}_b(\beta, s) = \sum_{\kappa=1}^{K} \bar{\phi}_{b\kappa}(\beta) \Psi_\kappa(s) \tag{24}$$

where $\bar{\phi}_{b\kappa}(\beta)$ is calculated from (17) with $\Phi(\mathbf{x}, s)$ replaced by $\phi_\kappa(\mathbf{x})$. Fig. 2 shows the resulting transfer function description. Single lines denote time dependent quantities, double lines denote time and space dependent quantities.

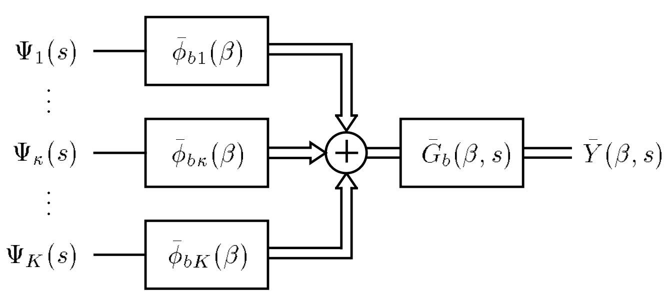

Figure 2: MD transfer function description

The only time dependent part is the transfer function with the simple structure of (13). It can be approximated by simple recursive systems as shown in [6, 7, 8]. The complexity of the spatial operator L and the boundary conditions f_b show up in the time independent terms $\bar{\phi}_{b\kappa}(\beta)$. They are computed in advance before the operation of the discrete system.

Thus rather general multidimensional systems with some or all of the advanced features from above can be effectively approximated by discrete structures, which are realizable by add, multiply, shift and delay operations and do not contain delay free loops.

7 CONCLUSIONS

We have discussed the development of transfer function models for multidimensional systems in a rather general framework. This extends an earlier reported method for setting up discrete simulation models for simple parabolic or hyperbolic problems. It has been shown that this approach now covers problems in three spatial dimensions, with possibly spatially varying coefficients, and with rather general differential operators. This allows to solve applications in heat and mass transport, mechanics, and electromagnetics by MD signal processing methods.

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