RECONSTRUCTION OF LOCALLY HOMOGENEOUS IMAGES

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ABSTRACT

The reconstruction of images involving large homogeneous zones from noisy data, given at the output of an observation system, is a common problem arising in various applications. A popular approach for its resolution is regularized estimation: the sought image is defined as the minimizer of an energy function combining a data-fidelity term and a regularization prior term. The latter term results from applying a set of potential functions (PFs) to the differences between neighbouring pixels and it can be seen as a Markovian energy.

We formalize and perform a mathematical study of the possibility to obtain images comprising either strongly homogeneous regions or weakly homogeneous zones, using regularized estimation. Our results reveal that the recovery of zones of either type in an estimated image depends *uniquely* on the smoothness at zero of the PFs involved in the prior term. These theoretical results are illustrated on the deblurring of an image.

1 INTRODUCTION

In various image reconstruction problems, a sought image $\widehat{\boldsymbol{x}}$ is recovered from observed data $\boldsymbol{y}$ as the minimizer of an energy function $\mathcal{E}_y$ [1, 5]. The latter combines closeness to data $\boldsymbol{y}$, measured by $\mathcal{L}$, and regularity with respect to (w.r.t.) prior constraint Φ, *via* $\beta > 0$:

$$\widehat{\boldsymbol{x}}=\mathcal{X}(\boldsymbol{y})=\arg\min_{\boldsymbol{x}} \mathcal{E}_y(\boldsymbol{x}), \quad \mathcal{E}_y(\boldsymbol{x})=\mathcal{L}_y(\boldsymbol{x})+\beta\Phi(\boldsymbol{x}). \quad (1)$$

$\mathcal{X}$ denotes estimator function. Data-fidelity is usually $\mathcal{L}_y(\boldsymbol{x})\propto -\ln p(\boldsymbol{y}|\boldsymbol{x})$. Linear-Gaussian models, $\boldsymbol{y}=\mathcal{A}\boldsymbol{x}+\boldsymbol{n}$ where $\mathcal{A}$ is an operator and $\boldsymbol{n}$ is white Gaussian noise, yield $\mathcal{L}_y(\boldsymbol{x}) = \|\mathcal{A}\boldsymbol{x}-\boldsymbol{y}\|^2$. Such models are used in image restoration, seismic imaging, non-destructive evaluation, X-ray tomography. Cite also the nonlinear models used in emission and transmission computed tomography [4]. In general, $\mathcal{L}_y$ can take different functional forms. In this work, $\mathcal{L}_y$ *is assumed to be continuously differentiable ($\mathcal{C}^1$-continuous) w.r.t. both $\boldsymbol{x}$ and $\boldsymbol{y}$.*

Typical images in these applications exhibit different kinds of local homogeneity. Such prior knowledge can be incorporated in estimator $\mathcal{X}$ by means of Φ. We consider the widely used class of regularizers Φ which are defined as the application of a set of *potential functions* (PFs) to the differences between neighbouring pixels; equivalently, Φ is the energy of a Markov random field. Although the interpretation of $\widehat{\boldsymbol{x}}$ as a MAP estimate [1, 5], the question of the link between the shape of Φ and the attributes of $\widehat{\boldsymbol{x}}$ is intricate and it remains open.

The problem that we set and resolve concerns the possibility for that a regularized estimator (1) yields images $\widehat{\boldsymbol{x}}$ containing regions which are either strongly homogeneous or weakly homogeneous. *Our results reveal that the recovery of regions of either type in an estimated image depends* uniquely *on the smoothness at zero of the PFs involved in Φ.* The proofs can be found in [9].

To our knowledge, this generic problem was not formalized previously. Several related works concern particular estimators [6, 7]. Recently, we studied the estimation of 1D signals containing strongly homogeneous zones from data obtained from a linear system corrupted by white Gaussian noise [8]. Now, we examine the estimation of images in the context of general observation systems: this topic is critically related to the spatial extent of the neighbourhood structure of images; our results hold for *arbitrary* neighbourhoods.

2 REGULARIZED IMAGE ESTIMATE

2.1 Markovian Energy on Differences

Let $\boldsymbol{x}$ be an $I \times J$ image whose $M = IJ$ sites are scanned column-by-column; so $\boldsymbol{x}$ is identified to a vector of $\mathbb{R}^M$; similarly, $\boldsymbol{y} \in \mathbb{R}^N$. Regularizer Φ is defined by the application of a set of PFs $(\varphi^q)_{q=1}^Q$ to a set of linear combinations of neighbouring pixels $\boldsymbol{d}_k^q\boldsymbol{x}$ [1, 7]:

$$\Phi(\boldsymbol{x})=\sum_{\mathcal{Q}} \varphi^q(\boldsymbol{d}_k^q\boldsymbol{x}) \quad \text{where} \quad \boldsymbol{d}_k^q\boldsymbol{x} = \sum_i x_{k-i}d_i^q. \quad (2)$$

Here, $\sum_{\mathcal{Q}}$ stands for $\sum_{(q,k)\in\mathcal{Q}}$. Usually, $\boldsymbol{d}_k^q\boldsymbol{x}$ are differences; *e.g.*, the first-order differences at pixel k are $\boldsymbol{d}_k^1\boldsymbol{x} = x_k - x_{k-I}$ and $\boldsymbol{d}_k^2\boldsymbol{x} = x_k - x_{k-1}$; *etc.* The set of all the cliques over $\boldsymbol{x}$ reads $\mathcal{Q} = \bigcup_q \bigcup_k C_k^q$ where $C_k^q = k - \{i : \boldsymbol{d}_i^q \neq 0\}$. Introduce now difference operator $\mathcal{D} = \{\boldsymbol{d}_k^q, \ 1 \leq q \leq Q \ \times \ 1 \leq k \leq M\}$.

Remark 1 *Difference operator $\mathcal{D}$ is of size $\#\{\mathcal{Q}\} \times M$, where $\#$ denotes cardinality. Typically, $\#\{\mathcal{Q}\} \gg M$, then Φ* cannot *be written as a sum of independent terms.*

(In contrast, $Q = 1$ for 1D signals and $\#\{\mathcal{Q}\} \leq M$—which situation is much easier to treat!)

PFs φ in (2) satisfy general requirements [1, 7]:

- PFs φ^q are symmetric and we set $\varphi^q(0) = 0$;
- PFs φ^q are increasing with $|t|$;
- PFs φ^q are $\mathcal{C}^1$-continuous except at several points.

PFs φ^q can be convex or nonconvex, smooth or nonsmooth. In order to simplify the presentation, the nonsmooth at zero PFs are supposed to be $\mathcal{C}^1$ for $t \neq 0$, while at zero they can be either nonsmooth with bounded derivatives, or discontinuous. Such PFs are the following [2, 3, 7]:

Generalized Gaussian	$\varphi(t) = \|t\|^\alpha,\ 1 \leq \alpha \leq 2$	
Truncated quadratic	$\varphi(t) = \min\{\alpha t^2, 1\}$	
Lorentzian	$\varphi(t) = \alpha t^2/(1+\alpha t^2)$	(3)
Concave	$\varphi(t) = \alpha\|t\|/(1+\alpha\|t\|)$	
"0-1"	$\varphi(0)=0,\ \varphi(t)=1$ if $t \neq 0$.	

2.2 Weak and Strong Local Homogeneity

The homogeneous regions in an image $\boldsymbol{x}$ are the locations of cliques (q,k) where the differences are weak.

Definition 1 *The set of strong homogeneity $\mathcal{J}$ of an image $\boldsymbol{x}$ with respect to operator $\mathcal{D}$ is the collection of couples (q,k) corresponding to zero-valued differences: $\mathcal{J} = \{\,(q,k) \in \mathcal{Q}\ :\ \boldsymbol{d}_k^q \boldsymbol{x} = 0\,\}$.*

(If $\mathcal{D}$ is first-order, $\mathcal{J}$ corresponds to constant regions.)

Remark 2 *From Remark 1, there exist many sets $\mathcal{J}$ which do not correspond to any nonzero image $\boldsymbol{x}$ (for which the system $\boldsymbol{d}_k^q \boldsymbol{x} = 0$, $(q,k) \in \mathcal{J}$ has no solution beyond $\boldsymbol{x} = \boldsymbol{0}$). We will consider only sets $\mathcal{J}$ which are consistent, i.e. which correspond to nonzero images.*

Definition 2 *The zones of $\boldsymbol{x}$ where the differences are weak ($\boldsymbol{d}_k^q \boldsymbol{x} \approx 0$) without being null are said to be weakly homogeneous.*

Now we focus on the behaviour of $\mathcal{J}$ w.r.t. $\boldsymbol{y}$.

Definition 3 *Let $\widehat{\boldsymbol{x}}$ have a nonempty strong homogeneity set $\mathcal{J}$. Given $\boldsymbol{y}$, estimator $\mathcal{X}$ is said to be locally strongly homogeneous if there exists $\xi > 0$, such that $\mathcal{J}(\boldsymbol{y}') = \mathcal{J}$ for any $\boldsymbol{y}' \in B(\boldsymbol{y};\xi)$.*

The images having the same $\mathcal{J}$ form a subspace $H_{\mathcal{J}}$:

$$H_{\mathcal{J}} = \{\,\boldsymbol{x}\ :\ \boldsymbol{d}_k^q \boldsymbol{x} = 0 \ \text{ for } \ (q,k) \in \mathcal{J}\,\}.$$

In general, $\dim[H_{\mathcal{J}}] \leq \#\{\mathcal{J}\}$; then a sub-system $\mathcal{J}^\circ \subset \mathcal{J}$ containing $\#\{\mathcal{J}^\circ\}$ linearly independent equations can be extracted and this determines a partial difference operator $\mathcal{D}_{\mathcal{J}}$. Equivalently, $H_{\mathcal{J}} = \{\boldsymbol{v}\ :\ \mathcal{D}_{\mathcal{J}}\boldsymbol{v} = \boldsymbol{0}\}$.

Definition 4 *Estimator $\mathcal{X}$ is strictly weakly homogeneous if for any $\boldsymbol{y}$, such that $\widehat{\boldsymbol{x}} = \mathcal{X}(\boldsymbol{y})$ has a nonempty set $\mathcal{J}(\boldsymbol{y})$, there exists $\xi > 0$ such that almost any $\boldsymbol{y}' \in B(\boldsymbol{y};\xi)$ leads to a solution $\widehat{\boldsymbol{x}}' = \mathcal{X}(\boldsymbol{y}')$ which is nowhere strongly homogeneous, $\mathcal{J}(\boldsymbol{y}') = \emptyset$.*

Small data variation are typically dues to noise.

2.3 Local Continuity of the Minimizers of $\mathcal{E}_y$

Definition 5 *$\mathcal{E}_y$ is said to be locally strictly unimodal in the vicinity of $\widehat{\boldsymbol{x}}$ if for any $\boldsymbol{v} \in \mathbb{R}^M$ there exists ξ_v such that $\mathcal{E}_y(\widehat{\boldsymbol{x}} + h\boldsymbol{v})$, $h \in [0, \xi_v[$, is strictly increasing with h.*

Theorem 1 *Let $\mathcal{E}_y$ be locally strictly unimodal in the vicinity of $\widehat{\boldsymbol{x}} = \mathcal{X}(\boldsymbol{y})$. Then $\mathcal{X}$ is locally continuous at $\boldsymbol{y}$.*

Henceforth, $\widehat{\boldsymbol{x}}$ denotes any minimizer where $\mathcal{E}_y$ is locally strictly unimodal, and $\mathcal{J}$ its set of strong homogeneity. Local strict unimodality is a very soft requirement.

3 SMOOTH REGULARIZATION & WEAK HOMOGENEITY

Suppose that *all the PFs in Φ are smooth at zero.* Let $R = \inf_{x;y} \operatorname{rank} D_{2,1}\mathcal{L}_y(\boldsymbol{x})$, where $D_{2,1}$ means differential w.r.t. the first and to the second argument. (*E.g.*, $R = \operatorname{rank} \mathcal{A}$ for the linear-Gaussian model.)

Theorem 2 *Let PFs $(\varphi^q)_{q=1}^Q$ be smooth at zero. Data $\boldsymbol{y}$ yielding a minimizer $\widehat{\boldsymbol{x}} = \mathcal{X}(\boldsymbol{y})$ with a large set of strong homogeneity $\mathcal{J}$, such that $\#\{\mathcal{J}^\circ\} > M - R$, belong to a set $\mathcal{M}_{\mathcal{J}}$ of dimension $\dim[\mathcal{M}_{\mathcal{J}}] < N$.*

Moreover, $\dim[\mathcal{M}_{\mathcal{J}}]$ decreases as long as $\#\{\mathcal{J}^\circ\}$ increases. At the same time, "interesting" reconstructions are justly images with $\#\{\mathcal{J}^\circ\} \gg M - R$. (Note that $R = M$ for a full-rank observation system.)

A set like $\mathcal{M}_{\mathcal{J}}$ exists for any strong homogeneity configuration and for any local minimizer of $\mathcal{E}_y$. The union of these sets contains all the data able to yield local minimizers having large strong homogeneity sets. However, this union is a set of measure zero in $\mathbb{R}^N$. The chance that $\boldsymbol{y}$ belong to it is *almost null*: we cannot expect that noisy data yield a solution involving a large set of strong homogeneity. An estimator, where the PFs are smooth at zero, is strictly weakly homogeneous.

4 NONSMOOTH REGULARIZATION

Henceforth, Φ involves $K \leq Q$ nonsmooth at zero PFs, $(\varphi^q)_{q=1}^K$, which can be either continuous or discontinuous. Often, $K = Q$; else, $(\varphi^q)_{q=K+1}^Q$ are smooth at zero.

The left and right derivatives of a nonsmooth at zero PF φ read: $\gamma = \dot{\varphi}_+(0) = \lim_{h\downarrow 0}\varphi(h)/h = -\dot{\varphi}_-(0)$. For the "0-1" PF, $\gamma = +\infty$. According to our assumptions, $\gamma > 0$ is finite if φ is continuous at zero.

The key property of a nonsmooth at zero PF, allowing $\mathcal{X}$ to be strongly homogeneous, is that it can be minorized locally, near to zero, by an increasing slope.

Proposition 1 *Let φ be a nonsmooth at 0 PF. Then:*

- *if φ is continuous at 0, for any $0 < \kappa < 1$ there exists $\theta > 0$ such that $\varphi(t) \geq (1-\kappa)\gamma|t|$ for $|t| < \theta$;*
- *if PF φ is discontinuous at zero, there exists $\tau > 0$ such that $\varphi(t) \geq \tau$ for any $t \neq 0$.*

4.1 Necessary Condition for a Minimum

Energy $\mathcal{E}_y$ is nonsmooth, and possibly discontinuous, on the union of hyperplanes $\cup_{\mathcal{Q}_K}[\boldsymbol{d}_k^q \boldsymbol{x} = 0]$, where $\mathcal{Q}_K = \{(q,k) \text{ for } 1 \leq q \leq K,\ k \in \mathcal{S}\}$ contains the indices of all the cliques regularized using nonsmooth at zero PFs.

Definition 6 *Let $\boldsymbol{v} \in \mathbb{R}^M$ be an arbitrary direction. The left and right directional derivatives of $\mathcal{E}_y$ at $\boldsymbol{x}$ along $\boldsymbol{v}$ are $\partial_{\boldsymbol{v}}^- \mathcal{E}_y(\boldsymbol{x}) = \lim_{h\downarrow 0}[\mathcal{E}_y(\boldsymbol{x} - h\boldsymbol{v}) - \mathcal{E}_y(\boldsymbol{x})]/(-h)$ and $\partial_{\boldsymbol{v}}^+ \mathcal{E}_y(\boldsymbol{x}) = \lim_{h\downarrow 0}[\mathcal{E}_y(\boldsymbol{x} + h\boldsymbol{v}) - \mathcal{E}_y(\boldsymbol{x})]/h$.*

Necessary conditions for local minima read [10]:

Theorem 3 *If $\mathcal{E}_y$ reaches a strict minimum at $\widehat{\boldsymbol{x}}$, then $\partial_{\boldsymbol{v}}^- \mathcal{E}_y(\widehat{\boldsymbol{x}}) \leq 0 \leq \partial_{\boldsymbol{v}}^+ \mathcal{E}_y(\widehat{\boldsymbol{x}})$, for any $\boldsymbol{v} \in \mathbb{R}^M$.*

Consider a strict minimizer $\widehat{\boldsymbol{x}}$ such that $\mathcal{J} \neq \emptyset$. Let $\mathcal{F}^{\mathcal{J}}$ collect the terms of $\mathcal{E}_y$ which are smooth at $\widehat{\boldsymbol{x}}$:

$$\mathcal{F}_y^{\mathcal{J}}(\widehat{\boldsymbol{x}}) = \mathcal{L}_y(\widehat{\boldsymbol{x}}) + \beta \Phi^{\mathcal{J}}(\widehat{\boldsymbol{x}}) \text{ where } \Phi^{\mathcal{J}}(\widehat{\boldsymbol{x}}) = \sum_{\mathcal{Q}\setminus\mathcal{J}} \varphi^q(\boldsymbol{d}_k^q \widehat{\boldsymbol{x}}).$$

Theorem 4 *Let $\widehat{\boldsymbol{x}}$ be a strict minimizer of $\mathcal{E}_y$ whose set of strong homogeneity $\mathcal{J}$ is nonempty. Then:*

$$D_1 \mathcal{F}_y^{\mathcal{J}}(\widehat{\boldsymbol{x}}) \cdot \boldsymbol{v} = 0 \quad \text{for any } \boldsymbol{v} \in H_{\mathcal{J}}, \tag{4}$$

$$|D_1 \mathcal{F}_y^{\mathcal{J}}(\widehat{\boldsymbol{x}}) \cdot \boldsymbol{v}| \leq \beta \sum_{\mathcal{J}} \gamma_q |\boldsymbol{d}_k^q \boldsymbol{v}| \quad \text{for any } \boldsymbol{v} \in H_{\mathcal{J}}^{\perp}, \tag{5}$$

where $H_{\mathcal{J}}^{\perp}$ is the orthogonal complement of subspace $H_{\mathcal{J}}$.

The specificity of the behaviour of estimators involving nonsmooth PFs is due to the possibility that different data satisfy the same inequality (5). The following lemmas are the king-pin for strong homogeneity of an estimator when $\#\{Q\} > M$ (*cf.* Remark 1).

Lemma 1 *Given $\mathcal{J}$, let $\boldsymbol{v} \in H_{\mathcal{J}}^{\perp}$ be arbitrary. Then: $\sum_{\mathcal{J}} \gamma_q |\boldsymbol{d}_k^q \boldsymbol{v}| \geq \omega\gamma\|\boldsymbol{v}\|$ where $\gamma = \min_{1\leq q\leq K} \gamma_q$ and $\omega^2 > 0$ is the smallest eigenvalue of $\mathcal{D}_{\mathcal{J}} \mathcal{D}_{\mathcal{J}}^T$.*

Lemma 2 *Let $\widehat{\boldsymbol{x}}$ satisfy (4-5), where $(\gamma_q)_{q=1}^Q$ are finite. If inequality (5) is strict, there exists $0 < \ell < 1$, such that $|D_1 \mathcal{F}_y^{\mathcal{J}}(\widehat{\boldsymbol{x}}) \cdot \boldsymbol{v}| \leq \ell\beta \sum_{\mathcal{J}} \gamma_q |\boldsymbol{d}_k^q \boldsymbol{v}|$ for any $\boldsymbol{v} \in H_{\mathcal{J}}^{\perp}$.*

4.2 A sufficient Condition for a Strict Minimum

Consider an image $\boldsymbol{x}$ such that $\mathcal{J} \neq \emptyset$ and the relevant $H_{\mathcal{J}}$, then $\boldsymbol{x} \in H_{\mathcal{J}}$. Let $\mathcal{E}_y^{\mathcal{J}}$ denote the restriction of $\mathcal{E}_y$ to $H_{\mathcal{J}}$, then $\mathcal{E}_y^{\mathcal{J}}$ is $\mathcal{C}^1$ on an open $H_{\mathcal{J}}$-neighbourhood of $\boldsymbol{x}$ (an open ball in $H_{\mathcal{J}}$). If $\widehat{\boldsymbol{x}}$ is a strict local minimizer of $\mathcal{E}_y$, then $\widehat{\boldsymbol{x}}$ is a strict local minimizer of $\mathcal{E}_y^{\mathcal{J}}$ as well.

Theorem 5 *Let PFs $(\varphi^q)_{q=1}^K$, $K \leq Q$ be nonsmooth at zero. Let $\widehat{\boldsymbol{x}}$ be a point in $\mathbb{R}^M$ such that:*

- *its set of strong homogeneity $\mathcal{J}$ is nonempty;*
- *$\widehat{\boldsymbol{x}}$ is a strict minimizer of $\mathcal{E}_y^{\mathcal{J}}$;*
- *inequality (5) is strict for any $\boldsymbol{v}$ orthogonal to $H_{\mathcal{J}}$.*

Then $\widehat{\boldsymbol{x}}$ is a strict local minimizer of $\mathcal{E}_y$.

So, the more Φ is irregular, the more $\widehat{\boldsymbol{x}}$ is regular.

5 STRONG HOMOGENEITY

5.1 Local behaviour of the estimator

Now we focus on the behaviour of the strict minimizers of $\mathcal{E}_y$ under small variations of $\boldsymbol{y}$.

Theorem 6 *Let Φ involve nonsmooth at zero PFs. Let $\widehat{\boldsymbol{x}} = \mathcal{X}(\boldsymbol{y})$ be such that*

- *its set of strong homogeneity $\mathcal{J}(\boldsymbol{y})$ is nonempty;*
- *inequality (5) is strict for any $\boldsymbol{v} \in H_{\mathcal{J}}^{\perp}$.*

Then, there exists $\xi > 0$ such that $\boldsymbol{y}' \in B(\boldsymbol{y};\xi)$ implies that $\widehat{\boldsymbol{x}}' = \mathcal{X}(\boldsymbol{y}') \in H_{\mathcal{J}}$.

Our main result is expressed in the following corollary.

Corollary 1 *Let Φ involve nonsmooth at zero PFs. Let $\widehat{\boldsymbol{x}} = \mathcal{X}(\boldsymbol{y})$ be a strict minimizer of $\mathcal{E}_y$, which has a nonempty strong homogeneity set $\mathcal{J}$. Then, there exists $\xi > 0$ such that $\boldsymbol{y}' \in B(\boldsymbol{y};\xi)$ implies $\mathcal{J}(\boldsymbol{y}') = \mathcal{J}$.*

Minimizer $\mathcal{X}$ keeps null the differences belonging to $\mathcal{J}$ for any data placed inside a ξ-ball surrounding $\boldsymbol{y}$. When PFs φ^q are discontinuous at 0, radius ξ is larger.

5.2 Partition of data space

We now turn to the organization of data space $\mathbb{R}^N$, induced by the strong homogeneity of estimator $\mathcal{X}$. Let

$$\mathcal{W} = \left\{ \begin{array}{l} \boldsymbol{y} \quad \text{such that} \\ \bullet\ \mathcal{X}_{\mathcal{J}}(\boldsymbol{y}) \in H_{\mathcal{J}} \text{ is a strict minimizer of } \mathcal{E}_y^{\mathcal{J}}, \\ \bullet\ |D_1 \mathcal{F}^{\mathcal{J}}(\mathcal{X}(\boldsymbol{y});\boldsymbol{y}) \cdot \boldsymbol{v}| \leq \beta \sum_{\mathcal{J}} \gamma_q |\boldsymbol{d}_k^q \boldsymbol{v}|\ \forall \boldsymbol{v} \in H_{\mathcal{J}}^{\perp}. \end{array} \right\}$$

Data of $\mathcal{W}$ yield minimizers which are different but have the same strong homogeneity set $\mathcal{J}$. This $\mathcal{W}$ is a *volume* in $\mathbb{R}^N$ since it contains an open ball of $\mathbb{R}^N$ (Corollary 1).

Suppose moreover that $\mathcal{W}$ corresponds to global minimizers of $\mathcal{E}_y$. Let $(\mathcal{J}_k)_{k=1}^Q$ be all the consistent configurations of $\mathcal{J}$ and $\mathcal{J}_0 = \emptyset$. A volume $\mathcal{W}_k$ can be associated to each $\mathcal{J}_k$; then $(\mathcal{W}_k)_{k=0}^P$ form a partition of $\mathbb{R}^N$. The probability that noisy data $\boldsymbol{y}$ are contained in a volume $\mathcal{W}_k$, $1 \leq k \leq Q$ (and hence to yield a solution strongly homogeneous over $\mathcal{J}_k$) is *strictly positive.* It depends on the extent of $\mathcal{W}_k$ and hence on parameters (α, β, γ).

6 NUMERICAL ILLUSTRATION

The original image (Fig. 1) has weakly and strongly homogeneous regions. Data are $\boldsymbol{y} = \boldsymbol{h} * \boldsymbol{x} + \boldsymbol{n}$, where $\boldsymbol{h}$ is a PSF and $\boldsymbol{n}$ is white Gaussian noise. The reconstructions below use the PFs given in (3) and first-order differences, so the strongly homogeneous regions are locally constant. In Figs. 2-7, the solution is shown on the left, the relevant PF in the middle and three sections of the solution (rows 35, 54 and 90) on the right.

In Fig. 2, a generalized Gaussian PF is used; it is smooth at 0 and $\widehat{\boldsymbol{x}}$ is weakly homogeneous. A modulus PF (Fig. 3) gives rise to locally constant patches. In Fig. 4, a Lorentizian PF yields a weakly homogeneous solution. A concave PF (Fig. 5) leads to strongly homogeneous regions. A truncated quadratic PF provides a solution (Fig. 6) with weakly homogeneous zones. For a "0-1" PF, $\widehat{\boldsymbol{x}}$ is composed of constant patches (Fig. 7).

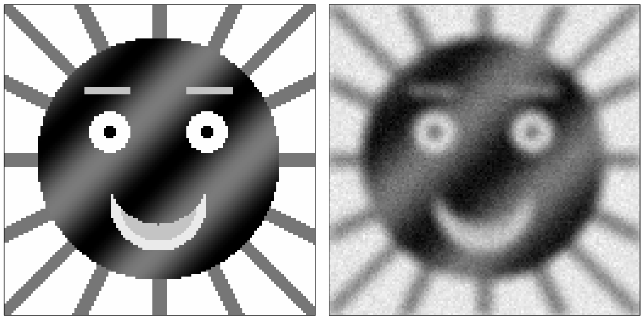

Fig. 1: Original image (left). Data (blurr+noise) (right).

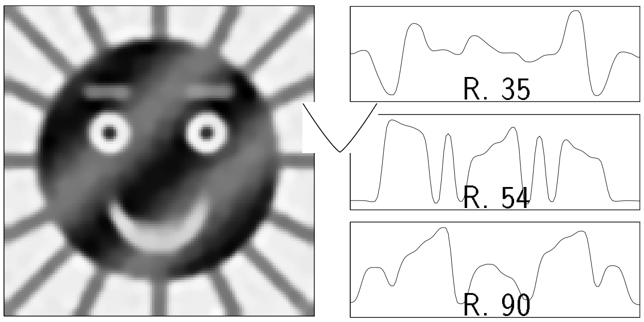

Fig. 2: Generalized Gaussian PF, $\alpha = 1.4$ *(smooth at 0)*.

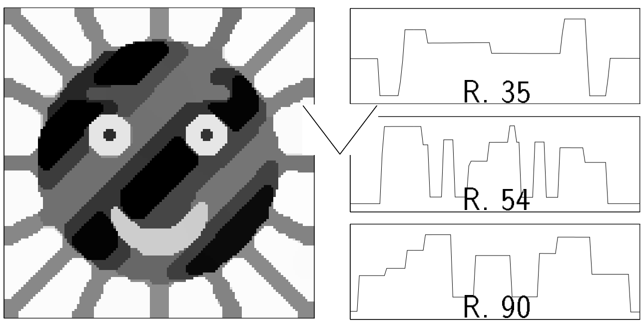

Fig. 3: Modulus PF *(nonsmooth at zero)*.

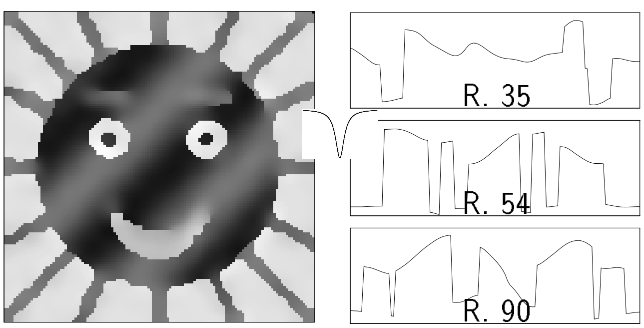

Fig. 4: Lorentzian PF *(smooth at zero)*.

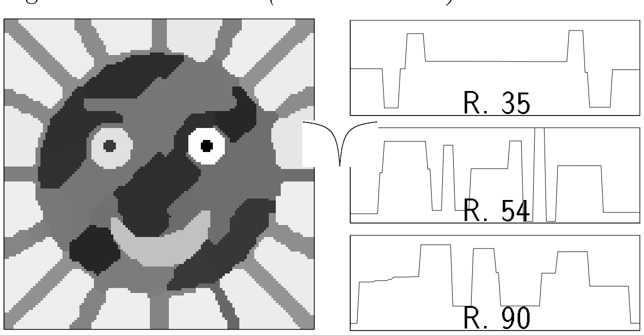

Fig. 5: Concave PF *(nonsmooth at zero)*.

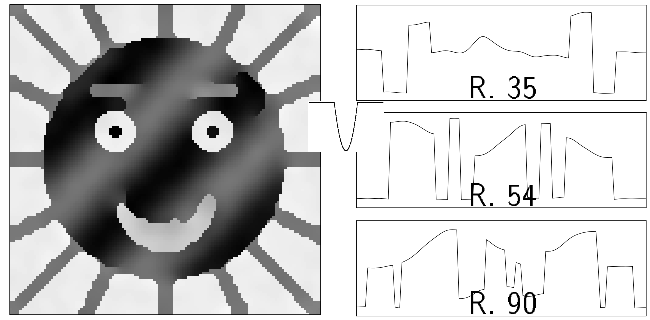

Fig. 6: Truncated quadratic PF *(smooth at zero)*.

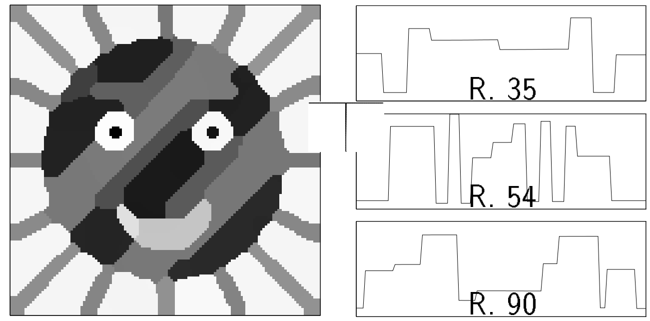

Fig. 7: "0-1" PF *(nonsmooth at zero)*.

7 CONCLUSION

We have set the problem of the recovery of either weakly homogeneous or strongly homogeneous regions in a regularized image estimate. We have shown that this recovery is uniquely determined by the smoothness at zero of the PFs involved in the regularization term.

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