CONSTRAINED PULSE SHAPE SYNTHESIS FOR DIGITAL COMMUNICATIONS

P. L. Combettes[†] *and P. Bondon*[‡]

[†]Department of Electrical Engineering, City College and Graduate School,
City University of New York, New York, NY 10031, USA.
[‡]CNRS – Laboratoire des Signaux et Systèmes
Plateau de Moulon, 91192 Gif-sur-Yvette, France.

ABSTRACT

The synthesis of optimal pulse shapes for digital data transmission over communication channels typically involves conflicting specifications. We consider the problem of finding a pulse satisfying some mandatory "hard" constraints while violating as little as possible the remaining "soft" constraints. The problem is formalized as that of minimizing a weighted sum of squared distances to the soft constraint sets over the intersection of the sets associated with the hard constraints. This constrained problem is analyzed and a numerical algorithm is proposed. Simulation results are presented.

1 INTRODUCTION

In digital communications, pulse shape design problems are always accompanied with various specifications [8]. For instance, for communications over power lines [9], the following constraints are pertinent:

1. Limited power.
2. Limited bandwidth.
3. Limited deviation from a nominal pulse.
4. Limited relative amplitude of sidelobes.
5. Periodic zero-crossings in the time-domain to avoid intersymbol interference.
6. Periodic zero-crossings in the frequency domain to avoid interferences with the harmonic noise present on power lines.
7. Linear phase.

The raised-cosine pulse [8] plays an important role in digital transmission systems and has been used extensively in modem design for both wire lines and radio systems. This pulse satisfies most of the above constraints, except that its frequency response is uniformly non zero in its band (see Figs. 1-2). Therefore, this pulse shape is not suitable for communications over power lines, where zero interference with the 60 Hz component of the power waveforms is desired.

Standard filter design techniques are computationally involved and not well suited to handle the wide variety of linear and nonlinear constraints that may arise in pulse shape design problems. On the other hand, if one calls l the length of the discrete-time pulse, it is not hard to see that all the above constraints (and indeed most of those arising in pulse shape design) yield closed convex sets in $\mathbb{R}^l$, say S_1, S_2,..., S_m. The design problem can then simply be formulated as that of finding a point in the intersection of these sets, i.e.,

$$\text{Find } \; a^\star \in S = \bigcap_{i=1}^{m} S_i. \tag{1}$$

This convex set theoretic formulation has been employed in numerous signal processing problems and efficient projection methods are available to solve it [2]. Unfortunately, in the present context, one is typically confronted with inconsistent constraints and $S = \varnothing$ in (1). In such instances, the traditional projection methods no longer operate and will either diverge or yield irrelevant solutions. Thus, when two constraint sets are present ($m = 2$), it was noted in [5] that the alternating projection scheme

$$a_{n+1} = P_1 \circ P_2(a_n), \tag{2}$$

where P_i is the projector onto S_i, converges to a solution in the set S_1 that lies closest to the set S_2. This algorithm was used in [6] to generate pulse shapes that satisfy exactly one constraint (i.e., that belong to the set S_1) and are closest to satisfying the second one (i.e., that are at minimum distance from the set S_2). A natural extension of (2) to m sets is the so-called POCS algorithm

$$a_{n+1} = P_1 \circ P_2 \circ \cdots \circ P_m(a_n). \tag{3}$$

However, when $S = \varnothing$, either this algorithm fails to converge or it converges to a point that can be guaranteed to lie only in S_1 and which fails to exhibit any degree

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of proximity with respect to the other sets (see [1] for a recent discussion).

An alternative approach, that can handle an arbitrary number of constraints, was proposed in [3]. There, a weighted sum of squared distances to the sets, namely

$$\Phi(a) = \frac{1}{2}\sum_{i=1}^{m} w_i d(a, S_i)^2 \tag{4}$$

where $(w_i)_{1\leq i\leq m} \subset]0,1]$ and $\sum_{i=1}^{m} w_i = 1$, was the objective to minimize. It was shown that the algorithm

$$a_{n+1} = a_n + \lambda_n \left(\sum_{i=1}^{m} w_i P_i(a_n) - a_n \right), \tag{5}$$

where $0 < \varepsilon \leq \lambda_n \leq 2 - \varepsilon$, converges to a minimizer of Φ, i.e., to a pulse shape which is optimal in a weighted least-squares sense. However, when $S = \emptyset$, this pulse cannot be guaranteed to lie in a prescribed set.

In summary, (2) is limited to two constraints and has the ability to enforce exactly one mandatory constraint while (5) is not limited in the number of constraints but cannot enforce any mandatory constraint exactly. In this paper, we propose to unify and extend these two approaches into a more general one, capable of producing pulse shapes satisfying exactly imperative specifications while best satisfying the remaining ones in a least-squares sense.

2 FORMALIZATION

Throughout, $\mathbb{R}^l$ is equipped with the usual euclidean distance d and $I = \{1, \dots, m\}$.

Let $I^\blacktriangle \subset I$ be the possibly empty set of indices associated with the hard constraints and let $I^\vartriangle = I \smallsetminus I^\blacktriangle$ be the set of indices associated with the remaining soft constraints. Now let $S^\blacktriangle = \bigcap_{i\in I^\blacktriangle} S_i$ be the (closed and convex) feasibility set relative to the hard constraints. In what follows, we assume $S^\blacktriangle \neq \emptyset$ and, by convention, take $S^\blacktriangle = \mathbb{R}^l$ if $I^\blacktriangle = \emptyset$. The pulse shape design problem can then be formulated as that of finding a pulse in $S^\blacktriangle$ which minimizes a weighted sum of the squares of the distance to the soft constraint sets, say

$$\Phi^\vartriangle(a) = \frac{1}{2}\sum_{i\in I^\vartriangle} w_i d(a, S_i)^2, \tag{6}$$

where $(w_i)_{i\in I^\vartriangle} \subset]0,1]$ and $\sum_{i\in I^\vartriangle} w_i = 1$. Thus, the problem reads

$$\text{Find } a^\star \in G \triangleq \{a \in S^\blacktriangle \mid (\forall b \in S^\blacktriangle)\ \Phi^\vartriangle(a) \leq \Phi^\vartriangle(b)\}. \tag{7}$$

The optimization problem (7) is the hard-constrained formulation of the inconsistent design problem (1). This formulation encompasses in particular the two approaches described in the introduction. In the former, $S^\blacktriangle = S_1$ and $\Phi^\vartriangle(a) = d(a, S_2)^2/2$, while in the latter $S^\blacktriangle = \mathbb{R}^l$ and $\Phi^\vartriangle(a)$ is given by (4).

It results from the convexity of the sets in (6) that $\Phi^\vartriangle$ is convex and continuous. Thus, (7) is a standard convex optimization problem and powerful tools are available to analyze it theoretically and solve it numerically. In particular, any minimum is global and therefore one does not have to contend with local minima that do not solve the problem.

We now turn to the question of the existence and the uniqueness of solutions to problem (7). Recall that a convex function $g : \mathbb{R}^l \to \mathbb{R}$ is strictly convex over a convex set $A \subset \mathbb{R}^l$ if, for every pair of distinct points $(a, b) \in A^2$, $g((a + b)/2) < (g(a) + g(b))/2$; moreover a set $A \subset \mathbb{R}^l$ is strictly convex if, for every pair of distinct points $(a, b) \in A^2$, $(a + b)/2$ lies in the interior of A [7].

Proposition 1 [4] If, for some $i \in I$, S_i is bounded, then (7) has a solution. If, in addition, one of the conditions below holds, then (7) has exactly one solution.

(i) $\Phi^\vartriangle$ is strictly convex over $S^\blacktriangle$.

(ii) $\Phi^\vartriangle$ has no unconstrained minimum over $S^\blacktriangle$ and the sets $(S_i)_{i\in I^\blacktriangle}$ are strictly convex.

Let us now investigate the question of solving numerically problem (7). While other methods [7] could be considered, we adopt a projected gradient approach that is convenient to implement, displays satisfactory convergence patterns, and will be seen to cover algorithms (2) and (5).

Hereafter, $\mathbb{N}$ denotes the set of nonnegative integers, and a_0 is a fixed point in $\mathbb{R}^l$. It is assumed that, for some $i \in I$, S_i is bounded (hence, Proposition 1 asserts that the solution set G is not empty). Moreover, for computational implementability, the projector $P^\blacktriangle$ onto $S^\blacktriangle$ is assumed to be numerically realizable. Finally, we let $\alpha^\star = \inf_{a\in S^\blacktriangle} \Phi^\vartriangle(a)$.

Proposition 2 [4] For every $n \in \mathbb{N}$, define

$$a_{n+1} = P^\blacktriangle\left(a_n + \lambda_n \left(\sum_{i\in I^\vartriangle} w_i P_i(a_n) - a_n \right) \right) \tag{8}$$

where $0 < \varepsilon \leq \lambda_n \leq 2 - \varepsilon$. Then:

(i) $(\forall n \in \mathbb{N})\quad \Phi^\vartriangle(a_{n+1}) \leq \Phi^\vartriangle(a_n)$ and $(\Phi^\vartriangle(a_n))_{n\geq 0}$ converges to $\alpha^\star$.

(ii) $(a_n)_{n\geq 0}$ has a convergent subsequence and the limit of any such subsequence is in G.

(iii) If (7) admits a unique solution $a^\star$, then the whole sequence $(a_n)_{n\geq 0}$ converges to $a^\star$.

When $I = \{1, 2\}$ and $I^\blacktriangle = \{1\}$, (8) with $\lambda_n = 1$ reduces to the alternating projections algorithm (2). When $I^\blacktriangle = \emptyset$, (8) takes the form of the parallel projection algorithm (5).

3 SIMULATION RESULTS

A pulse shape for digital communications over 60 Hz power lines is synthesized under the specifications:

C1. The lines have a limited bandwidth of 300Hz. Hence, the Fourier transform of the pulse must be zero beyond 300Hz. S_1 is a vector subspace.

C2. The Fourier transform of the pulse is zero at 60Hz to avoid interferences with power waveforms. S_2 is a vector subspace.

C3. The maximum distance squared from the pulse to the raised cosine pulse is $1/2$. S_3 is a closed ball.

C4. The pulse has periodic zero crossings every 2.73 ms to avoid intersymbol interference. S_4 is a vector subspace.

C5. The pulse is normalized so that its maximum value is 1. S_5 is an affine subspace.

C6. The absolute amplitude of the sidelobes of the pulse is limited to 3% of that of the main lobe. S_6 is a lower dimensional parallelogram.

The projectors $(P_i)_{1\leq i\leq 6}$ onto these sets are easily obtained and thus not derived here. Since the pulse of Figs. 1-2 satisfies C1, C3, C4, and C5, a simple scheme would be to project it onto S_2. As seen on Figs. 3-4, this approach is not acceptable: while the spectral properties C1 and C2 are enforced, the remaining ones are destroyed. Another way to obtain a signal in $S_1 \cap S_2$ is to apply POCS (3). As seen earlier and illustrated in Figs. 5-6, this also results in a poor quality pulse. Let us now consider the proposed approach. In this problem the hard constraint is that physically imposed by the lines, i.e., C1. We therefore take $I^{\blacktriangle} = \{1\}$ and $I^{\vartriangle} = \{2,3,4,5,6\}$. Note that the problem does admit a solution thanks to Proposition 1 since S_3 is bounded. Figs. 7-8 show the pulse generated by (8).

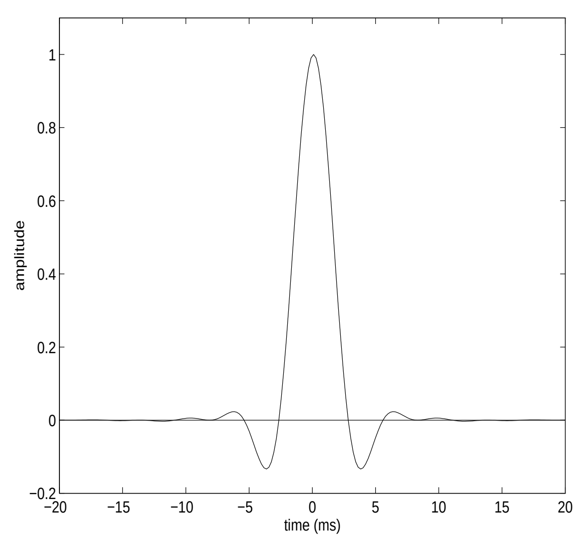

Fig. 1: Raised cosine pulse.

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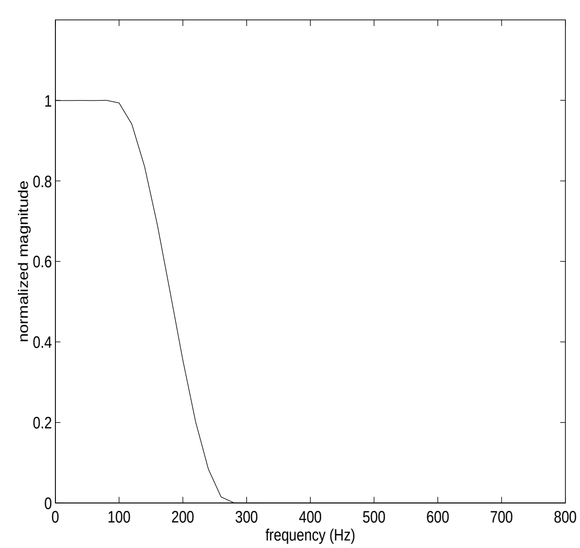

Fig. 2: Fourier magnitude of the pulse of Fig. 1.

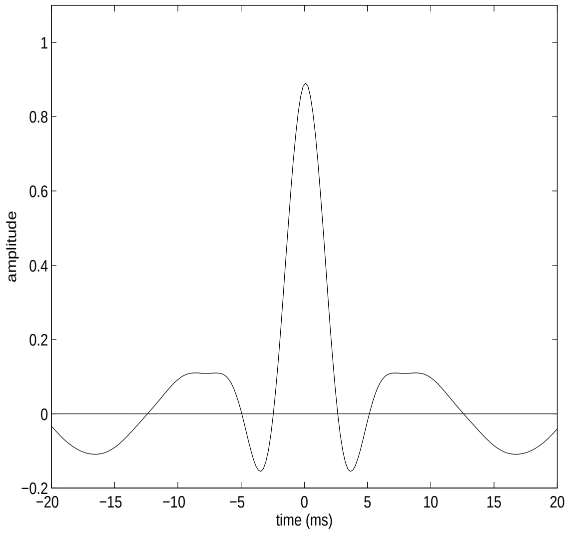

Fig. 3: Projection of the raised cosine pulse.

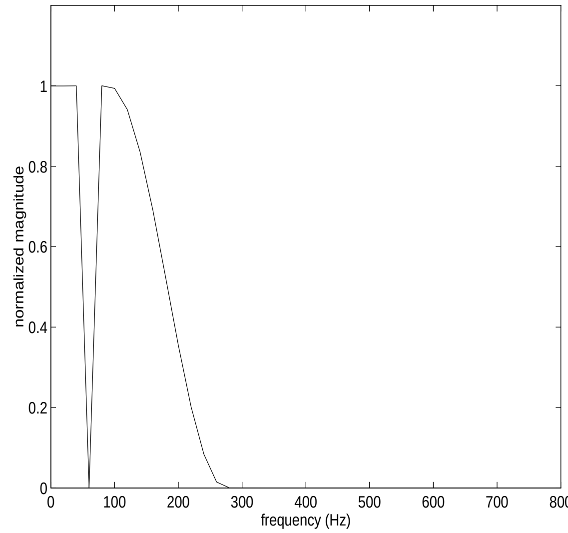

Fig. 4: Fourier magnitude of the pulse of Fig. 3.

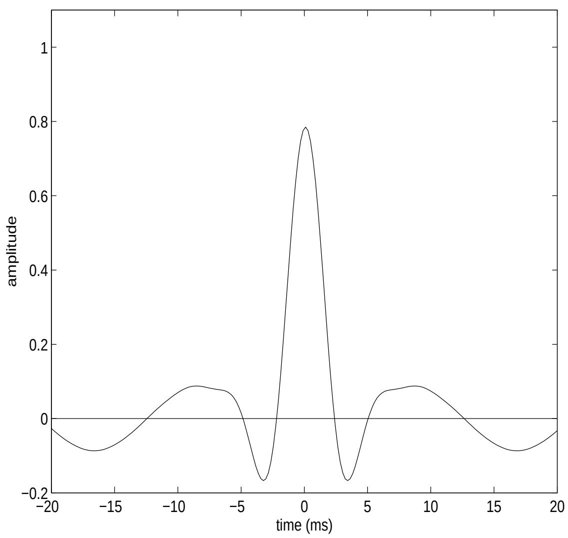

Fig. 5: Pulse generated by POCS.

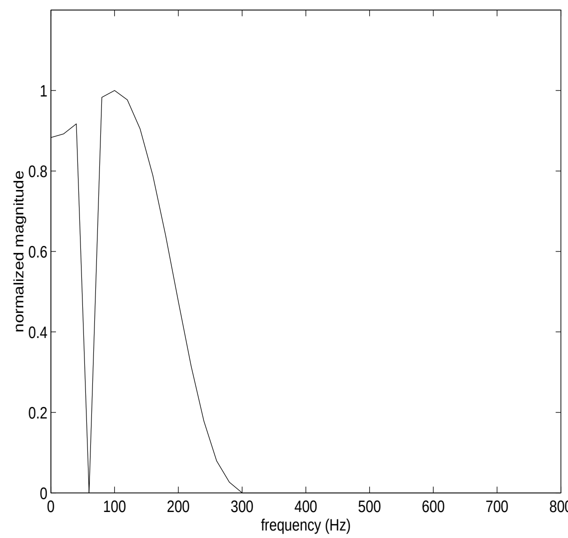

Fig. 6: Fourier magnitude of the pulse of Fig. 5.

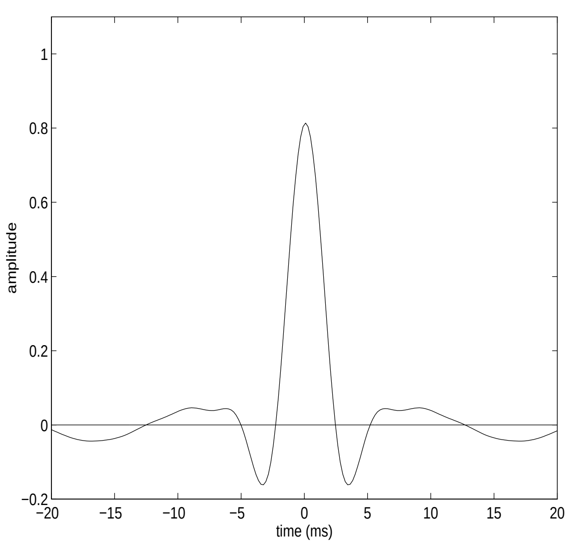

Fig. 7: Hard-constrained pulse.

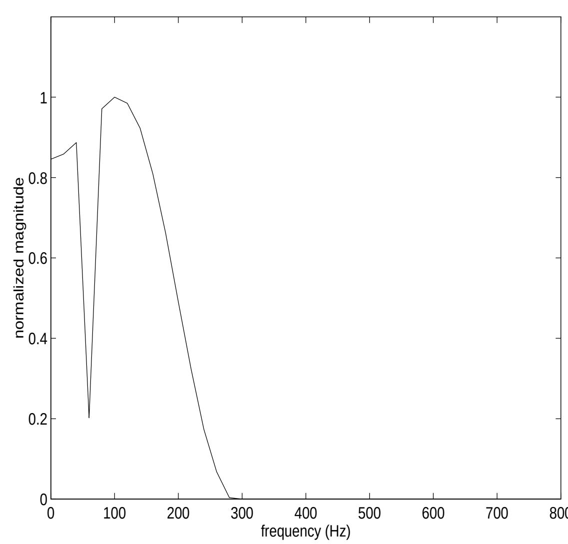

Fig. 8: Fourier magnitude of the pulse of Fig. 7.