FILTER CATEGORIES PRESERVING DESIRABLE SIGNAL CHARACTERISTICS

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ABSTRACT

This paper considers the problem of characterizing nonlinear digital filters via their qualitative input-output behavior. For example, it is known that (non-causal) symmetric median filters preserve arbitrary monotone sequences. Behavioral characterizations ask converse questions like: what can be said about the class of all filters $\mathcal{F}$ that preserve monotonicity? The primary point of this paper is to illustrate that category theory provides a natural framework for examining such questions.

1 GENERAL PROBLEM FORMULATION

One important motivation for the design of digital filters is the recovery of distorted or noise-corrupted versions of signals $\{x_k\}$ that belong to some set S. In the design of linear filters, this set is traditionally defined by the frequency-domain specification of a pass-band [7]. This basic notion has been extended to the description of nonlinear digital filters through the use of signal characterizations like local monotonicity [12, 14, 15]. A closely related problem is the behavioral characterization of discrete-time dynamic models in the empirical modeling of complex physical systems [11]. As a specific example, some physical systems are known to exhibit monotone responses to monotone input sequences, and this observation provides the basis for defining classes of behaviorally-compatible models [9]. This paper considers the following problem: given a specified set S of "desirable signals," characterize those digital filters that preserve this behavior—i.e., those filters $\mathcal{F}$ such that $\mathcal{F}[S] \subset S$. This specification defines a *category* [3] of filters and thus establishes a very general framework for examining questions relating desirable behavior of this type to specific classes of filter structures. In particular, one of the advantages of this approach is that filter categories may be defined either behaviorally or structurally; intersecting these categories then provides a basis for examining the questions of interest here.

2 BASIC NOTION OF A CATEGORY

A category is a collection of *objects* (e.g., mathematical entities like vector spaces), together with a collection of *morphisms* relating those objects (e.g., linear mappings between vector spaces). In addition, there must be an identity morphism associated with each object and a composition law $\circ$ relating morphisms [3, p. 13]. A typical example is the category **Vect** of linear vector spaces, where the class of objects is the set of all linear vector spaces R^n, the morphisms are the $n \times m$ matrices taking R^m into R^n, the identity morphisms are the $n \times n$ identity matrices, and the composition law corresponds to ordinary matrix multiplication.

If A and B are objects in $\mathcal{O}$, the set of morphisms relating these objects is typically written $\hom(A, B)$ and individual morphisms f in this set may be conveniently denoted as $f : A \to B$. A fundamental requirement on the composition law is that if $f \in \hom(A, B)$ and $g \in \hom(B, C)$, then $g \circ f$ must exist and belong to the set $\hom(A, C)$. Also, the composition law must be associative: if $f : A \to B$, $g : B \to C$, and $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$. Defining requirements for the identity morphism id_A are that $f \circ id_A = f$ for all $f \in \hom(A, X)$ and $id_A \circ g = g$ for all $g \in \hom(Y, A)$.

3 DIGITAL FILTER CATEGORIES

In the categories of interest in this paper, the objects are data sequences $\{x_k\}$ and the morphisms are time-invariant digital filters that transform one data sequence into another. Identity morphisms correspond to the identity system $y_k = x_k$, and composition of morphisms corresponds to the cascade connection of digital filters. Because these filters may be viewed as sequence transformations and such transformations are associative, it follows that any collection of digital filters will constitute a category if it includes the identity filter and is closed under cascade connections [10, 11].

3.1 Mallows Filter Categories

Mallows considered the smoothing behavior of filters $\mathcal{F}$ satisfying the following five conditions [8]:

A1. time-invariance: $u_k \to u_{k-j} \Rightarrow y_k \to y_{k-j}$

A2. location-invariance: $\mathcal{F}[x_k + c] = \mathcal{F}[x_k] + c$

A3. centrality: $\mathcal{F}[\mathbf{0}] = \mathbf{0}$

A4. finite memory: $\mathcal{F}[x_k] = \Phi(x_k, \ldots, x_{k-q})$

A5. finite variance: $\mathrm{var}\{\mathcal{F}[x_k]\} < \infty$.

Here, we restrict consideration to deterministic data sequences and consequently replace A5 with the *bounded-input, bounded-output* (BIBO) stability condition:

A5'. given M, $|x_k| \leq M$ for all k implies the existence of finite N such that $|\mathcal{F}[x_k]| \leq N$ for all k.

Restrictions A1, A2, A3, and A5' are *behavioral* in that they describe input-output behavior, whereas restriction A4 is *structural*, defining an explicit form for the input-output relationship (i.e., this condition excludes recursive filter structures).

Note that the identity system satisfies all of the restrictions A1 through A4 and A5'. Further, it is not difficult to show that if $\mathcal{F}$ and $\mathcal{G}$ both satisfy any of these conditions, so does the cascade connection $\mathcal{F} \circ \mathcal{G}$. This result follows immediately from the definitions for the four behavioral restrictions A1, A2, A3, and A5', whereas for the structural restriction A4, the cascade connection is given explicitly as:

$$\begin{aligned} y_k &= \Phi(x_k, \ldots, x_{k-q}) \\ z_k &= \Psi(y_k, \ldots, y_{k-r}) \\ \Rightarrow z_k &= \Psi[\Phi(x_k, \ldots, x_{k-q}), \ldots, \Phi(x_{k-r}, \ldots, x_{k-r-q})] \\ &\equiv \Gamma(x_k, \ldots, x_{k-q-r}). \end{aligned}$$

Consequently, all five of these restrictions define digital filter categories, which will be denoted $\mathbf{Mallows}_j$ for $j = 1, 2, ..., 5$ in subsequent discussions. Similarly, the category of nonlinear filters satisfying all five of these conditions will be denoted $\mathbf{Mallows}_{12345}$.

3.2 Signal-preserving Filter Categories

Another collection of categories of particular interest here are the categories $\mathbf{F}_S$ of filters preserving some specified set S of desirable input sequences, such as:

- the set $\mathcal{B}_M$ of bounded sequences: $|x_k| \leq M$,
- the set $\mathcal{V}_M$ of bounded variation sequences: $\sum_{k=-\infty}^{\infty} |x_k - x_{k-1}| \leq M$ [13],
- the set $\mathcal{E}_\alpha$ of exponentially decaying sequences: $|x_k| \leq M\alpha^k$ for all $k \geq 0$,
- the set $\mathcal{M}$ of monotone sequences,
- the set $\mathcal{M}_n$ of locally monotone or n-monotone sequences [15] (also called lomo $- n$ sequences [12, 14]),
- the set $\mathcal{Q}$ of quasi-monotonic sequences [6], defined as sequences that become monotone under certain averaging operations.

For any pair of sequences $\{x_k\}$ and $\{y_k\}$ in this class, the morphism set $\hom(\{x_k\}, \{y_k\})$ contains all time-invariant digital filters $\mathcal{F}$ such that $\mathcal{F}[x_k] = y_k$ and composition of morphisms again corresponds to cascade connection. As in the four behaviorally-defined Mallows categories, closure of these filter classes under composition follows immediately from the invariance requirement. Specifically, note that if $\mathcal{F}[S] \subset S$ and $\mathcal{G}[S] \subset S$, then $\mathcal{G} \circ \mathcal{F}[S] = \mathcal{G}[\mathcal{F}[S]] \subset \mathcal{G}[S] \subset S$.

3.3 Homopolymer and copolymer categories

Multi-layer and iterative structures arise naturally in nonlinear signal processing, as in the context of iterated application of nonlinear filters [2, 4] and neural networks [16]. These constructions are roughly analogous to polymerization, the chemical process by which small molecules are combined into long chains with physical properties very different from the starting materials. This observation motivates the introduction of the following structurally-defined filter categories. First, suppose $\mathcal{F}$ is any digital filter and define the *homopolymer category* $\mathbf{Ho}[\mathcal{F}]$ as follows. The objects in this category consist of all real-valued data sequences and the morphisms consist of all possible repeated compositions of $\mathcal{F}$, denoted:

$$\mathcal{F}^k = \mathcal{F} \circ \mathcal{F} \circ \cdots \circ \mathcal{F},$$

where k is the number of compositions appearing on the right-hand side of this expression. Further, in analogy with the usual algebraic notation, define $\mathcal{F}^0$ to be the identity system $y_k = x_k$. As a specific example, suppose $\mathcal{M}_K$ is the symmetric median filter $\mathcal{M}_K$, defined by the input-output relation:

$$y_k = \text{median}\,\{x_{k-K}, \ldots, x_k, \ldots, x_{k+K}\}, \qquad (1)$$

for some $K > 0$. The homopolymer category $\mathbf{Ho}[\mathcal{M}_K]$ consists of all possible iterations of this median filter.

More general structures, somewhat analogous to multi-layer neural networks, result if we consider the *copolymer category* $\mathbf{Co}[\mathcal{F}, \mathcal{G}]$ of two filters $\mathcal{F}$ and $\mathcal{G}$, defined as follows. As in the homopolymer categories just defined, the objects here are all possible real-valued sequences and the morphisms relating these objects are all possible cascade connections of $\mathcal{F}$ and $\mathcal{G}$, together with the identity system as before. Finally, a still more general construction is the copolymer category formed from two *categories* $\mathbf{C}$ and $\mathbf{D}$. This category is denoted $\mathbf{Co}[\mathbf{C}, \mathbf{D}]$ and is constructed analogously: objects are real-valued data sequences, morphisms include all filters from either of the categories $\mathbf{C}$ or $\mathbf{D}$, and composition of morphisms corresponds to the cascade connection of these filters. The advantage of these structural categories is that they provide a means of building up complex nonlinear digital filters from simpler building blocks whose qualitative behavior is easier to characterize.

4 SUBCATEGORIES

To describe relationships between filter structure and behavior, the following concept is useful. A *subcategory* $\mathbf{D}$ of a category $\mathbf{C}$ satisfies [3, p. 39]:

1. every object of $\mathbf{D}$ is also an object of $\mathbf{C}$
2. for all objects X, Y of $\mathbf{D}$, the set $\hom_{\mathbf{D}}(X, Y)$ is a subset of $\hom_{\mathbf{C}}(X, Y)$
3. composition of morphisms is the same in $\mathbf{C}$ and $\mathbf{D}$
4. the identity morphisms are the same in $\mathbf{C}$ and $\mathbf{D}$, for all objects in $\mathbf{D}$.

As a specific example, suppose $\mathbf{C}$ is a category of filters whose objects are all real-valued sequences and $\mathcal{F}$ is any morphism in $\mathbf{C}$. It follows immediately that the homopolymer category $\mathbf{Ho}[\mathcal{F}]$ is a subcategory of $\mathbf{C}$. Similarly, if $\mathcal{F}$ and $\mathcal{G}$ are both morphisms in $\mathbf{C}$, it follows that the copolymer category $\mathbf{Co}[\mathcal{F}, \mathcal{G}]$ is also a subcategory of $\mathbf{C}$. Finally, note that if $\mathbf{C}$ and $\mathbf{D}$ are both subcategories of some larger category $\mathbf{E}$, then the copolymer category $\mathbf{Co}[\mathbf{C}, \mathbf{D}]$ is also a subcategory of $\mathbf{E}$. Further, note that $\mathbf{C}$ and $\mathbf{D}$ are each subcategories of $\mathbf{Co}[\mathbf{C}, \mathbf{D}]$.

Given a structural filter category $\mathbf{C}$ (e.g., the nonlinear FIR filter category $\mathbf{Mallows}_4$) and a behavior-preserving filter category $\mathbf{D}$ (e.g., $\mathbf{F}_S$ for some set of desirable signals like the set $\mathcal{M}$ of monotone sequences), define the *joint subcategory* $\mathbf{Joint}(\mathbf{C}, \mathbf{D})$ as follows [11, ch. 7]. The objects on which this new category are based is the intersection of the object classes on which $\mathbf{C}$ and $\mathbf{D}$ are based, and the morphism set $\hom_J(\{x_k\}, \{y_k\})$ in the joint subcategory is the intersection of the morphism sets $\hom_C(\{x_k\}, \{y_k\})$ for the structural category $\mathbf{C}$ and $\hom_D(\{x_k\}, \{y_k\})$ for the behavior-preserving category $\mathbf{D}$. Note that $\mathbf{Joint}(\mathbf{C}, \mathbf{D})$ is a subcategory of both parent categories $\mathbf{C}$ and $\mathbf{D}$ and its morphisms correspond to all filters whose structure is defined by the category $\mathbf{C}$ with the behavior defined by the category $\mathbf{D}$.

Again, it is instructive to consider the iterated and multi-layer filter categories $\mathbf{Ho}[\mathcal{F}]$ and $\mathbf{Co}[\mathcal{F}, \mathcal{G}]$ here. First, suppose $\mathcal{F}$ represents a morphism in the signal-preserving filter category $\mathbf{F}_S$, implying $\mathcal{F}[S] \subset S$. It follows immediately that $\mathcal{F}^k[S] \subset S$ for all k, implying $\mathbf{Ho}[\mathcal{F}]$ is a subcategory of $\mathbf{F}_S$, provided we restrict the objects of $\mathbf{Ho}[\mathcal{F}]$ to the sequence set S. Similarly, if $\mathcal{F}$ belongs to the joint subcategory $\mathbf{Joint}(\mathbf{C}, \mathbf{D})$, it follows that $\mathcal{F}$ is a morphism in both categories $\mathbf{C}$ and $\mathbf{D}$, implying the homopolymer category $\mathbf{Ho}[\mathcal{F}]$ is a subcategory of both $\mathbf{C}$ and $\mathbf{D}$, again provided the objects of $\mathbf{Ho}[\mathcal{F}]$ are restricted to the common objects of $\mathbf{C}$ and $\mathbf{D}$. Consequently, it follows that this object-restricted verison of $\mathbf{Ho}[\mathcal{F}]$ is also a subcategory of $\mathbf{Joint}(\mathbf{C}, \mathbf{D})$. The same results follow for the copolymer category $\mathbf{Co}[\mathcal{F}, \mathcal{G}]$ if $\mathcal{F}$ and $\mathcal{G}$ are two different morphisms in $\mathbf{Joint}(\mathbf{C}, \mathbf{D})$.

5 AN EXAMPLE

To illustrate the practical utility of the results just presented, consider the following example. It is not difficult to show that the symmetric median filter $\mathcal{M}_K$ defined by Eq. (1) satisfies all five of Mallows' nonlinear smoothing criteria, implying $\mathcal{M}_K$ is a morphism in the category $\mathbf{Mallows}_{12345}$. Hence, the homopolymer category $\mathbf{Ho}[\mathcal{M}_K]$ of iterated median filters is a subcategory of $\mathbf{Mallows}_{12345}$. More interesting, however, are the nonlinear smoothing filters obtained by via cascade connection of $\mathcal{M}_K$ with other filter structures. In particular, suppose $\mathcal{G}$ is a linear FIR filter, also defined on the same symmetric data window, i.e.:

$$y_k = \sum_{j=-K}^{K} g_j x_{k-j}. \tag{2}$$

Without restrictions, this filter satisfies Mallows criteria A1, A3, A4, and A5', but not generally A2. To satisfy this criterion, it is necessary to impose the constraint that the filter weights $\{g_j\}$ sum to 1. If we do impose this restriction, it follows that $\mathcal{G}$ is also a morphism in the category $\mathbf{Mallows}_{12345}$; hence, by the results presented above, the copolymer category $\mathbf{Co}[\mathcal{F}, \mathcal{G}]$ is a subcategory of $\mathbf{Mallows}_{12345}$. This result is interesting because the filters in this copolymer category represent generalizations of many of the popular *block-oriented nonlinear model structures* considered in the biomedical engineering [5] and process control [11] literature. For example, the *Hammerstein model* consists of a static nonlinearity $\mathcal{N}$ followed in cascade by a linear dynamic system $\mathcal{L}$; replacing $\mathcal{N}$ with $\mathcal{M}_K$ results in a nonlinear smoothing filter in the copolymer category $\mathbf{Co}[\mathcal{M}_K, \mathcal{G}]$. A possible practical advantage of this composite filter is that it provides both impulsive noise rejection and linear smoothing.

Another useful characteristic of the standard median filter is that its root set includes all monotone sequences [1]. Further, it is not difficult to show that the linear filter $\mathcal{G}$ defined by Eq. (2) preserves monotonicity if and only if $g_i \geq 0$ for all i [9]. In particular, note that:

$$y_{k+1} - y_k = \sum_{j=-K}^{K} g_j [x_{k+1-j} - x_{k-j}]. \tag{3}$$

Since $x_{k+1} - x_k \geq 0$ for all k for an increasing sequence, it follows that $\{y_k\}$ is increasing if and only if $g_j \geq 0$ for all j; by analogous reasoning, the same result holds for decreasing sequences. Combining these results, it follows that if $g_j \geq 0$ for all j, the copolymer category $\mathbf{Co}[\mathcal{M}_K, \mathcal{G}]$ is a subcategory of the monotone signal-preserving category $\mathbf{F}_{\mathcal{M}}$. Further, since all sequences in $\mathcal{M}$ are invariant under median filtering, the restriction of $\mathbf{Co}[\mathcal{M}_K, \mathcal{G}]$ to monotone sequences is equivalent to the *linear* model category $\mathbf{Ho}[\mathcal{G}]$. Conversely, for non-monotone sequences, the behavior of the filters in this copolymer category is no longer linear, in general.

Finally, another interesting property of the median filter is the following [13]: if the input sequence $\{x_k\}$ belongs to the set $\mathcal{V}_M$ of bounded variation sequences for fixed M, so does the output sequence $\{y_k\}$. Also, it follows from Eq. (3) that if $\{x_k\} \in \mathcal{V}_M$, a sufficient condition for $\{y_k\} \in \mathcal{V}_M$ is:

$$\sum_{j=-K}^{K} |g_j| \leq 1. \tag{4}$$

Hence, if $\mathcal{G}$ is a morphism in both $\mathbf{Mallows}_{12345}$ (implying the coefficients g_j sum to 1) and the category $\mathbf{F}_{\mathcal{M}}$ of monotonicity-preserving filters (implying $g_j \geq 0$ for all j), condition (4) follows, so $\mathcal{G}$ also preserves bounded variation sequences. Hence, the copolymer category $\mathbf{Co}(\mathcal{M}_K, \mathcal{G})$ is a subcategory of $\mathbf{Mallows}_{12345}$, the monotonicity-preserving category $\mathbf{F}_{\mathcal{M}}$, and the variation-preserving category $\mathbf{F}_{\mathcal{V}}$.

6 SUMMARY

In the framework described here, nonlinear digital filter design involves three questions, all of which are independently interesting. First is the specification of a set S of desirable signals, a topic addressed in the references [6, 12, 14, 15]. In the context of filter categories, this specification defines the objects on which behaviorally-defined categories are based. In addition, note that if the set S consists of a single sequence, the behavioral requirement $\mathcal{F}[S] \subset S$ reduces to the restriction that this sequence be a root of $\mathcal{F}$. The second interesting question is the specification of a class $\mathcal{C}$ of useful filter structures; even without the category restriction (i.e., closure under cascade connection), the search for novel nonlinear filter structures is one of the major efforts in signal processing research. Adding the category restriction can lead to the examination of some interesting new structures like the copolymer categories introduced here, extending the notion of block-oriented nonlinear models [5, 11] to the domain of nonlinear digital filters. The third question is probably the one of greatest ultimate interest: how to characterize the morphisms in the joint subcategory defined by these two parent categories, one behavioral and the other structural. Again, the copolymer category construction introduced here becomes quite interesting because it provides a simple, systematic approach to the construction of cascade filter structures satisfying various behavioral restrictions. In particular, as the example in Sec. 5 illustrates, this construct provides a simple way of combining known or easily derivable characterizations of linear FIR filters with known properties of well-characterized nonlinear FIR filters like the median filter $\mathcal{M}_K$.

References

[1] J. Astola, P. Heinonen, and Y. Neuvo, "On root structures of median and median-type filters," *IEEE Trans. Signal Proc.*, vol. 35, 1987, pp. 1199–1201.

[2] J.A. Bangham, "Properties of a Series of Nested Median Filters, Namely the Data Sieve," *IEEE Trans. Signal Proc.*, vol. 41, 1993, pp. 31–42.

[3] T.S. Blyth, *Categories*, Longman, 1986.

[4] M. Charif-Chefchaouni and D. Schonfeld, "On the Convergence and Roots of Order-Statistics Filters," *IEEE Trans. Signal Proc.*, vol. 43, 1995, pp. 1990–1993.

[5] H. Chen, "Modeling and identification of parallel nonlinear systems: Structural classification and parameter estimation methods," *Proc. IEEE*, vol. 83, 1995, pp. 39–66.

[6] A. Ehrenfeucht, J. Haemer, and D. Haussler, "Quasi-monotonic sequences: theory, algorithms, and applications," *SIAM J. Algebraic Discrete Meth.*, vol. 8, 1987, pp. 410–429.

[7] R.W. Hamming, *Digital Filters*, 2nd ed., Prentice-Hall, 1983.

[8] C.L. Mallows, "Some Theory of Nonlinear Smoothers," *Ann. Statist.*, v. 8, 1980, pp. 695–715.

[9] R.K. Pearson, "Empirical models that exhibit monotone responses to monotone inputs," *Proc. ECC'99 European Control Conf.*, Karlsruhe, September, 1999 (CD-ROM).

[10] R.K. Pearson, "Categories of nonlinear dynamic models," in *System Theory: Modeling, Analysis, and Control*, T.E. Djaferis and I.C. Schick (eds.), ch. 29, pp. 391–403, 2000.

[11] R.K. Pearson, *Discrete-time Dynamic Models*, Oxford University Press, 1999.

[12] A. Restropo and A.C. Bovik, "Locally Monotone Regression," *IEEE Trans. Signal Proc.*, vol. 41, 1993, pp. 2796–2810.

[13] A. Restropo and L. Chacon, "A Smoothing Property of the Median Filter," *IEEE Trans. Signal Proc.*, vol. 42, 1994, pp. 1533–1555.

[14] A. Restropo and A.C. Bovik, "On the Statistical Optimality of Locally Monotonic Regression," *IEEE Trans. Signal Proc.*, vol. 42, 1994, pp. 1548–1550.

[15] C.H. Rowher and L.M. Toerien, "Locally monotone robust approximation of sequences," *J. Comput. Applied Math.*, vol. 36, 1991, pp. 399–408.

[16] D.E. Rumelhart, J.L. McClelland, and the PDP Research Group, *Parallel Distributed Processing*, M.I.T. Press, 1988.