DATA DECORRELATION BY WAVELET TRANSFORM

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ABSTRACT

The Wavelet Transform is not optimal for image compression. The coefficients on the same level of decomposition preserve a residual correlation which may harm the efficiency of the encoding algorithm. In the case of entropic codes, this effect can be reduced by using an appropriate coefficients scanning and long enough contexts for the conditional probabilities. Such an approach needs the knowledge of the residual correlation. This paper proposes a set of formulas for the evaluation of the coefficients correlation, based on the image and wavelet auto- correlations. The wavelet type and its length is shown to be unimportant for data decorrelation as proven by similar results on various cases. According to the proposed formulas, in the case of images with separable autocorrelation, the transformation of a coordinate preserves the other coordinate correlation. For classes of signals like images, with autocorrelation matching mathematical models, general encoding procedures could be provided based on these conclusions.

Keywords : image compression, multiresolution analysis, wavelets, correlation.

1 INTRODUCTION

The lossy compression of correlated data is done, usually, in three steps : transform, quantization, encoding. The transform decorrelates the data and compacts its energy, the quantization reduces the entropy and, finally, by encoding, the data is compressed.

The decorrelation is crucial for the coding step. It is known that the entropic coding can provide, at best, a code with an average length equal to data entropy [3]. In the case of correlated data, this inferior limit can be approached only if conditional probabilities are taken into account. For high correlated sequences such an approach is unacceptable because of codebooks complexity. It is, therefore, necessary to previously decorrelate the data by an appropriate transform. Our paper focuses on the decorrelation aspect in the particular case of the multiresolution analysis with wavelets.

The Wavelet Transform is not optimal regarding decorrelation. Although the statistical dependency is reduced in the transformed space, the coefficients preserve a residual correlation that must be taken into account by the coding algorithm if high compression ratios are desired. This correlation can be measured after the transform has been performed or can be evaluated in advance. For this latter option the paper proposes formulas based on the wavelet properties and the correlation of previous finer approximation.

2 THE RESIDUAL CORRELATION OF WAVELET COEFFICIENTS

The approach currently used for multiresolution analysis with wavelets is the iterative algorithm of Mallat [1]. In the 1D case, for its first step, the algorithm splits the signal to be analyzed, \( x(k) \), in two halves: the approximation \( A_1 \) and the details \( D_1 \). At the second step, only the approximation \( A_1 \) is further split in two halves: the coarser approximation \( A_2 \) and the second level details \( D_2 \). The algorithm may proceed this way until the approximation is reduced to a single sample.

The approximation \( A_m \) and the details \( D_m \) on the \( m \)-th level of decomposition, are obtained by convolving the previous finer approximation \( A_{m-1} \) with two quadrature mirror filters, \( \tilde{h}(n) \) and \( \tilde{g}(n) \) [1]:

\[
A_m(k) = \sum_n 2^{-\frac{m}{2}} \tilde{h}(2k-n)A_{m-1}(n)
\]

(1)

\[
D_m(k) = \sum_n 2^{-\frac{m}{2}} \tilde{g}(2k-n)A_{m-1}(n)
\]

(2)

\( \tilde{h}(n) \) is the scale function and \( \tilde{g}(n) \) is the wavelet.

In a compression application, the details are scanned and encoded level by level. The details on each decomposition level are not completely decorrelated. We are going now to establish a formula for the details residual autocorrelation.
The $m$-th level details autocorrelation $R_d^{(m)}$ is given by [2]:
\[
R_d^{(m)}(j) = E[D_m(k)D_m(k + j)]
\]
(3)

Table 1  Horizontal correlation after row processing

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp.</td>
<td>$\rho_d^{(1)}(j)$</td>
<td>0.36</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>$\rho_d^{(2)}(j)$</td>
<td>0.94</td>
<td>0.86</td>
<td>0.79</td>
</tr>
<tr>
<td>Th.</td>
<td>$\rho_d^{(1)}(j)$</td>
<td>0.33</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>$\rho_d^{(2)}(j)$</td>
<td>0.94</td>
<td>0.86</td>
<td>0.79</td>
</tr>
<tr>
<td></td>
<td>$\rho_{hor}(j)$</td>
<td>0.97</td>
<td>0.93</td>
<td>0.89</td>
</tr>
</tbody>
</table>

By introducing (2) in (3) we get:
\[
R_d^{(m)}(j) = E\left[\sum_n 2^2 \mathbb{g}(2k-n)A_{m-1}(n)\right] = \\
= E\sum_n 2^2 g(n)A_{m-1}(n + 2k)
\]
(4)

The correlation coefficient of the previous finer approximation and the wavelet autocorrelation.

A similar formula can be obtained for the correlation coefficient of $m$-th level approximation, $\rho_a^{(m)}$:
\[
\rho_a^{(m)}(j) = \frac{1}{1+v} \left\{ \rho_a^{(m-1)}(-2j); R_h \right\}
\]
(10)

where $\sigma_m$ and $\sigma_{m-1}$ satisfy the following equation (Appendix):
\[
\sigma_m^2 = (1-v)\sigma_{m-1}^2
\]
(8)

By introducing (8) in (7), we obtain:
\[
\rho_a^{(m)}(j) = \frac{1}{1+v} \left\{ \rho_a^{(m-1)}(-2j); R_h \right\}
\]
(11)

The formulas (10) and (11) are tested on ‘Lena’ image. By using a Daubechies wavelet with four coefficients, a single level decomposition is performed on each row and the horizontal correlation coefficient is measured on the two halves. Table 1 shows the experimental values and the theoretical ones evaluated with (10) and (11). The two sets of values are practically equal.

The correlation coefficient of the previous finer approximation, i.e., image horizontal correlation $\rho_{hor}$, is shown on the last table row. By wavelet transforming, the correlation has decreased in the details half where only the neighboring pixels are still correlated. Their correlation coefficient is approximately 0.3. On the approximation half, the correlation has remained almost unchanged.

3 IMAGE CASE
The images multiresolution analysis consists in applying
the equations (1,2), alternately, on rows and on columns [1]. Since row processing may change the vertical
correlation, it is necessary to compute it prior to column
decomposition. Consequently, we have derived the following two formulas allowing the computation of the
cross correlation on the details and approximation halves:

\[
\rho_{d}^{(m)}(r,c) = \frac{1}{1-v} \langle R^c_{r}: \rho_{a}^{(m-1)}(r,-2c) \rangle
\]  
(12)

\[
\rho_{a}^{(m)}(r,c) = \frac{1}{1+v} \langle R^c_{r}: \rho_{a}^{(m-1)}(r,-2c) \rangle
\]  
(13)

where \( \rho_{d}^{(m)} \) is the cross correlation of the details half, \( \rho_{a}^{(m)} \)
the cross correlation of the approximation half and \( \rho_{a}^{(m-1)} \)
the cross correlation of the image previous finer
approximation. By \( r \) and \( c \) we denote the row and the
column lags. The proof is similar to (3-11). For \( c = 0 \), (12)
and (13) give the vertical correlation.

Table 2 shows the vertical correlation after row
processing in the first iteration. For the approximation half,
the experimental and the theoretical values are equal. Slight
differences appear for the details half where the estimation
error is enhanced by the parameter \( 1-v \) which has very
small values for correlated data. The vertical correlation
coefficient of the previous finer approximation, i.e. image
vertical correlation \( \rho_{vert}(r) \), is shown on the last table row.
It is interesting to note that by row transformation the
vertical correlation is practically unchanged on the
approximation half. A certain decrease is noticed for the
details half where the correlation coefficient of neighboring
pixels is reduced from 0.98 to 0.6 – 0.7.

A hypothesis often verified by images is the
separability of the cross correlation:

\[
\rho(r,c) = \rho_{vert}(r)\rho_{hor}(c)
\]  
(14)

In such cases the details cross correlation after the row
processing in the first iteration, is given by:

\[
\rho_{d}^{(1)}(r,c) = \frac{1}{1-v} \langle R^c_{r}: \rho_{vert}(r)\rho_{hor}(k-2c) \rangle = \\
\rho_{vert}(r)\rho_{hor}(k-2c) = \rho_{vert}(r)\rho_{d}^{(1)}(0,c)
\]  
(15)

which shows that row processing leaves unchanged the
vertical correlation:

\[
\rho_{d}^{(1)}(r;0) = \rho_{vert}(r)\rho_{d}^{(1)}(0,0) = \rho_{vert}(r)
\]  
(16)

A similar result can be obtained for the approximation
half. Since, according to (15), the crosscorrelation preserves
its separability, this result may be extended for all the
decomposition levels.

By using (10-11) and (12-13), the coefficient of
correlation can be computed, iteratively, for all the
decomposition levels of the image multiresolution analysis.
Figure 1 shows the correlation coefficient for the details on
the first level of decomposition.
The maps obtained for the vertical, horizontal, and diagonal details suggest that different encoding strategies should be used for the transformed data.

4 CONCLUSIONS

The correlation formulas proposed in this paper may represent a useful tool in wavelet based compression applications. The knowledge of the correlation is important for the coding stage of the compression scheme. It shows how to scan the data and which context should be used for the conditional probabilities. For classes of signals with correlation matching mathematical models, a general encoding procedure could be elaborated based on the formulas proposed in this paper. The images are well appropriated for such procedures since their correlation is often a decreasing exponential function.

In the case of images with separable correlation, according to (15), the transformation of a coordinate does not affect the other coordinate correlation. This result is a strong indication for the necessity of multidimensional transform for images and multichannel image compression.

According to (10-11), the correlation of the transformed data depends on the wavelet by means of $g_R$.

The similarity of various wavelets $g_R$ curves plotted in Figure 2 shows that the wavelet type is not decisive for data decorrelation.

Appendix

The variance of the $m$-th level details is given by [3]:

$$\sigma^2_m = E[D_m^2] - E[D_m]^2$$  \hspace{1cm} (17)

By using (2) and taking the expectation on index $k$, we get:

$$E[D_m^2] = E[\sum_n 2^2 g(n-2k)A_{m-1}(n)]^2 =$$

$$= E[\sum_n 2^2 g(n)A_{m-1}(n+2k)]^2 =$$

$$\sum_n 2g^2(n)E[A_{m-1}(n+2k)^2] +$$

$$+ 2\sum_n 2g(n)g(n+1)E[A_{m-1}(n+2k)A_{m-1}(n+1+2k)] +$$

$$+ 2\sum_n 2g(n)g(n+2)E[A_{m-1}(n+2k)A_{m-1}(n+2+2k)] + \ldots$$ \hspace{1cm} (18)

$$E[D_m] = E[\sum_n 2^2 g(n-2k)A_{m-1}(n)]^2 =$$

$$= E[\sum_n 2^2 g(n)A_{m-1}(n+2k)]^2 =$$

$$\sum_n 2g^2(n)E[A_{m-1}(n+2k)^2] +$$

$$+ 2\sum_n 2g(n)g(n+1)E[A_{m-1}(n+2k)A_{m-1}(n+1+2k)] +$$

$$+ 2\sum_n 2g(n)g(n+2)E[A_{m-1}(n+2k)A_{m-1}(n+2+2k)] + \ldots$$ \hspace{1cm} (19)

By subtracting the previous two expressions we obtain:

$$\sigma^2_D = \sigma^2(2\sum_n g^2(n) + 2\sum_n g(n+1)\rho(1) +$$

$$+ 2\sum_n g(n+2)\rho(2) + \ldots$$ \hspace{1cm} (20)

Since $\sum_n g^2(n) = \frac{1}{2}$, it follows:

$$\sigma^2_D = \sigma^2(R_s:\rho)$$ \hspace{1cm} (21)

References

