CAN STOCHASTIC RESONANCE BE USED IN DETECTION?

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ABSTRACT

This paper deals with stochastic resonance and its application in sine detection. The nonlinear physical phenomenon of stochastic resonance generally occurs in bistable systems excited by a random noise plus a sine. Such systems force cooperation between the input noise and the input sine: Provided a fine tuning between the noise amplitude and the dynamics, the system reacts periodically. The interesting fact is that the local output signal-to-noise ratio presents a maximum when plotted against the input noise amplitude. In this paper we recall the main results for the discrete-time nonlinear AR(1) systems. We then show how stochastic resonance can be used to detect small noisy sine and that the classical incoherent detector can be improved in some non-Gaussian contexts.

1 INTRODUCTION

Stochastic Resonance (SR) is a physical phenomenon recently discovered [1] in the climatic dynamics. This phenomenon occurs generally in bistable dynamical systems excited by random noise and a sine. These systems are generally able to make cooperate noise and sine such that the sine is amplified by the noise. The interesting fact is that the output Signal-to-Noise Ratio (SNR) at the frequency of the sinusoid presents a maximum when plotted against the amplitude of the input noise (see Figures 1 - 3). This fact has led several researchers to examine the ability of SR to detect [8] or amplify small amplitude periodical signal [3].

The theory of SR in continuous time systems is difficult. However, under some assumptions, several approximate theories exist that explain the phenomenon [2, 5]. We have examined in [9] SR in a discrete time context, through the nonlinear AR(1) model:

$$\left\{ \begin{array}{lcl} x_n & = & \Phi(x_{n-1}) + b_n + \varepsilon_n \\ y_n & = & c \, \mathrm{sign}(x_n) \end{array} \right. \quad (1)$$

where b_n is an independent identically distributed (iid) noise of even probability density function (pdf), ε_n is a weak sine of amplitude ε and of frequency λ_0 and where $\Phi(x)$ is taken odd and bistable, where $\pm c$ are the two stable equilibrium points. In this paper we will recall the essential theoretical results and we will present some further results on the simple case $\Phi = c\,\mathrm{sign}$. Finally we will envisaged to use SR to detect a small sine corrupted by an additive noise.

2 MAIN RESULTS ON SR IN DISCRETE TIME

In (1) the iid property of b_n implies that x_n and then y_n are Markovian but non homogeneous: the study is then difficult. Nevertheless, under the hypothesis that the sine is small compared to the noise, we can evaluate the pdf of x_n [9, 10]. Then the probability vector of y_n and the transition matrix from y_{n-1} to y_n can easily be evaluated. Due to the Markov property the transition matrix from y_n to y_{n+k} is simply the product of the one-step transition matrices. Then using the transition matrix from y_n to y_{n+k} and the probability vector of y_n, the zero-cyclic correlation function of signal y_n, $\Gamma(k) = \langle E[y_{n+k} y_n] \rangle_n$ (where $\langle . \rangle_n$ represent an average in n), can be calculated: $\Gamma(k) \approx c^2 \beta^k + \frac{\varepsilon^2 |\chi(\lambda_0)|^2}{2} \cos(2\pi k \lambda_0)$. Parameter β and the susceptibility $\chi(\lambda)$ depends on the system and on the input noise (see [9]). χ is a kind of transfert function but the nonlinearity of the system is contained in this parameter.

For the simple system $\Phi = c\,\mathrm{sign}$, the parameters are simply $\beta = F_b(c) - F_b(-c)$ and $\chi(\lambda) = \frac{2 c f_b(c)}{1 - \beta \exp(-2\imath\pi\lambda)}$ (f_b and F_b are respectively the probability and the cumulative density function of b_n). Then we can evaluate the local output SNR at frequency λ_0 which is $R_{\lambda_0} = \frac{\varepsilon^2 f_b(c)^2}{1-\beta^2}$. This SNR is dependent on the pdf of the noise and is plotted against the input noise amplitude σ in figure 1 when the input noise is Gaussian, in figure 2 when the input noise is uniform, and in figure 3 when the input noise is binary. This figures exhibit the SR phenomenon: there exists an optimal input noise amplitude that maximize the local output SNR. We have shown in [9, 10] that the optimal input noise amplitude σ is relied on the system parameter c by relation

$$\frac{c}{\sigma} = \alpha \quad (2)$$

where $\alpha = 1.575$ for the Gaussian noise, $\alpha = \sqrt{3}$ for the uniform noise and $\alpha = 1$ for the binary noise.

Figures 4, 5 and 6 depict the gain of the signal that is the ratio between the output and the input local SNR. It can be seen on these figures that SR do not improve the local SNR in Gaussian context but hugely improve the local SNR in uniform and binary context.

3 ON THE USE OF SR IN SINE DETECTION

In this section we propose to use SR to solve the problem of the detection of a small sine corrupted by an additive noise. Some researchers has already used SR in this context but their attempts have ended in failure. This was due to the fact that they had envisaged the use of SR in Gaussian context: nevertheless it is not possible to improve the classical detector in this context as we will see further.

The problem is the following: N points of a signal r_n are observed and we have to decide whether the sine is present or not

$$\begin{cases} \mathrm{H}_0: & r_n = b_n & n = 0, \ldots, N-1 \\ \mathrm{H}_1: & r_n = b_n + \varepsilon_n & n = 0, \ldots, N-1 \end{cases}$$

where b_n is an iid noise and where ε_n is a sine of amplitude ε, of frequency λ_0 and of random phase φ_0 uniformly distributed on $[0\,,\,2\pi[$. The Neyman-Pearson approach consist on maximizing the good detection probability P_d for a given false alarm probability P_fa. This leads to the likelihood ratio test (LRT). When the sine frequency λ_0 is known, and when the noise is Gaussian and white, this strategy leads to the incoherent detector (see [7])

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} r_n \mathrm{e}^{-2\imath\pi n\lambda_0} \right| \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \eta \qquad (3)$$

If the left hand side which is the modulus of the Fourier Transform (FT) of r_n at frequency λ_0 is greater than a threshold η, we decide that the sine is present, and that it is not present if the modulus of the FT of r_n is lower than η. The threshold is taken to obtain a fixed false alarm probability. It can then be shown [7] that the detection probability is connected to the false alarm probability by

$$P_\mathrm{d} = Q\left(\sqrt{-2\,N\,R_{\lambda_0}}\,,\,\sqrt{-2\,\ln(P_\mathrm{fa})}\right) \qquad (4)$$

where Q is known as the Marcum function [7] and R_{λ_0} is the local SNR at frequency λ_0 of the sine. For a given x, $Q(x,.)$ is a growing function, hence in the Neyman-Pearson strategy, the better the local SNR, the better the performance of the detector. It can be shown that if the noise is neither Gaussian nor white (4) holds (since the central limit theorem for $\displaystyle\frac{1}{N}\sum_{n=0}^{N-1} r_n \mathrm{e}^{-2\imath\pi n\lambda_0}$ is valid under mild conditions). To improve the incoherent detector, we then have to improve the local SNR of the observed signal. We have previously seen that in uniform or binary iid context SR can lead to the improvement of the local SNR (contrary to any linear filtering). We have also seen that in Gaussian context there is no improvement: SR cannot be envisaged to improve detection in this context. This result is not surprising because the incoherent detector is optimal in the Gaussian iid context.

In the previous section we have recalled that for the $\Phi = c\,\mathrm{sign}$ system the optimal noise amplitude is simply linked on the parameter c of the system. Reversing the problem, for a given noise amplitude, the idea is to adapt the parameter c using (2). Because the relation between c and σ at the optimum depends on the kind of noise, a solution is to use several stochastic resonators $c\,\mathrm{sign}$ for different parameters c. If the noise amplitude is not known, σ can be replaced by an estimated $\widehat{\sigma} = \sqrt{\frac{1}{N}\sum_{n=0}^{N-1} r_n^2 - \left(\frac{1}{N}\sum_{n=0}^{N-1} r_n\right)^2}$: if σ is "badly" estimated (as an example when the sine is present), the fact that several resonators are used will permit a kind of correction. We can then computed the FT of each output. At last, because the power spectral densities (psd) at λ_0 of the noise part of each output of the resonators are not the same, it is necessary to normalize each resonator output. We show in [10, 9] that the psd of the noise part of the output of a resonator is given by $S(\lambda) = \frac{1-\beta^2}{1+\beta^2-2\cos(2\pi\lambda)\beta}$ where β is a jump parameter and can easily be estimated even if the sine is present in the input. Hence the psd of the noise part of each output can be estimated. This leads to the scheme of detection illustrate figure 7. Figures 8 and 9 depicts the estimated receiver operational characteristics (ROC) of the SR detector in the uniform and binary contexts and for a tiny input SNR, compared to the ROC of the incoherent detector.

When the frequency of the sine is not known, the LRT is generalized: in the Gaussian iid context (3) is evaluated for several frequencies (using the FFT) and the maximum result against the frequency is chosen. This generalization can also be used in our scheme: the maximum seek on parameter c is completed by a maximum seek on the frequency.

4 DISCUSSION

In the ROC curves of figure 8 and 9 it can be seen that SR has greatly improved the performance of the incoherent detector. It is important to notice that the SR detector has the same performance when the noise amplitude is known and when it is unknown: the possible estimation error of σ is compensated by the fact that several stochastic resonators are used. At least, notice that the detection is studied for a weak input SNR: the results on SR are valid under this assumption and in this case the ROC of the incoherent detector is the worst.

Now, we currently work to evaluate the theoretical performance of our detector, some bounds of the false alarm probability are given in [10]. We also must compare the performance of our detector to that of the locally optimum detectors (LOD) or other detectors [4]. An other futur way of investigation is to try to replace the parallelized scheme by an adaptive one: optimal parameter c is seek adaptively. This idea is inspired by the work of Mitaim *et al.* in [6] where for a given system the optimal noise amplitude is seek adaptively.

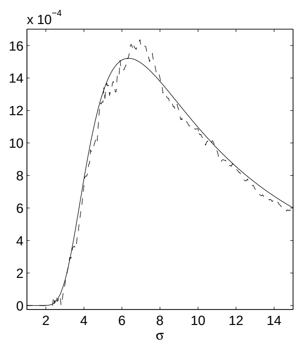

Figure 1: *Theoretical (solid line) and experimental (dashed line) SNR versus σ. b_n is Gaussian, $c = 10$, $\lambda_0 = .02$ and $\varepsilon = 1$.*

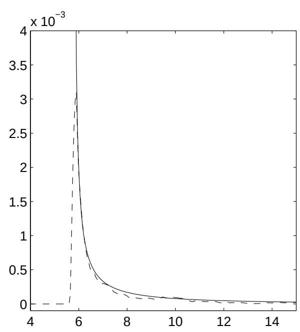

Figure 2: *Theoretical (solid line) and experimental (dashed line) SNR versus σ. b_n is uniform, $c = 10$, $\lambda_0 = .02$ and $\varepsilon = .25$.*

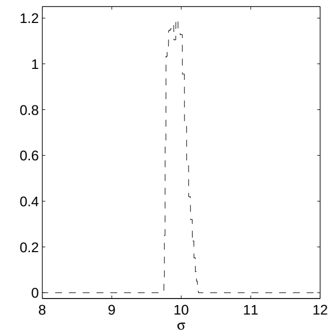

Figure 3: *Experimental SNR versus σ. b_n is binary, $c = 10$, $\lambda_0 = .02$ and $\varepsilon = 1$.*

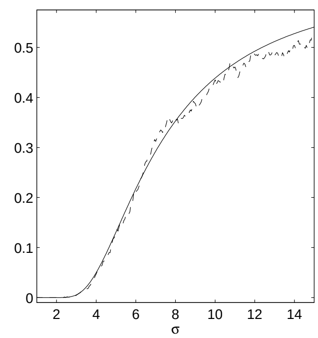

Figure 4: *Theoretical (solid line) and experimental (dashed line) gain versus σ. b_n is Gaussian, $c = 10$, $\lambda_0 = .02$ and $\varepsilon = 1$.*

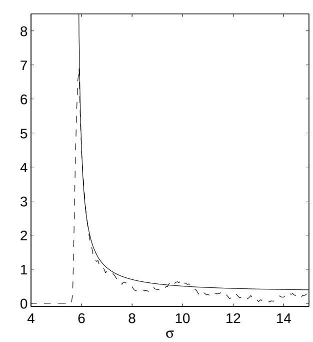

Figure 5: *Theoretical (solid line) and experimental (dashed line) gain versus σ. b_n is uniform, $c = 10$, $\lambda_0 = .02$ and $\varepsilon = .25$.*

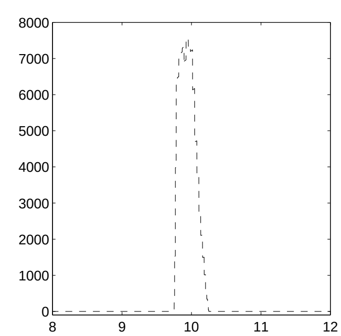

Figure 6: *Experimental gain versus σ. b_n is binary, $c = 10$, $\lambda_0 = .02$ and $\varepsilon = 1$.*

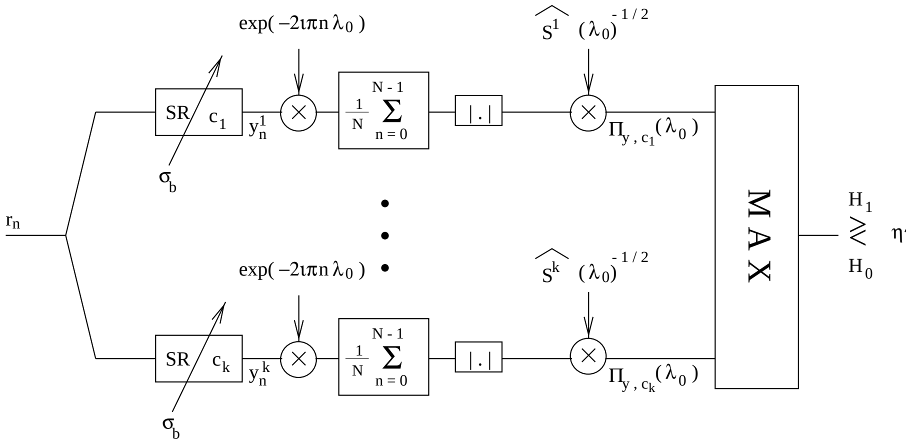

Figure 7: *Strategy of detection using SR.*

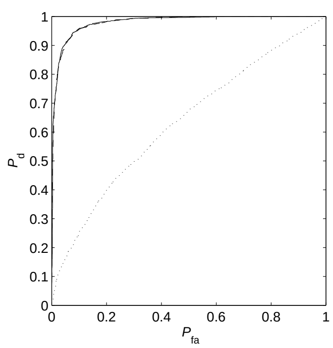

Figure 8: *ROC curve in the uniform iid context: solid line, performance of our detector when σ is unknown; dashed line when σ is known; dotted line incoherent detector using r_n (σ known). Amplitude of the sine $\varepsilon = 0.25$ and global SNR of -35 dB.*

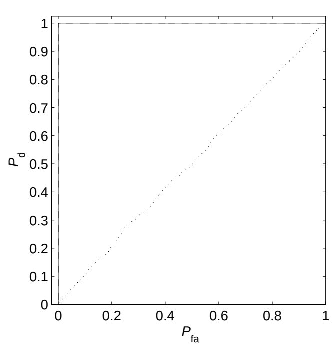

Figure 9: *ROC curve in the binary iid context: solid line, performance of our detector when σ is unknown; dotted line incoherent detector using r_n (σ known). Amplitude of the sine $\varepsilon = 0.05$ and global SNR of -49 dB.*

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