TIME-FREQUENCY AND TIME-SCALE CANONICAL REPRESENTATIONS OF DOUBLY SPREAD CHANNELS

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ABSTRACT

A general technique for the generation of canonical channel models and demonstrate the application of the technique to time-frequency and time-scale integral kernel operators is developed. As an example, the derivation of Sayeed/Aazhang's time-frequency canonical channel characterization that forms the basis for the time-frequency RAKE receiver is shown. Then, a canonical time-scale channel model for wideband communication is developed.

1. INTRODUCTION

The linear time-varying channel is characterized by the time-varying impulse response $h(t,\tau)$ which denotes the response of the channel at time t to an impulse at time $t-\tau$. The channel input-output relationship is

$$y(t) = \int h(t,\tau)x(t-\tau)\mathrm{d}\tau. \tag{1}$$

Taking the Fourier transform of $h(t,\tau)$ with respect to the first argument, we obtain the spreading function $S(\cdot,\tau) = \mathscr{F}\{h(\cdot,\tau)\}$ which has channel input-output relationship

$$y(t) = \iint S(\theta,\tau)x(t-\tau)e^{j2\pi\theta t}\mathrm{d}\tau\mathrm{d}\theta. \tag{2}$$

In [1], Sayeed and Aazhang expanded this channel model to form a canonical time-frequency channel model

$$y(t) = \sum_{n=0}^{N}\sum_{k=-K}^{K} x\left(t-\frac{n}{W}\right)e^{j2\pi kt/T}\hat{S}\left(\frac{k}{T},\frac{n}{W}\right) \tag{3}$$

where N, K, W, and T depend on the channel and signal characteristics. The channel can thus be thought of as combining a discrete set time delayed and frequency shifted versions on the input signal. This channel characterization is associated with narrowband signaling environments.

Our goal in this paper is to develop a similar decomposition for a channel characterization consistent with wideband signaling,

$$y(t) = \iint \mathscr{L}(a,b)\frac{1}{\sqrt{|a|}}x\left(\frac{t-b}{a}\right)\mathrm{d}a\mathrm{d}b, \tag{4}$$

where $\mathscr{L}(a,b)$ is the wideband spreading function.

In Section 2 we review the derivation of the time-frequency canonical channel model. In Section 3 we restate the channel decomposition in a general setting. In Section 4 we prove the main result which allows us to generate canonical channel models. In Section 5 we revisit the time-frequency model and use the thereom to determine the decomposition. Finally, in Section 6 we derive the decomposition for the time-scale canonical channel model.

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2. CANONICAL TIME-FREQUENCY MODEL

We begin with the derivation of the canonical model associated with the standard RAKE receiver. The classic expression of the sampling theorem for a signal $X(\nu)$ with support $(-W/2,W/2)$ is

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{W}\right)\frac{\sin\left(\pi W\left(t-\frac{n}{W}\right)\right)}{\pi W\left(t-\frac{n}{W}\right)}. \tag{5}$$

An alternative formulation of the sampling theorem from [2] is

$$x(t-\tau) = \sum_{n=-\infty}^{\infty} x\left(t-\frac{n}{W}\right)\frac{\sin\left(\pi W\left(\tau-\frac{n}{W}\right)\right)}{\pi W\left(\tau-\frac{n}{W}\right)}. \tag{6}$$

Following [2], substituting (6) into the time-varying impulse response channel characterization (1), we obtain

$$\begin{aligned} y(t) &= \int h(t,\tau)x(t-\tau)\mathrm{d}\tau && \text{(7a)}\\ &= \sum_{n=-\infty}^{\infty} x\left(t-\frac{n}{W}\right)\underbrace{\left[\int h(t,\tau)\frac{\sin\left(\pi W\left(\tau-\frac{n}{W}\right)\right)}{\pi W\left(\tau-\frac{n}{W}\right)}\mathrm{d}\tau\right]}_{=h_n(t)} && \text{(7b)}\\ &\approx \sum_{n=0}^{L:=\lceil T_m/W\rceil} x\left(t-\frac{n}{W}\right)h_n(t) && \text{(7c)} \end{aligned}$$

where the approximation is made based on the assumption that the channel is causal and has finite multipath spread, T_m. That is, $h(t,\tau)=0, \forall\tau<0, \tau>T_m$. Under this assumption, the approximation (7c) corresponds to $h_n(t)$ for which the mainlobe of the sinc function overlaps with the support of the time-varying impulse response. The tapped-delay line in (7c) forms the basis for the classic RAKE receiver, where each of the $h_n(t)$ are usually assumed to be independent.

Now, we move to the Time-Frequency RAKE which was originally derived in [1]. Alternative, but similar models are explored in [3, 4, 5]. The path we take in this derivation is essentially the same as that in [1]. We look at only the $(0,T)$ portion of the received waveform, that is, $y(t)1_{(0,T)}(t)$. Starting from (7b), we insert the $(0,T)$ portion assumption and obtain

$$y(t)1_{(0,T)}(t) = \sum_{n=-\infty}^{\infty} x\left(t-\frac{n}{W}\right)\left[\int h(t,\tau)1_{(0,T)}(t)\mathrm{sinc}\left(W\left(\tau-\frac{n}{W}\right)\right)\mathrm{d}\tau\right] \tag{8}$$

Now we expand the $h(t,\tau)1_{(0,T)}(t)$ term as a Fourier series,

$$\begin{aligned} h(t,\tau)1_{(0,T)}(t) &= \sum_{k=-\infty}^{\infty}\frac{1}{T}\left[\int_0^T h(t',\tau)e^{-j2\pi kt'/T}\mathrm{d}t'\right]e^{j2\pi kt/T} && \text{(9a)}\\ &= \sum_{k=-\infty}^{\infty}\frac{1}{T}\underbrace{\left[\int_{-\infty}^{\infty} h(t',\tau)1_{(0,T)}(t')e^{-j2\pi kt'/T}\mathrm{d}t'\right]}_{\int_{-\infty}^{\infty} S(\theta,\tau)T\mathrm{sinc}\left(\left(\frac{k}{T}-\theta\right)T\right)e^{-j\pi(k-T\theta)}\mathrm{d}\theta}e^{j2\pi kt/T} && \text{(9b)} \end{aligned}$$

which is valid for $t \in (0,T)$.

Plugging (9b) into (8) we obtain,

$$y(t) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x\left(t - \frac{n}{W}\right) e^{j2\pi kt/T} \hat{S}\left(\frac{k}{T}, \frac{n}{W}\right) \tag{10}$$

where,

$$\hat{S}(\theta,\tau) := \iint S(\theta',\tau') \mathrm{sinc}\left((\tau-\tau')W\right) \mathrm{sinc}\left((\theta-\theta')T\right) e^{-j\pi(\theta-\theta')T} \mathrm{d}\theta' \mathrm{d}\tau' \tag{11}$$

(10) is valid for the $(0,T)$ received portion of bandlimited signals.

Under the path scatterer interpretation we assume that the channel introduces a maximum delay spread of T_m and maximum Doppler spread of B_d, that is, $S(\theta,\tau)$ has support in $(-B_d, B_d) \times (0,T_m)$. In the smoothed version of $S(\theta,\tau)$ in (11), if we consider only the terms in (10) where the mainlobe of the smoothing kernel (which has size $(-1/T,1/T)$-by-$(-1/W,1/W)$) overlaps with the support of $S(\theta,\tau)$, we need only sum over $n = 0,\ldots,N$ where $N = \lceil WT_m \rceil$ and $k = -K,\ldots,K$ where $K = \lceil TB_d \rceil$. We thus obtain the canonical representation of the time-frequency channel model,

$$y(t) = \sum_{n=0}^{\lceil WT_m \rceil} \sum_{k=-\lceil TB_d \rceil}^{\lceil TB_d \rceil} x\left(t - \frac{n}{W}\right) e^{j2\pi kt/T} \hat{S}\left(\frac{k}{T}, \frac{n}{W}\right) \tag{12}$$

3. RESTATEMENT

The double sum time-frequency channel formulation (10) was obtained by assuming,

- the input signal is bandpass with bandwidth W, and
- the output signal is analyzed only for $t \in (0,T)$.

With these assumptions in mind, we define the following two projection operators,

$$P_T x(t) \doteq 1_{[0,T]}(t) x(t) \tag{13}$$

and,

$$Q_W x(t) \doteq \mathscr{F}^{-1}\{1_{[-W/2,W/2]}(\omega)\mathscr{F}\{x(t)\}(\omega)\}, \tag{14}$$

and using the following two operators, translation operator,

$$T_\tau x(t) \doteq x(t-\tau), \tag{15}$$

and modulation operator,

$$M_\nu x(t) \doteq x(t) e^{j2\pi\nu t}, \tag{16}$$

we can rewrite (10) as,

$$P_T \mathscr{N}_S Q_W = \sum_{m,n} c_{m,n} P_T M_{\frac{1}{T}}^m T_{\frac{1}{W}}^n Q_W \tag{17}$$

where the $c_{m,n} = \hat{S}\left(\frac{m}{T}, \frac{n}{W}\right)$ and $\mathscr{N}_S$ is the narrowband channel operator,

$$\mathscr{N}_S\{x\}(t) \doteq \iint S(\theta,\tau) x(t-\tau) e^{j2\pi\theta t} \mathrm{d}\tau \mathrm{d}\theta. \tag{18}$$

Restating the channel operator in this setting, we can ask what general properties of the operators allow us to express the channel operator as a double summation of transformed input waveforms. In the next section, we determine properties of the operators used in the expansion that are sufficient conditions for the existence of such an expansion. Our goal is to develop an analogous time-scale canonical channel model. That is, in Section 6 we propose projections P and Q such that,

$$P \mathscr{W}_L Q = \sum_{m,n} c_{m,n} P D_{a_0}^m T_{t_0}^n Q \tag{19}$$

for some choice of dilation and translation spacing parameters (a_0 and t_0), where the $c_{m,n}$ depend on $\mathscr{L}$, and D is the dilation operator,

$$D_a x(t) \doteq \frac{1}{\sqrt{|a|}} x\left(\frac{t}{a}\right), \tag{20}$$

for the wideband channel operator,

$$\mathscr{W}_{\mathscr{L}}\{x\}(t) \doteq \iint \mathscr{L}(a,b) \frac{1}{\sqrt{|a|}} x\left(\frac{t-b}{a}\right) \mathrm{d}a \mathrm{d}b. \tag{21}$$

4. GENERALIZATION

For the statement of the general theorem, we require the following definition.

Definition 1 (paired-up operators). *P and U are paired-up operators with generator e_0 iff,*

1. *P is an orthogonal projection in $L^2(\mathbb{R})$*
2. *U is unitary in $L^2(\mathbb{R})$*
3. *$PU = UP$*
4. *$\exists e_0 \in Ran P$ s.t. $\{U^m e_0 : m \in \mathbb{Z}\}$ is an orthonormal basis for $Ran P$*

Using two different pairs of paired-up operators, the following theorem gives a sufficient condition for the channel expansion.

Theorem 1. *If (P,U) and (Q,V) are both paired-up operators with generator elements e_0 and f_0 respectively, H is a bounded operator, and $\exists c_{m,n}$ such that*

$$\sum_{m,n} c_{m,n} \left\langle V^{n+k} f_0, U^{l-m} e_0 \right\rangle = \left\langle HV^k f_0, U^l e_0 \right\rangle, \quad \forall k,l, \tag{22}$$

then,

$$PHQ = P\left(\sum_{m,n} c_{m,n} U^m V^n\right) Q \tag{23}$$

Proof. First we expand out PQ using the orthonormal basis and unitary properties of the paired-up operators,

$$P = \sum_m \langle \cdot, U^m e_0 \rangle U^m e_0 \tag{24}$$

and

$$Q = \sum_n \langle \cdot, V^n f_0 \rangle V^n f_0, \tag{25}$$

we derive,

$$\begin{aligned} PQx &= \sum_m \langle Qx, U^m e_0 \rangle U^m e_0 && \text{(26a)} \\ &= \sum_m \left\langle \sum_n \langle x, V^n f_0 \rangle V^n f_0, U^m e_0 \right\rangle U^m e_0 && \text{(26b)} \\ &= \sum_{m,n} \langle x, V^n f_0 \rangle \langle V^n f_0, U^m e_0 \rangle U^m e_0. && \text{(26c)} \end{aligned}$$

We use this to determine,

$$\begin{aligned} P\left(\sum_{m,n} c_{m,n} U^m V^n\right) Qx &= \sum_{m,n} c_{m,n} U^m PQV^n x && \text{(27a)} \\ &= \sum_{m,n} c_{m,n} U^m \left(\sum_{k,l} \left\langle V^l f_0, U^k e_0 \right\rangle \left\langle V^n x, V^l f_0 \right\rangle U^k e_0\right) && \text{(27b)} \\ &= \sum_{m,n,k,l} c_{m,n} \left\langle V^l f_0, U^k e_0 \right\rangle \left\langle x, V^{-n} V^l f_0 \right\rangle U^m U^k e_0 && \text{(27c)} \\ &= \sum_{u,s} \left(\sum_{m,n} c_{m,n} \langle V^{n+u} f_0, U^{s-m} e_0 \rangle\right) \langle x, V^u f_0 \rangle U^s e_0 && \text{(27d)} \end{aligned}$$

where the commuting property of paired-up operators was used in (27a), (26c) was used in moving from (27a) to (27b), and the unitary property of V was used in moving from (27b) to (27c). Now, looking to the LHS of (23), we use expand using the orthonormal basis and obtain,

$$\begin{aligned} PHQx &= \sum_s \langle HQx, U^s e_0 \rangle U^s e_0 && (28a)\\ &= \sum_s \left\langle H\left(\sum_u \langle x, V^u f_0\rangle V^u f_0\right), U^s e_0 \right\rangle U^s e_0 && (28b)\\ &= \sum_{s,u} \langle x, V^u f_0\rangle \langle HV^u f_0, U^s e_0\rangle U^s e_0 && (28c)\\ &= \sum_{u,s} h_{u,s} \langle x, V^u f_0\rangle U^s e_0. && (28d)\end{aligned}$$

Given H, we then compute,

$$h_{u,s} \doteq \langle HV^u f_0, U^s e_0\rangle \tag{29}$$

which we use to solve,

$$\sum_{m,n} c_{m,n} \langle V^{n+u} f_0, U^{s-m} e_0\rangle = h_{u,s}, \quad \forall u,s \tag{30}$$

for $c_{m,n}$. These $c_{m,n}$ satisfy (23). □

4.1 Solving the coefficient equation

We now discuss the form of the solution to (22). We define

$$a_{k,l} \doteq \left\langle V^k f_0, U^l e_0 \right\rangle \tag{31}$$

and define

$$\tilde{c}_{m,n} \doteq c_{n,-m} \tag{32}$$

which allows us to express (22) as,

$$\begin{aligned} h_{u,s} &= \sum_{m,n} c_{m,n} \langle V^{n+u} f_0, U^{s-m} e_0\rangle && (33a)\\ &= \sum_{m,n} \langle V^{u-n} f_0, U^{s-m} e_0\rangle \tilde{c}_{n,m} && (33b)\\ &= (a \star \tilde{c})_{u,s} && (33c)\end{aligned}$$

where

$$(a\star\tilde{c})_{u,s} \doteq \sum_{k,l} a_{u-k,s-l}\tilde{c}_{k,l} = \sum_{k,l} a_{k,l}\tilde{c}_{u-k,s-l} \tag{34}$$

Expressing h, a, and $\tilde{c}$ in the Z-transform domain,

$$\begin{aligned} A(z_1,z_2) &\doteq \textstyle\sum_{k,l} z_1^k z_2^l a_{k,l} = \sum_{k,l} z_1^k z_2^l \left\langle V^k f_0, U^l e_0\right\rangle && (35)\\ H(z_1,z_2) &\doteq \textstyle\sum_{k,l} z_1^k z_2^l h_{k,l} = \sum_{k,l} z_1^k z_2^l \left\langle HV^k f_0, U^l e_0\right\rangle && (36)\\ \tilde{C}(z_1,z_2) &\doteq \textstyle\sum_{k,l} z_1^k z_2^l \tilde{c}_{k,l} && (37)\end{aligned}$$

we can write (33c) as,

$$H = A\tilde{C} \tag{38}$$

and solve for $\tilde{C}$

$$\tilde{C}(z_1,z_2) = \frac{H(z_1,z_2)}{A(z_1,z_2)}. \tag{39}$$

In terms of $c_{m,n}$, this is,

$$c_{m,n} = Z^{-1}\left(\frac{H(z_1,z_2)}{A(z_1,z_2)}\right)_{-n,m} \tag{40}$$

where

$$Z^{-1}\left(F(z_1,z_2)\right)_{m,n} = \int_0^1 \mathrm{d}\theta_1 \int_0^1 \mathrm{d}\theta_2 e^{-j2\pi\theta_1 m} e^{-j2\pi\theta_2 n} F\left(e^{j2\pi\theta_1}, e^{j2\pi\theta_2}\right) \tag{41}$$

We can express (40) as a convolution of coefficients by defining

$$\hat{A}(e^{j2\pi\theta_1}, e^{j2\pi\theta_2}) \doteq \frac{1}{A(e^{j2\pi\theta_1}, e^{j2\pi\theta_2})} \tag{42}$$

and

$$\hat{a}_{m,n} \doteq \int_0^1 \mathrm{d}\theta_1 \int_0^1 \mathrm{d}\theta_2 e^{-j2\pi\theta_1 m} e^{-j2\pi\theta_2 n} \hat{A}\left(e^{j2\pi\theta_1}, e^{j2\pi\theta_2}\right), \tag{43}$$

and we can obtain the $c_{m,n}$ using

$$c_{m,n} = \tilde{c}_{-n,m} = (\hat{a}\star h)_{n,-m}. \tag{44}$$

We will use (44) to determine the coefficients in practice.

4.2 Coefficient calculation

Thus, to calculate the coefficients $c_{m,n}$,

1. calculate $h_{k,l}$ via (29),
2. calculate $a_{m,n}$ via (31),
3. use $a_{m,n}$ to obtain $A(e^{j2\pi\theta_1}, e^{j2\pi\theta_2})$ via (35),
4. use $A(e^{j2\pi\theta_1}, e^{j2\pi\theta_2})$ to obtain $\hat{a}_{m,n}$ vua (42) and (43), and
5. use $h_{k,l}$ and $\hat{a}_{m,n}$ to obtain $c_{m,n}$ via (44).

5. REVISITING TIME-FREQUENCY

The example we have seen so far of the application of this theorem had,

- $(P,U,e_0) = (P_T, M_{\frac{1}{T}}, \frac{1}{\sqrt{T}} 1_{[0,T]}(t))$
- $(Q,V,f_0) = (Q_W, T_{\frac{1}{W}}, \sqrt{W}\operatorname{sinc}(Wt))$

for the operator $H = \mathcal{N}_S$ of the form,

$$Hx(t) = \iint S(\theta,\tau) e^{j2\pi\theta t} x(t-\tau)\mathrm{d}\theta\mathrm{d}\tau. \tag{45}$$

Modulation and translation operators were a natural fit with our channel description, $\mathcal{N}_S$, which describes the channel as a (continuous) summation of time and frequency shifts of the input signal. We highlight only the results of the calculations listed in Section 4.2. For more detailed steps, consult [6].

$$h_{k,l} = \sqrt{\frac{W}{T}} \iiint \mathrm{d}\theta\mathrm{d}\tau\mathrm{d}t 1_{[0,T]}(t) e^{j2\pi t(\theta - \frac{l}{T})} \operatorname{sinc}(Wt - k - W\tau) S(\theta,\tau) \tag{46}$$

$$a_{m,n} = \sqrt{\frac{W}{T}} \int_0^T e^{-j2\pi\frac{nt}{T}} \operatorname{sinc}(Wt-m)\mathrm{d}t \tag{47}$$

For $\theta_1, \theta_2 \in [0,1]$,

$$A(e^{j2\pi\theta_1}, e^{j2\pi\theta_2}) = \begin{cases} \sqrt{WT} e^{j2\pi WT\theta_1\theta_2} & : \theta_1 \in \left(0,\frac{1}{2}\right) \\ \sqrt{WT} e^{j2\pi WT(\theta_1 - 1)\theta_2} & : \theta_1 \in \left(\frac{1}{2},1\right) \end{cases} \tag{48}$$

$$\hat{a}_{m,n} = \frac{1}{\sqrt{WT}} \int_0^1 \mathrm{d}\theta_2 e^{-j2\pi\theta_2 n} \operatorname{sinc}(WT\theta_2 + m) = \frac{1}{WT} a_{-m,n} \tag{49}$$

$$c_{m,n} = \iint S(\theta,\tau) e^{j\pi(T\theta+m)} \operatorname{sinc}(T\theta+m)\operatorname{sinc}(n+W\tau)\mathrm{d}\theta\mathrm{d}\tau \tag{50}$$

which are precisely the coefficients in (12).

6. TIME-SCALE CANONICAL MODEL

We now develop the time-scale canonical characterization. For other possible extensions to time-scale, see the approach in [7], [8], and [9] using wavelet packet modulation.

6.1 The scale projection

We use the following projection operator in scale space,

$$P = R_1^{-1}\left(1_{\left[0,\frac{1}{\ln a_0}\right]} \oplus 0\right) R_1 \tag{51}$$

where,

$$R_1 \doteq (\mathscr{F} \oplus \mathscr{F}) R \tag{52}$$

where,

$$R_1 : x \stackrel{R}{\rightarrow} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \stackrel{\mathscr{F}\oplus\mathscr{F}}{\longrightarrow} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{53}$$

for,

$$x_1(t) = e^{\frac{t}{2}} x(e^t) \quad x_2(t) = e^{\frac{t}{2}} x(-e^t) \tag{54}$$

and,

$$R_1^{-1} : \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \stackrel{\mathscr{F}^{-1}\oplus\mathscr{F}^{-1}}{\longrightarrow} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \stackrel{R^{-1}}{\rightarrow} x \tag{55}$$

$$x(t) = \frac{1}{\sqrt{|t|}}\left(x_1(\ln(t))1_{(0,\infty)} + x_2(\ln(-t))1_{(-\infty,0)}\right) \tag{56}$$

where, $x, x_1, x_2, X_1, X_2 \in L^2(\mathbb{R})$.

6.2 The scale generator

Using the characteristic function in scale space (Ω_1, Ω_2), $\Omega_1 = \left[-\frac{1}{2\ln a_0}, \frac{1}{2\ln a_0}\right]$, $\Omega_2 = \emptyset$, leads to the generator,

$$e_0(t) = \begin{cases} \frac{1}{\sqrt{\ln a_0}} \frac{1}{\sqrt{t}} \operatorname{sinc}\left(\frac{\ln|t|}{\ln a_0}\right) & : \ t > 0 \\ 0 & : \ t < 0 \end{cases} \tag{57}$$

It can be shown that $(P, U, e_0) = (R_1^{-1}\left(1_{[0,\frac{1}{\ln a_0}]} \oplus 0\right) R_1, D_{a_0}, e_0)$ are paired-up.

6.3 Time-scale paired-up operators

For the time-scale model, we use the following paired-up operators,

- $(P, U, e_0) = (R_1^{-1}\left(1_{[0,\frac{1}{\ln a_0}]} \oplus 0\right) R_1, D_{a_0}, e_0(t)$ from (57))
- $(Q, V, f_0) = (Q_{\frac{1}{t_0}}, T_{t_0}, \frac{1}{\sqrt{t_0}}\operatorname{sinc}(\frac{t}{t_0}))$

to decompose the wideband channel corresponding to the operator $H = \mathscr{W}_{\mathscr{L}}$ of the form,

$$Hx(t) = \iint \mathscr{L}(a,b) \frac{1}{\sqrt{|a|}} x\left(\frac{t-b}{a}\right) \mathrm{d}a\mathrm{d}b \tag{58}$$

into a discrete double summation,

$$P\mathscr{W}_L Q = \sum_{m,n} c_{m,n} P D_{a_0}^m T_{t_0}^n Q. \tag{59}$$

Again, we highlight only the results of the calculations for the steps in Section 4.2. For more details, consult [6].

$$h_{u,s} = \frac{1}{\sqrt{t_0 \ln a_0}} \int \frac{\mathrm{d}a}{\sqrt{|a|}} \int \mathrm{d}b \mathscr{L}(a,b) \int_0^\infty \mathrm{d}t \frac{1}{\sqrt{t}} \operatorname{sinc}\left(\frac{t-b}{at_0} - u\right) \operatorname{sinc}\left(\frac{\ln t}{\ln a_0} - s\right) \tag{60}$$

$$a_{m,n} = \sqrt{\frac{1}{t_0 \ln a_0}} \int_0^\infty \frac{1}{\sqrt{t}} \operatorname{sinc}\left(\frac{t}{t_0} - m\right) \operatorname{sinc}\left(\frac{\ln|t|}{\ln a_0} - n\right) \mathrm{d}t \tag{61}$$

For $\theta_1, \theta_2 \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, in distributional sense,

$$A(\theta_1, \theta_2) = \sqrt{\frac{1}{t_0 \ln a_0}} t_0^{\frac{1}{2} + j2\pi\frac{\theta_2}{\ln a_0}} \int_0^\infty t^{-\frac{1}{2} + j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1 t} \mathrm{d}t \tag{62}$$

$$\hat{a}_{m,n} = \sqrt{\ln a_0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \frac{t_0^{-j2\pi\frac{\theta_2}{\ln a_0}} e^{-j2\pi\theta_1 m} e^{-j2\pi\theta_2 n}}{\int_0^\infty t^{-\frac{1}{2} + j2\pi\frac{\theta_2}{\ln a_0}} e^{j2\pi\theta_1 t} \mathrm{d}t} \tag{63}$$

$$c_{m,n} = \iint \mathrm{d}a\mathrm{d}b \mathscr{L}(a,b) \operatorname{sinc}\left(n - \frac{b}{at_0}\right) \operatorname{sinc}\left(\frac{\ln(a)}{\ln a_0} - m\right) \tag{64}$$

The canonical time-scale model is then,

$$y(t) = \sum_{m,n} \frac{c_{m,n}}{a^{m/2}} x\left(\frac{t - nba^m}{a^m}\right) \tag{65}$$

for the $c_{m,n}$ defined in (64).

7. SUMMARY

Both time-frequency and time-scale integral kernel operators are often used to model time-varying communication channels. Sayeed and Aazhang have developed a canonical time-frequency representation of the doubly spread channel which has proved useful for the exploitation of the diversity of such channels. We developed a generalization of this canonical model and showed their time-frequency canonical model as an application of this generalization, which was also applied in a time-scale setting to derive a time-scale canonical description of the channel. We hope that further study of this time-scale description will yield similar benefits for wideband signals that Sayeed and Aazhang demonstrated in the narrowband setting.

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