BALANCED CAPACITY OF WIRELINE MULTIACCESS CHANNELS

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ABSTRACT

This paper analyzes the multiuser capacity of Gaussian wireline multiaccess channels. These channels are frequency selective because of the multiple reflections and the losses in the cables. The uplink (multiple access channel) and downlink (broadcast channel) capacity regions are known to be identical for a total power constraint. The concept of balanced capacity is introduced, and algorithms for the computation of the balanced multiuser capacity are proposed for an arbitrary number of users.

1. INTRODUCTION

The wireline multiaccess channel is an example of multiuser channel with ISI. Some practical applications are the outdoor powerline channel and the CATV network. The uplink corresponds to a multiple access channel, whose capacity region is well known in the two-user [1] and K-user [2] cases. An iterative algorithm [3] is necessary to compute the optimal power spectrum allocation. This difficulty can be reduced if the constraint is put on the total transmitted power and not on that of each individual user. In that case, the downlink capacity is the same as the uplink capacity [4]. To ensure fair rates for the various users present in the network, additional constraints should be imposed. The balanced rates constraint is proposed instead of the usual common rates constraint [5]. In the present paper, we first analyze in detail the optimal total power allocation for a Gaussian memoryless channel with K users. The problem of spectrum allocation in frequency-selective channels is then addressed with a constraint on the total transmitted power or on the total transmitted power spectral density. Finally, an iterative algorithm is proposed to compute the maximum balanced rates, and the multiuser diversity gain is computed for a 20-user access network.

2. WIRELINE MULTIACCESS CHANNELS

The investigated multiaccess channel is composed of a main cable and a number of derivations whose terminations are connected to the user modems. These modems are called *network terminations* in this paper, according to the DSL terminology. For an access network with K users, the modems are denoted by NT_k with $k \in \{1, \cdots, K\}$. The end of the distribution cable is connected to the *line termination* (LT) which is supposed to provide a common access to the broadband services. All of the K NT's are about to establish a duplex connection with the LT at a given time. Figure 1 illustrates the access network topology, where d_k and d'_k denote the length of the different cable segments. Unlike the DSL access networks (where each user has its own twisted pair), the physical medium has to be shared by all the users who wish to establish a connection simultaneously. A major feature of the wave propagation in these networks is that multiple reflections may happen on the cable derivations and unmatched terminations. The resulting channel responses $H_k(\omega)$ are multipath. Each path is characterized by its weight a_{lk} and its length L_{lk}, and the channel frequency responses are :

$$H_k(\omega) = \sum_l a_{lk} \exp\left[-\gamma(\omega) L_{lk}\right]. \tag{1}$$

The weights a_{lk} do not depend on the frequency for standard cable derivations and terminations (matched or open). For lossy cables, each path is also frequency selective. A convenient model for the propagation factor $\gamma(\omega)$ is :

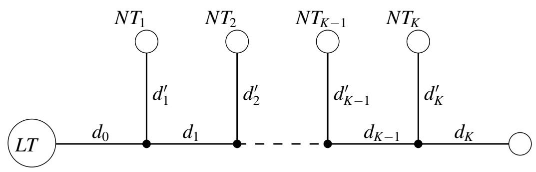

Figure 1: Wireline multiaccess network

$$\gamma(\omega) = \frac{\omega\sqrt{\kappa}}{c}\sqrt{(j+\delta_d)\left[(1+j)\delta_{c1}/\sqrt{f}+j\right]} \tag{2}$$

where $c/\sqrt{\kappa}$ is the light velocity in the cable dielectric, δ_d is the dielectric loss angle and δ_{c1} is the conductor loss angle at 1 Hz (representing the skin effect in the conductors). The resulting channel responses $H_k(\omega)$ are highly frequency selective. Furthermore, the channel attenuation generally increases very rapidly when the user modem is far from the LT. The *uplink* (data transmission from the NT's to the LT) corresponds to a *multiple access channel* (MA), while the *downlink* (data transmission from the LT to the NT's) corresponds to a *broadcast channel* (BC). The additive noise is supposed to be Gaussian and white, with a double-sided power spectral density $N_0/2$. In a practical system, constraints are put on the bandwidth and power of the transmitted signals. We assume in the sequel a transmission bandwidth $[0, B]$. Concerning the power constraint for the MA channel, it makes sense to consider the set of K transmitters as a 'distributed' transmitter with a single constraint on the power-sum. For wireline communications, it is also common to put a constraint on the power spectral density (PSD).

3. MULTIUSER CAPACITY

The multiuser capacity region is defined as the set of achievable rates $\{R_k\}$ at which the receiver(s) may decode information from the transmitter(s) without error. For the Gaussian K-user channel, it is a convex region in the K-dimensional space. This region reflects the tradeoff among the individual data rates of the different users competing for the limited resources. The boundary of the capacity region can be traced out by means of a set of relative priority coefficients α_k with $\sum_k \alpha_k = 1$. Each boundary point of the capacity region maximizes the linear combination of the user rates $R_\alpha = \sum_k \alpha_k R_k$. To obtain a practical characterization of the access network capacity, it is useful to put forward some specific points of the boundary :

- The *single user rates* R_k^1 are the maximum achievable rates in the single user communication scenario.
- The *maximum sum rate* is for $\alpha_k = 1/K$. It generally results in unfair situations where the users with the best channels have a much higher rate than the others, which is not desirable in practical applications.
- The *maximum common rate* [5] is such that $R_1 = \cdots = R_K$. When the single user rates are very different, the common rate constraint is generally a waste of resources as it forces the users

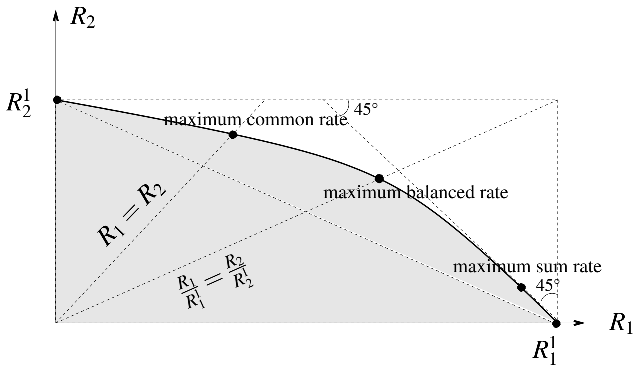

Figure 2: Two-user capacity region

with the best channels to lower their rate dramatically to reach the level of the weakest channels.

- The *maximum balanced rates* are such that $\frac{R_1}{R_1^1} = \cdots = \frac{R_K}{R_K^1}$. In other words, each user transmits at a rate R_k which is in the same proportion w.r.t. the potential rate R_k^1 offered by its own channel. The relative cost implied by the coexistence with the other users is the same for all the users. As a lower bound on the balanced rate, time sharing across the users can be considered, with time slots of equal duration for each user. This strategy gives a set of rates $R_k = \frac{R_k^1}{K}$. A smart power and spectrum management allows higher balanced rates, of course.

Figure 2 gives an example of a convex capacity region and the corresponding boundary points for $K = 2$.

3.1 Memoryless channel

In the memoryless scenario, the channels are fully defined by the set of attenuation factors $\{h_k^2\}$ and the AWGN variance $\sigma^2 = N_0 B$. Without loss of generality, we assume that *the users are ordered in ascending order of the priority coefficients* $\{\alpha_k\}$. For a given set of transmission powers $\{P_k\}$, the capacity region for the MA channel is known to be [2] :

$$\left\{ \{R_k\} : \sum_{k \in \mathrm{S}} R_k \leq \mathrm{C}\left(\frac{\sum_{k\in\mathrm{S}} P_k h_k^2}{\sigma^2} \right) \forall \mathrm{S} \subset \{1, \cdots, K\} \right\} \tag{3}$$

where $\mathrm{C}(x) = B\log(1+x)$. This region is defined by $2^K - 1$ constraints, each corresponding to a nonempty subset S of users. It has $K!$ vertices in the positive quadrant. Each vertex is achievable by a successive decoding using one of the $K!$ possible orderings of the users. In the special case $K = 2$, the capacity region is a pentagon whose two useful vertices correspond to the decoding orders $(1,2)$ and $(2,1)$ respectively. For the choice of priority coefficients $\{\alpha_k\}$, the maximum of R_α is obtained at a single vertex that corresponds to the decoding order $\{1, \cdots, K\}$. In other words, the messages with the lower priorities are decoded first and subtracted from the received signal before decoding the messages with higher priorities. The maximum weighted rate is then given by:

$$R_\alpha = \sum_{k=1}^{K} \alpha_k \mathrm{C}\left(\frac{P_k h_k^2}{\sigma^2 + \sum_{l=k+1}^{K} P_l h_l^2} \right). \tag{4}$$

In the BC channel, the decoding strategy is different for each receiver. Here the decoding order is not dictated by the user relative priorities, but by the relative channel gains. A given receiver is able to decode the messages intended to the users with lower channel gains (and thus transmitted at a lower rate) before decoding its own message. The remaining messages are considered as noise. For a given set of transmission powers $\{P_k\}$, the capacity region has a single vertex that maximizes the weighted rate R_α for any set $\{\alpha_k\}$ [6] :

$$R_\alpha = \sum_{k=1}^{K} \alpha_k \mathrm{C}\left(\frac{P_k h_k^2}{\sigma^2 + \left(\sum_{h_k^2 < h_l^2} P_l\right) h_k^2} \right). \tag{5}$$

We denote by $\{P_k^*\}$ and $\{P_k^\circ\}$ the optimal power allocations that maximize R_α for the MA (eq. (4)) and BC (eq. (5)) channels respectively, with a given constraint $\bar{P}$ on the total transmitted power. When no power is assigned ($P_k = 0$), the marginal rate gain for each user is given by $D_k(0)$ with $D_k(P) = \frac{\alpha_k h_k^2/\sigma^2}{1+P h_k^2/\sigma^2}$. Some power P_{k_1} is firstly allocated to the user k_1 with the maximum $\alpha_k h_k^2$. The marginal gain for the remaining users becomes $D_k(P_{k_1})$. The power P_{k_1} is then increased until a second user k_2 (with $\alpha_{k_2} > \alpha_{k_1}$ and $h_{k_2}^2 < h_{k_1}^2$) reaches a marginal gain equal to that of user k_1 (unless $\bar{P}$ is reached before). When this happens, the common marginal gain can be shown to be $(\alpha_{k_2} h_{k_2}^2/\sigma^2) F_{k_1,k_2}$ where the correction factor $F_{i,j}$ is defined as $F_{i,j} = \frac{1-\alpha_i/\alpha_j}{1-h_i^{-2}/h_j^{-2}}$. Extending this analysis, it can be shown [7] that the optimal solution satisfies :

$$R_\alpha = B \int_0^{\bar{P}} \max_{k\in\{1,\cdots,K\}} D_k(x)dx, \tag{6}$$

in the MA and BC scenarios. The maximum rates are found to be identical for the MA and BC channels. These two channels have thus the same capacity region when the constraint is put on the transmitted power sum. However, the power distributions corresponding to a given boundary point of the capacity region are distinct. At the end of the allocation process, a subset $S_J = \{k_1, \cdots, k_J\} \subset \{1, \cdots, K\}$ of $J \leq K$ users is obtained, who get a non-zero fraction of the total power $\bar{P}$. The users in this subset are sorted in *decreasing order of the channel gains* h_k^2 and in *increasing order of the priority coefficients* α_k.

Algorithm 1 *Computation of the subset* S_J *:*

- *Initialization* $(J = 1)$ *:*

$$k_1 = \arg\max_{k\in[1,K]} \left(\alpha_k h_k^2\right). \tag{7}$$

- *While* $k_J < K$*:*
  - *Compute the next candidate :*

$$k^* = \arg\max_{k\in[k_J+1,K]} \left(\alpha_k h_k^2 F_{k_J,k}\right). \tag{8}$$

  - *If there is enough power :*

$$\left(1 + \frac{\bar{P} h_{k^*}^2}{\sigma^2}\right) \geq F_{k_J,k^*}^{-1} \tag{9}$$

    then: $J = J+1$, $k_J = k^*$.
  - *Else: stop.*

The maximum user rates are then obtained as follows :

$$\begin{pmatrix} R_{k_1}^* \\ \cdots \\ R_{k_j}^* \\ \cdots \\ R_{k_J}^* \end{pmatrix} = B\log \begin{pmatrix} \frac{\alpha_{k_1} h_{k_1}^2}{\alpha_{k_2} h_{k_2}^2 F_{k_1,k_2}} \\ \cdots \\ \frac{\alpha_{k_j} h_{k_j}^2 F_{k_{j-1},k_j}}{\alpha_{k_{j+1}} h_{k_{j+1}}^2 F_{k_j,k_{j+1}}} \\ \cdots \\ \left[1 + \frac{\bar{P} h_{k_J}^2}{\sigma^2}\right] F_{k_{J-1},k_J} \end{pmatrix} \tag{10}$$

The associated MA and BC optimal powers are :

$$\begin{pmatrix} P^{\circ}_{k_1} \\ \cdots \\ P^{\circ}_{k_j} \\ \cdots \\ P^{\circ}_{k_J} \end{pmatrix} = \frac{1}{\alpha_{k_J}} \left(\bar{P} + \frac{\sigma^2}{h^2_{k_J}} \right) \underline{\Pi}\left(\underline{\alpha}, \underline{h}\right) - \begin{bmatrix} 0 \\ \cdots \\ 0 \\ \cdots \\ \frac{\sigma^2}{h^2_{k_J}} \end{bmatrix} \tag{11}$$

with

$$\underline{\Pi}\left(\underline{\alpha}, \underline{h}\right) = \begin{bmatrix} h^{-2}_{k_1}\left(\alpha_{k_1} h^2_{k_1} - \alpha_{k_2} h^2_{k_2} F_{k_1,k_2}\right) \\ \cdots \\ h^{-2}_{k_j}\left(\alpha_{k_j} h^2_{k_j} F_{k_{j-1},k_j} - \alpha_{k_{j+1}} h^2_{k_{j+1}} F_{k_j,k_{j+1}}\right) \\ \cdots \\ h^{-2}_{k_J}\left(\alpha_{k_J} h^2_{k_J} F_{k_{J-1},k_J}\right) \end{bmatrix},$$

and

$$\begin{pmatrix} P^{*}_{k_1} \\ \cdots \\ P^{*}_{k_j} \\ \cdots \\ P^{*}_{k_J} \end{pmatrix} = \sigma^2 \begin{pmatrix} G_{k_1,k_2} \\ \cdots \\ G_{k_j,k_{j+1}} - G_{k_{j-1},k_j} \\ \cdots \\ -G_{k_{J-1},k_J} \end{pmatrix} + \begin{pmatrix} 0 \\ \cdots \\ 0 \\ \cdots \\ \bar{P} \end{pmatrix} \tag{12}$$

with $G_{i,j} = \frac{\alpha_i h_i^2 - \alpha_j h_j^2}{(\alpha_j - \alpha_i) h_i^2 h_j^2}$. Algorithm 1 ensures that the obtained rates and powers are positive.

3.2 Frequency-selective channel

The frequency-selective channels $H_k(\omega)$ can be discretized by dividing the frequency spectrum into a large number N of frequency bins of width df. As N increases to infinity, a piece-wise constant channel model converges to the actual channels. Each frequency bin $n \in [1,N]$ corresponds to a multiuser memoryless Gaussian channel with channel gains $h^2_{kn} = |H_k(\omega_n)|^2$ and a bandwidth B/N. The problem is now to find the K optimal power spectra P_{kn} corresponding to a given boundary point of the capacity region. The user rates R_k are computed by summing the partial rates R_{kn} corresponding to the N parallel subchannels.

The solution is made of two steps:

- Find the solution along the frequency axis, i.e. compute the optimal spectral allocation of the power sum $\bar{P}_n = \sum_{n=1}^{N} P_{kn}$. For a constraint on the total transmitted PSD $\sum_k \gamma(\omega) \leq \bar{\gamma}(\omega)$, the power sum in each frequency bin is trivially given by $\bar{P}_n = \bar{\gamma}(\omega_n) df$. For a constraint on the total power sum $\sum_n \bar{P}_n = \bar{P}$, the solution should satisfy :

$$\frac{\partial R_{\alpha n}}{\partial \bar{P}_n} \begin{cases} = \mu & \bar{P}_n > 0 \\ < \mu & \bar{P}_n = 0 \end{cases} \tag{13}$$

  where μ is a positive Lagrange multiplier. Using (6), the partial derivative is $\max_k \{D_{kn}(\bar{P}_n)\}$, which provides the multiuser water-filling solution :

$$\mu \bar{P}_n = \max_{k \in [1,K]} \left(1 - \left[\frac{\mu}{h^2_{kn}/\sigma^2} + (1 - \alpha_k) \right] \right)_+ . \tag{14}$$

  A simple binary search must be performed to find the right μ that satisfies the total power constraint.
- Compute the optimal power allocation $\{P_{kn}\}$ among the users in each frequency bin with $\sum_{k=1}^{K} P_{kn} = \bar{P}_n$, by using the results of section 3.1. In each frequency bin, some power is allocated to a specific subset $S_J(n)$ of $J(n)$ users.

3.3 Maximum balanced rates

To obtain a specific boundary point of the capacity region such that the corresponding user rates satisfy $\frac{R_1^*}{\beta_1} = \cdots = \frac{R_K^*}{\beta_k}$ for a predefined set of normalization coefficients $\{\beta_k\}$, the right set of priority coefficients $\{\alpha_k\}$ has to be found iteratively. Writing $\underline{\alpha} = [\alpha_1, \cdots, \alpha_K]^T$, the following system must be solved :

$$\underline{F}\left(\underline{\alpha}\right) = \begin{pmatrix} \frac{R_2^*}{\beta_2} - \frac{R_1^*}{\beta_1} \\ \vdots \\ \frac{R_K^*}{\beta_K} - \frac{R_1^*}{\beta_1} \\ \sum_k \alpha_k - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{15}$$

A Newton-Raphson iterative search is proposed to find the solution :

Algorithm 2 *Maximum balanced rates*

- *Start with* $\underline{\alpha}^{(0)} = [1/R_1^1, \cdots, 1/R_K^1]^T / \sum_k (1/R_k^1)$, *set* $i = 0$.
- *Compute the user rates* $\{R_k^*\}$, *the normalized rates* $\{R_k^*/\beta_k\}$, *the average normalized rate* μ_{R_β} *and the standard deviation* σ_{R_β}.
- *While* $\sigma_{R_\beta}/\mu_{R_\beta} > \varepsilon$:
  - *Update the set of priority coefficients as follows:*

$$\underline{\Delta}_\alpha^{(i)} = -\left(\frac{\partial \underline{F}}{\partial \underline{\alpha}}\right)^{-1}_{\underline{\alpha} = \underline{\alpha}^{(i)}} \underline{F}\left(\underline{\alpha}^{(i)}\right) \tag{16}$$

$$\underline{\alpha}^{(i+1)} = \underline{\alpha}^{(i)} + \lambda_i \underline{\Delta}_\alpha^{(i)}. \tag{17}$$

  - *Update* $\{R_k^*\}$, $\{R_k^*/\beta_k\}$, μ_{R_β} *and* σ_{R_β}, *set* $i = i+1$.

  where ε *is a tolerance parameter and the sequence* $\lambda_i \leq 1$ *is used to ensure convergence in the first few iterations.*

The computation of the update direction $\underline{\Delta}_\alpha^{(i)}$ requires the knowledge of the Jacobian matrix sum $\frac{N}{B} \sum_n \frac{\partial R_n}{\partial \underline{\alpha}}$. From (10), each term in this sum appears to be a symmetric tridiagonal matrix $\underline{\underline{T}}_n\left(\underline{\alpha}\right)$ with the following entries on the three main diagonals :

$$\begin{aligned} \underline{\underline{T}}_{k_j,k_j} &= \frac{\alpha_{k_{j+1}} - \alpha_{k_{j-1}}}{(\alpha_{k_j} - \alpha_{k_{j-1}})(\alpha_{k_{j+1}} - \alpha_{k_j})} \\ \underline{\underline{T}}_{k_j,k_{j+1}} &= \frac{-1}{\alpha_{k_{j+1}} - \alpha_{k_j}} = \underline{\underline{T}}_{k_{j+1},k_j} \end{aligned} \tag{18}$$

where $\alpha_{k_0} = \alpha_{k_{J+1}} = 0$. In particular, each Jacobian matrix is zero on the rows and columns corresponding to users who are not in the subset $S_J(n)$.

For a variable total power spectrum, a correction term must be included in the expression of the Jacobian matrices, to account for the variability of $\bar{P}_n$ with the user relative priorities. This term has non-zero entries on its k_J^{th} row :

$$\left(\frac{h^2_{k_J n}/\sigma^2}{1 + \bar{P}_n h^2_{k_J n}/\sigma^2} \right) \frac{\partial \bar{P}_n}{\partial \alpha_k}. \tag{19}$$

Let us define N_k as the number of frequency bins n such that $k_J(n) = k$, with $\sum_k N_k = N$. From the solution of the water-filling problem, the partial derivative $\frac{\partial \bar{P}_n}{\partial \alpha_k}$ can be shown to be :

$$\frac{\partial \bar{P}_n}{\partial \alpha_k} = \frac{1}{\mu} \left(\delta_{k_J,k} - \frac{\alpha_{k_J} N_k}{\sum_{l=1}^{K} \alpha_l N_l} \right). \tag{20}$$

4. RESULTS

A regular-pattern wireline access network is considered with $K = 20$ derivations, identical cable segments of length $d_k = d'_k = 15$ m, and ideal matched terminations. The lossy cables parameters are $\kappa = 2.3$, $\delta_d = 0.02$ and $\delta_{c1} = 85\sqrt{\text{Hz}}$. The top of Figure 3 gives the channel frequency responses $|H_k(\omega)|^2$ in the bandwidth $[0, 10]$ MHz. Through the combined effect of cable losses and multiple reflections on the cable derivations, the channel gains for the remote users can go below -100 dB at some frequencies. The noise level is chosen as -120 dBm/Hz, and two kinds of power constraints are put on the transmission power: a maximum transmission PSD-sum of $\bar{\gamma} = -60$ dBm/Hz ('flat-PSD'), or a maximum transmission power-sum of $\bar{P} = \bar{\gamma}B = 10$ mW ('waterfilling-PSD').

Figure 4 gives the evolution of the single-user capacities R_k^1 for the two kinds of power constraints. In a multiuser scenario with a subset $S_{K^*} \subset \{1, \cdots, K\}$ of K^* active users, the maximum balanced rates are $R_k = g\,\frac{R_k^1}{K^*}$ where g can be interpreted as a 'multiuser diversity gain'. This gain increases with K^*, and is much larger when the user channels in S_{K^*} are very different. Figure 5 presents some examples of multiuser gains associated with various sets S_{K^*}.

Figure 3 also gives the results of the optimal MA and BC power allocations (the user PSDs are normalized with respect to $\bar{\gamma}$) for $S_4 = \{5, 10, 15, 20\}$ and a water-filling PSD. The normalized optimal PSD-sum is also given (upper lines). As expected, the users with the weakest channels concentrate their transmission power in the best parts of the frequency spectrum while the users with the best channels (and lowest priorities) can make use of the remaining frequencies. The remote users require more power in the BC channel (where they are decoded first) than in the MA channel (where they are decoded last). The lower part of the figure illustrates the spectral efficiency in bits/s per Hz.

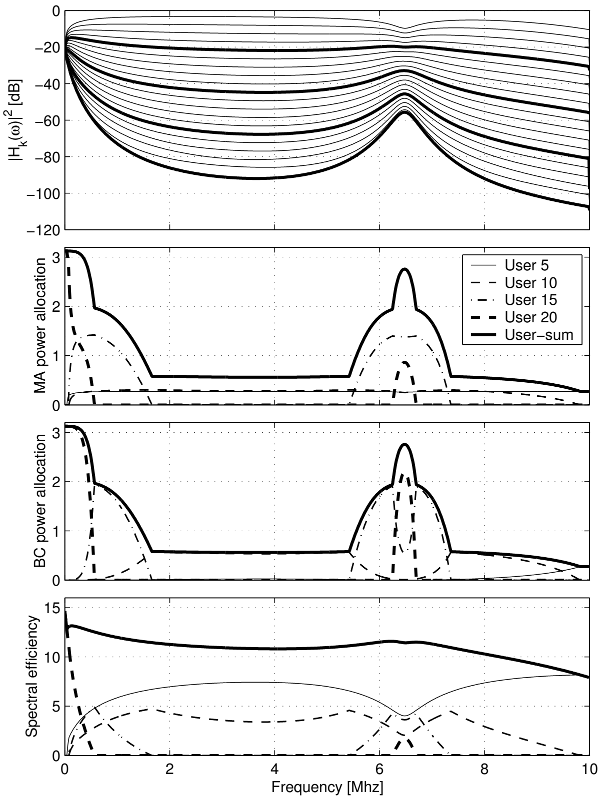

Figure 3: Channel frequency responses ($K = 20$ users), power allocations and spectral efficiency for $S_4 = \{5, 10, 15, 20\}$

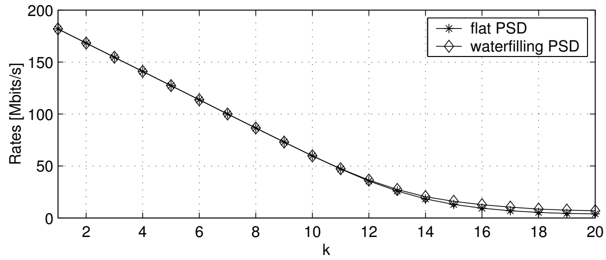

Figure 4: Single user rates vs. user indek k

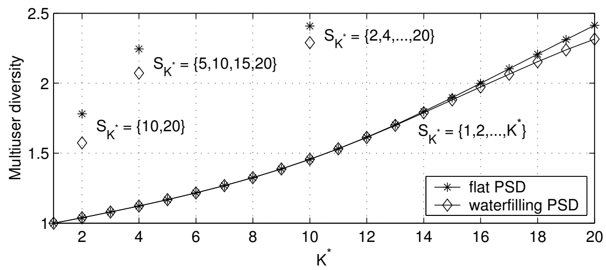

Figure 5: Multiuser diversity vs. number of active users K^*

5. CONCLUSION

This paper proposes a simple method to compute the maximum balanced rates of a K-user network, with a constraint on the total transmitted power (or PSD). Both the uplink (multiple access channel) and the downlink (broadcast channel) power allocations are computed, and the resulting user rates are the same in both configurations. The method basically involves three steps : a multiuser water-filling step (total power spectrum allocation), a power allocation among the users in each frequency bin, and the iterative computation of priority coefficients providing balanced user rates. Results are provided for a wireline access network with $K = 20$ users.

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