A NEW EXPRESSION OF THE ASYMPTOTIC PERFORMANCES OF MAXIMUM LIKELIHOOD DOA ESTIMATION METHOD WITH MODELING ERRORS

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ABSTRACT

This paper provides a new analytic expression of the RMS (Root Mean Square) error and bias of the Maximum Likelihood (ML) Direction Of Arrival (DOA) estimator in the presence of steering vectors modeling errors. The reference [4] proposes a first order approximation of these performances which is adapted to small modeling errors. In order to take into account larger modeling errors and provide tools for designing experimental set-up, a more accurate and easily usable derivation of these performances is necessary. For such an investigation, the DOA estimation errors are written as an hermitean form with a stochastic vector composed by the modeling errors. Finally, a closed form expression between the performances (bias and RMS error) and statistical moments of the model error are deduced from the statistics of the hermitean form. Simulations confirm the theoretical results.

1. INTRODUCTION

It is well-known that high resolution methods [1][2] are sensitive to inadequacies of the reception model. Despite its great interest for designing an experimental set-up, few works were related to the analytical quantification of the degradation introduced by modeling errors of the reception network. On the assumption of small modeling errors B.Friedlander [4] presented a first order analysis of the ML algorithm. However by confronting his theoretical results with simulations in the presence of larger modeling errors, the proposed performances do not take into account well the RMS error obtained by simulations; the closed form expressions of [4] are generally optimistic. A more accurate relation between larger modeling errors and the DOA estimation performances is necessary for the design of a DOA estimation system, like specifications of the material (receivers, antennas , …) for a performance target. The purpose of this paper is to provide a new expression of the RMS error and bias of the DOA estimation errors of the ML algorithms in the presence of modeling errors [3][5] with a second order analysis.

The covariance matrix of the signals is assumed to be known : asymptotic case. The modeling errors are random variables without assumptions on their probability law. It is shown that the DOA errors can be written as a compact hermitean form of multi-variate complex random variables (steering vectors modeling errors) whose statistical properties are well-known. This key result allows easily the ML performances derivation : the bias depends on the second order statistics of the steering vectors errors and the RMS error on the fourth order whatever the probability density function of the model errors.

2. SIGNAL MODELING AND PROBLEM FORMULATION

A noisy mixture of M narrow-band sources with DOAs θ_m ($1\leq m\leq M$) is assumed to be received by an array of N sensors. The associated observation vector, $\mathbf{x}(t)$, whose components $x_n(t)$ ($1\leq n\leq N$) are the complex envelopes of the signals at the output of the sensors, is thus given by

$$\mathbf{x}(t)=\sum_{m=1}^{M}\tilde{\mathbf{a}}(\theta_m{}^0)\ s_m(t)+\mathbf{n}(t)=\widetilde{\mathbf{A}}(\underline{\theta}_0)\ \mathbf{s}(t)+\mathbf{n}(t), \qquad (1)$$

where $\tilde{\mathbf{a}}(\theta)$ is the steering vector of a source in the direction θ, $\tilde{\mathbf{A}}(\underline{\theta}_0)=[\tilde{\mathbf{a}}(\theta_1{}^0)\ldots\tilde{\mathbf{a}}(\theta_M{}^0)]$, $\underline{\theta}_0=[\ \theta_1{}^0\ldots\ \theta_M{}^0]$, $\mathbf{n}(t)$ is the supposed spatially white noise vector and $s_m(t)$ is the complex envelope of the m^{th} source. By noting $\mathbf{a}(\theta)$ the steering vector used in the algorithm, the modeling errors $\mathbf{e}_m$ of the m^{th} source is defined by:

$$\mathbf{e}_m=\tilde{\mathbf{a}}(\theta_m{}^0)-\mathbf{a}(\theta_m{}^0)=\tilde{\mathbf{a}}_m-\mathbf{a}_m, \qquad (2)$$

where $\tilde{\mathbf{a}}(\theta_m)=\tilde{\mathbf{a}}_m$ and $\mathbf{a}(\theta_m)=\mathbf{a}_m$. In the sequel, $(.)^{\mathrm{T}}$ is the transposition operator. In these conditions the matrix $\tilde{\mathbf{A}}(\underline{\theta}_0)$ is a function of $\mathbf{E}$ with $\mathbf{E}=[\mathbf{e}_1\ \ldots\ \mathbf{e}_M]$ and can be noted $\tilde{\mathbf{A}}(\underline{\theta}_0,\mathbf{e})=\tilde{\mathbf{A}}(\underline{\theta}_0)$ where $\mathbf{e}=vec(\mathbf{E})=[\mathbf{e}_1{}^{\mathrm{T}}\ \ldots\ \mathbf{e}_M{}^{\mathrm{T}}]^{\mathrm{T}}$ and:

$$\tilde{\mathbf{A}}(\underline{\theta}_0,\mathbf{e})=\mathbf{A}(\underline{\theta}_0)+\mathbf{E}. \qquad (3)$$

In particular $\mathbf{A}(\underline{\theta}_0)=\tilde{\mathbf{A}}(\underline{\theta}_0,\mathbf{e}=0)=[\mathbf{a}_1\ \ldots\ \mathbf{a}_M]=\mathbf{A}_0$ .This modeling error can be adapted to all kinds of distortion such as sensors error position[3] and mutual coupling effects [5].

The estimation problem under consideration is to estimate the M DOA parameters $\theta_1{}^0,\ldots,\theta_M{}^0$ with the ML algorithm where $\mathbf{R}_x(\mathbf{e})=\mathrm{E}[\mathbf{x}(t)\ \mathbf{x}(t)^{\mathrm{H}}]$ and M are assumed to be known (E[.] is the expectation mean and $^{\mathrm{H}}$ defines the conjugate-transpose). $\mathbf{R}_x(\mathbf{e})$ can be expressed as:

$$\mathbf{R}_x(\mathbf{e})=\tilde{\mathbf{A}}(\underline{\theta}_0,\mathbf{e})\ \mathbf{P}\ \tilde{\mathbf{A}}(\underline{\theta}_0,\mathbf{e})^{\mathrm{H}}+\sigma^2\,\mathbf{I}_N, \qquad (4)$$

where $\mathbf{P}$= E[$\mathbf{s}(t)$ $\mathbf{s}(t)^{\mathrm{H}}$] and $\mathbf{I}_N$ is the NxN identity matrix. However, $\tilde{\mathbf{A}}(\underline{\theta}_0,\mathbf{e})$ must be full rank. The conditional ML algorithm [2] assumed that the signal of the input sources are deterministic and the noise $\mathbf{n}$(t) is Gaussian with E[$\mathbf{n}(t)$ $\mathbf{n}(t)^{\mathrm{H}}$]=$\sigma^2$ $\mathbf{I}_N$. Thus the unknown parameters are $\underline{\theta}_0$ and σ^2. The associated likelihood criterion leads up to the minimization of the criterion $c_{ML}(\underline{\theta}, \mathbf{e})$ with $\underline{\theta}$:

$$c_{ML}(\underline{\theta}, \mathbf{e}) = trace(\Pi(\underline{\theta})\, \mathbf{R}_x(\mathbf{e})), \tag{5}$$

where $\underline{\theta}$=[$\theta_1 \dots \theta_M$] , $trace(\mathbf{A})$ is the trace of the matrix $\mathbf{A}$ and $\Pi(\underline{\theta})$ is the projector such that :

$$\Pi(\underline{\theta}) = \mathbf{I}_N - \mathbf{A}(\underline{\theta})\, \mathbf{A}(\underline{\theta})^{\#}, \tag{6}$$

where $\mathbf{A}(\underline{\theta})$=[$\mathbf{a}(\theta_1)\dots\mathbf{a}(\theta_M)$], $\mathbf{a}(\theta)$ is the array manifold used in the algorithm and $^{\#}$ defines the Moore-Penrose pseudo-inverse such as: $\mathbf{A}^{\#}\mathbf{A}=\mathbf{I}_M$. As $\Pi(\underline{\theta}_0)$ $\mathbf{A}(\underline{\theta}_0)=\mathbf{0}$ due to the orthogonal property of $\Pi(\underline{\theta})$, according to (4) the ML algorithm criterion checks :

$$c_{ML}(\underline{\theta},\mathbf{e=0}) \geq c_{ML}(\underline{\theta}_0,\mathbf{e=0})= \sigma^2(N\text{-}M), \tag{7}$$

where in this case $\mathbf{a}_m=\tilde{\mathbf{a}}_m$ for ($1\leq m\leq M$). When $\mathbf{a}_m\neq\tilde{\mathbf{a}}_m$, the criterion becomes $c_{ML}(\underline{\theta},\mathbf{e})> \sigma^2(N\text{-}M)$ and presents a local minimum in the direction $\dot{\underline{\theta}}_0=[\dot{\theta}_1 \dots \dot{\theta}_M]$ different from $\underline{\theta}_0$. The random variable $\Delta\theta_m=\dot{\theta}_m-\theta_m$ defines the DOA estimation error of the direction of the m^{th} source; one seeks to evaluate its bias E[$\Delta\theta_m$] and its RMS error noted RMS_m : RMS_m^2=E[$\Delta\theta_m^2$]. The purpose of this work is to provide the moments of the random variable $\Delta\theta_m=\dot{\theta}_m-\theta_m$ up to the second order in ||$\mathbf{e}$|| instead of first order as in [4].

3. LINK BETWEEN MODEL ERRORS AND DOA ESTIMATION ERRORS

The purpose of this section is to give a relation between the random variables $\Delta\theta_m$ and the modeling error vector $\mathbf{e}=[\mathbf{e}_1^{\mathrm{T}}\dots\mathbf{e}_M^{\mathrm{T}}]^{\mathrm{T}}$. According to (4) and (5) the ML criterion in $\underline{\theta}=\underline{\theta}_0$ is a function of $\mathbf{e}$:

$$c_{ML}^{opt}(\mathbf{e}) = trace(\mathbf{\Pi}_0\, \mathbf{R}_x(\mathbf{e})\,), \tag{8}$$

where $\mathbf{\Pi}_0= \mathbf{I}_N - \mathbf{A}_0\, \mathbf{A}_0^{\#}$. After a second order Taylor expansion in $\underline{\theta}$ of $c_{ML}(\underline{\theta},\mathbf{e})$ around $\underline{\theta}=\underline{\theta}_0$, the expression of the DOA estimation error $\Delta\underline{\theta}=[\Delta\theta_1 \dots\Delta\theta_M]^{\mathrm{T}}$ becomes according to [2]:

$$\Delta\underline{\theta} = -\mathbf{H}(\mathbf{e})^{-1}\, \nabla_0(\mathbf{e}), \tag{9}$$

where the gradient vector $\nabla(\mathbf{e})$ checks:

$$\nabla_0(\mathbf{e}) = -2\Re(diag(\mathbf{A}_0^{\#}\, \mathbf{R}_x(\mathbf{e})\, \mathbf{\Pi}_0\, \dot{\mathbf{A}}_0)), \tag{10}$$

and where $\Re\{\ \}$ indicates the real part , $\dot{\mathbf{A}}_0=[\dot{\mathbf{a}}_1 \cdots \dot{\mathbf{a}}_M]$, $\dot{\mathbf{a}}_m$ is the first derivative of the steering vector $\mathbf{a}(\theta)$ at θ_m and $diag(\mathbf{A})$ is a vector composed by the diagonal elements of the matrix $\mathbf{A}$. The Hessien matrix $\mathbf{H}(\mathbf{e})$ checks :

$$\mathbf{H}(\mathbf{e}) = 2\Re((\ \dot{\mathbf{A}}_0^{\mathrm{H}}\, \mathbf{\Pi}_0\, \dot{\mathbf{A}}_0) \odot (\ \mathbf{A}_0^{\#}\, \mathbf{R}_x(\mathbf{e})\, (\mathbf{A}_0^{\#})^{\mathrm{H}})^{\mathrm{T}}), \tag{11}$$

where $\odot$ is the Hadamard product. Using the previous assumption, we now show that the elements of the vector $\Delta\underline{\theta}$ can be provided in hermitean forms of vector $\boldsymbol{\varepsilon}=[\boldsymbol{\mu}^{\mathrm{T}}\ \mathbf{e}^{\mathrm{H}}]^{\mathrm{T}}$ with $\boldsymbol{\mu}=[1\ \mathbf{e}^{\mathrm{T}}]^{\mathrm{T}}$. Using (3)(4) and (10) the gradient vector becomes:

$$\nabla_0(\mathbf{e}) = -2\Re(diag(\mathbf{P}\, \mathbf{E}^{\mathrm{H}}\, \mathbf{\Pi}_0\, \dot{\mathbf{A}}_0)) - 2\Re(diag(\mathbf{A}_0^{\#}\, \mathbf{E}\, \mathbf{P}\, \mathbf{E}^{\mathrm{H}}\, \mathbf{\Pi}_0\, \dot{\mathbf{A}}_0)). \tag{12}$$

Using expressions (3)(4) and (11) the hessien matrix $\mathbf{H}(\mathbf{e})$ can be rewritten as following:

$$\mathbf{H}(\mathbf{e}) = \mathbf{H_0} + \mathbf{dH}(\mathbf{e})$$

$$\text{with } \mathbf{H_0} = 2\Re((\ \dot{\mathbf{A}}_0^{\mathrm{H}}\, \mathbf{\Pi}_0\, \dot{\mathbf{A}}_0) \odot (\mathbf{P}+\sigma^2(\mathbf{A}_0^{\mathrm{H}}\, \mathbf{A}_0)^{-1})^{\mathrm{T}}),$$

$$\text{and } \mathbf{dH}(\mathbf{e}) = 2\Re((\ \dot{\mathbf{A}}_0^{\mathrm{H}}\, \mathbf{\Pi}_0\, \dot{\mathbf{A}}_0) \odot (\ \mathbf{U}(\mathbf{e})\, \mathbf{P}\, \mathbf{U}(\mathbf{e})^{\mathrm{H}} - \mathbf{P})^{\mathrm{T}})$$

$$\text{and } \mathbf{U}(\mathbf{e}) = \mathbf{I}_M + \mathbf{A}_0^{\#}\, \mathbf{E}. \tag{13}$$

It is shown in appendix A that the second order Taylor expansion of expression (9) around $\mathbf{e}$=0 verifies:

$$\Delta\underline{\theta} = -\,\mathbf{H_0}^{-1}\, \nabla_0(\mathbf{e}) + \mathbf{H_0}^{-1}\, \Delta\mathbf{H}(\mathbf{e})\, \mathbf{H_0}^{-1}\, \nabla_1(\mathbf{e}) + o(\|\mathbf{e}\|^2) \tag{14}$$

where :

$$\Delta\mathbf{H}(\mathbf{e}) = 4\Re((\ \dot{\mathbf{A}}_0^{\mathrm{H}}\, \mathbf{\Pi}_0\, \dot{\mathbf{A}}_0) \odot (\ \mathbf{A}_0^{\#}\, \mathbf{E}\, \mathbf{P})^{\mathrm{T}}) \tag{15}$$

and:

$$\nabla_1(\mathbf{e}) = -2\Re(diag(\mathbf{P}\, \mathbf{E}^{\mathrm{H}}\, \mathbf{\Pi}_0\, \dot{\mathbf{A}}_0)) \tag{16}$$

In reference [4] the authors used the following first order Taylor expansion of $\Delta\underline{\theta}$:

$$\Delta\underline{\theta}_{\mathrm{F}} = -\,\mathbf{H_0}^{-1}\, \nabla_1(\mathbf{e}) + o(\|\mathbf{e}\|) \tag{17}$$

In order to transform expression (14) of $\Delta\underline{\theta}$ in hermitean forms, the following relations are used:

$$\mathbf{u}^{\mathrm{H}}\, \mathbf{E}^{\mathrm{H}}\, \mathbf{v} = \mathbf{e}^{\mathrm{H}}\, (\mathbf{u}^*\otimes\mathbf{v}) = \boldsymbol{\mu}^{\mathrm{H}}\, \mathbf{Q}_1(\mathbf{u},\mathbf{v})\, \boldsymbol{\mu}$$

$$\text{with } \mathbf{Q}_1(\mathbf{u},\mathbf{v}) = \begin{bmatrix} 0 & 0 \\ \mathbf{u}^*\otimes\mathbf{v} & \mathbf{0} \end{bmatrix} \text{ and } \mathbf{q}_1(\mathbf{u},\mathbf{v}) = \mathbf{u}^*\otimes\mathbf{v} \tag{18}$$

with $\boldsymbol{\mu}=[1\ \mathbf{e}^{\mathrm{T}}]^{\mathrm{T}}$ and:

$$\mathbf{u}^{\mathrm{H}}\, \mathbf{E}\, \mathbf{P}\, \mathbf{E}^{\mathrm{H}}\, \mathbf{v} = \boldsymbol{\mu}^{\mathrm{H}}\, \mathbf{Q}_2(\mathbf{u},\mathbf{v})\boldsymbol{\mu}$$

$$\text{where } \mathbf{Q}_2(\mathbf{u},\mathbf{v}) = \begin{bmatrix} 0 & \mathbf{0}^{\mathrm{H}} \\ \mathbf{0} & \mathbf{P}^{\mathrm{T}}\otimes(\mathbf{v}\,\mathbf{u}^{\mathrm{H}}) \end{bmatrix} \tag{19}$$

Let us note $(\mathbf{A}_0^{\#})^{\mathrm{H}}=[\mathbf{g}_1\dots\ \mathbf{g}_M]$ and $\mathbf{P}=[\mathbf{p}_1\dots\ \mathbf{p}_M]$; the m^{th} component of $\nabla_0(\mathbf{e})$ becomes according to (12)(18) and (19):

$$\nabla_0(\mathbf{e})_m = \boldsymbol{\mu}^{\mathrm{H}}\, \mathbf{Q}_{\nabla m}\, \boldsymbol{\mu} \text{ with } \mathbf{Q}_{\nabla m} = -\mathbf{Q}_{\nabla m0} - \mathbf{Q}_{\nabla m0}^{\mathrm{H}}$$

$$\text{and } \mathbf{Q}_{\nabla m0} = \mathbf{Q}_1(\mathbf{g}_m, \mathbf{\Pi}_0\dot{\mathbf{a}}_m) + \mathbf{Q}_2(\mathbf{p}_m, \mathbf{\Pi}_0\dot{\mathbf{a}}_m) \tag{20}$$

Thus the m^{th} component $\Delta\underline{\theta}_m^{\ 1}$ of $\Delta\underline{\theta}^{1}$= $-\,\mathbf{H_0}^{-1}\nabla_0(\mathbf{e})$ becomes:

$$\Delta\underline{\theta}_m^{\ 1} = \boldsymbol{\mu}^{\mathrm{H}}\, \mathbf{Q}_m^{\ 1}\, \boldsymbol{\mu} \text{ with } \mathbf{Q}_m^{\ 1} = -\sum_{i=1}^{M} \mathbf{H}_0^{-1}(m,i)\ \mathbf{Q}_{\nabla i} \tag{21}$$

where $\mathbf{H_0}^{-1}(i,j)$ is the ij^{th} element of $\mathbf{H_0}^{-1}$. The m^{th} component $\mathbf{b}_m$ of $\mathbf{b}=\mathbf{H_0}^{-1}\ \nabla_1(\mathbf{e})$ becomes according to (16) and (18):

$$\mathbf{b}_m = -2\Re(\ \mathbf{e}^{\mathrm{H}}\, \mathbf{q}_m)$$

$$\text{with } \mathbf{q}_m = \sum_{i=1}^{M} \mathbf{H}_0^{-1}(m,i)\ (\mathbf{p}_i^*\otimes(\mathbf{\Pi}_0\, \dot{\mathbf{a}}_i)) \tag{22}$$

The m^{th} row and i^{th} column of $\mathbf{B}=4\mathbf{H_0}^{-1}\Re((\mathbf{A}_0^{\mathrm{H}}\, \mathbf{\Pi}_0\, \mathbf{A}_0)\odot(\mathbf{A}_0^{\#}\, \mathbf{E}\, \mathbf{P})^{\mathrm{T}})$ then becomes according to (15) and (18):

$$\mathbf{B}_{mi} = 4\Re(\mathbf{q}_{mi}^{\ \mathrm{H}}\, \mathbf{e}),$$

$$\text{with } \mathbf{q}_{mi} = \sum_{k=1}^{M} \mathbf{H}_0^{-1}(m,k)\ (\mathbf{p}_k^*\otimes \mathbf{g}_i)\, (\dot{\mathbf{a}}_k^{\mathrm{H}}\, \mathbf{\Pi}_0\, \dot{\mathbf{a}}_i)^*. \tag{23}$$

Knowing that 4 $\Re(a)\,\Re(b)$= 2 $\Re(a\,b)$ + 2 $\Re(a\,b^*)$, one then has $\mathbf{B}_{mi}\, \mathbf{b}_i = -4\,(\Re(\mathbf{e}^{\mathrm{H}}\, \mathbf{q}_i\, \mathbf{q}_{mi}^{\ \mathrm{H}}\, \mathbf{e}) + \Re(\ \mathbf{e}^{\mathrm{H}}\, \mathbf{q}_i\, \mathbf{q}_{mi}^{\ \mathrm{T}}\, \mathbf{e}^*))$ and the

m^{th} component $\Delta\underline{\theta}_m^2$ of $\Delta\underline{\theta}^2 = \mathbf{B}\,\mathbf{b} = \mathbf{H_0}^{-1}\,\Delta\mathbf{H}(\mathbf{e})\,\mathbf{H_0}^{-1}\,\nabla_1(\mathbf{e})$ is:

$$\Delta\underline{\theta}_m^2 = \boldsymbol{\varepsilon}^{\mathrm{H}}\,\mathbf{Q}_m^2\,\boldsymbol{\varepsilon} \quad \text{with} \quad \boldsymbol{\varepsilon}=[1\ \mathbf{e}^{\mathrm{T}}\ \mathbf{e}^{\mathrm{H}}]^{\mathrm{T}} \tag{24}$$

and:

$$\mathbf{Q}_m^{\ 2} = -2\sum_{i=1}^{M}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{q}_i\mathbf{q}_{mi}^{\ \mathrm{H}} & \mathbf{q}_i\mathbf{q}_{mi}^{\ \mathrm{T}} \\ 0 & \mathbf{q}_{mi}^{\ *}\mathbf{q}_i^{\ \mathrm{H}} & \mathbf{q}_i^{\ *}\mathbf{q}_{mi}^{\ \mathrm{T}} \end{bmatrix} \tag{25}$$

According to (14) (21) and (24) we finally obtain that $\Delta\underline{\theta} = \Delta\underline{\theta}^1 + \Delta\underline{\theta}^2$ and that the m^{th} component $\Delta\underline{\theta}_m$ of $\Delta\underline{\theta}$ verifies:

$$\Delta\underline{\theta}_m = \boldsymbol{\varepsilon}^{\mathrm{H}}\,\mathbf{Q}_m\,\boldsymbol{\varepsilon} \quad \text{with} \quad \mathbf{Q}_m = \begin{bmatrix} \mathbf{Q}_m^{\ 1} & 0 \\ 0 & \mathbf{0} \end{bmatrix} + \mathbf{Q}_m^{\ 2} \tag{26}$$

with $\boldsymbol{\varepsilon}=[1\ \mathbf{e}^{\mathrm{T}}\ \mathbf{e}^{\mathrm{H}}]^{\mathrm{T}}$. This last expression is the key equation of this paper.

4. BIAS AND RMS ERROR OF ML

The purpose of this section is to provide the moments of an hermitean form $\Delta\theta_m = \underline{\boldsymbol{\varepsilon}}^{\mathrm{H}}\,\mathbf{Q}_m\,\underline{\boldsymbol{\varepsilon}}$ in order to derive the bias and the RMS error of the ML algorithm. Using that :

$$\mathrm{E}[\mathbf{x}^{\mathrm{H}}\,\mathbf{Q}\,\mathbf{x}] = \mathrm{E}[\,trace(\,\mathbf{Q}\,\mathbf{x}\,\mathbf{x}^{\mathrm{H}}\,)] = trace(\mathbf{Q}\,\mathrm{E}[\,\mathbf{x}\,\mathbf{x}^{\mathrm{H}}]), \tag{27}$$

the bias of the m^{th} source is given by the following expression according to (26):

$$\mathrm{E}[\Delta\theta_m] = trace(\mathbf{Q}_m\,\mathbf{R}_{\boldsymbol{\varepsilon}}) \quad \text{with} \quad \mathbf{R}_{\boldsymbol{\varepsilon}} = \mathrm{E}[\,\boldsymbol{\varepsilon}\,\boldsymbol{\varepsilon}^{\mathrm{H}}] \tag{28}$$

Using the equality $(AB)\otimes(CD)=(A\otimes C)(B\otimes D)$:

$$\begin{aligned}(\mathbf{x}^{\mathrm{H}}\,\mathbf{Q}\,\mathbf{x}\,)^2 &= (\mathbf{x}^{\mathrm{H}}\,\mathbf{Q}\,\mathbf{x}\,)\otimes(\mathbf{x}^{\mathrm{H}}\,\mathbf{Q}\,\mathbf{x}\,) \\ &= (\mathbf{x}^{\mathrm{H}}\otimes\mathbf{x}^{\mathrm{H}})((\,\mathbf{Q}\,\mathbf{x})\otimes(\,\mathbf{Q}\,\mathbf{x})) \\ &= (\mathbf{x}^{\mathrm{H}}\otimes\mathbf{x}^{\mathrm{H}})((\,\mathbf{Q}\otimes\mathbf{Q})\,(\mathbf{x}\otimes\mathbf{x})) = (\mathbf{x}^{\otimes 2})^{\mathrm{H}}\,\mathbf{Q}^{\otimes 2}\,\mathbf{x}^{\otimes 2}\end{aligned} \tag{29}$$

where $\mathbf{u}^{\otimes 2} = \mathbf{u}\otimes\mathbf{u}$. According to (26) (27) and (29), the RMS error of the m^{th} source is thus given by the following expression:

$$\mathrm{RMS}_m = \sqrt{trace(\mathbf{Q}_m\,\mathbf{R}_\varepsilon^4)} \quad \text{with} \quad \mathbf{R}_{\boldsymbol{\varepsilon}}^4 = \mathrm{E}[\,\boldsymbol{\varepsilon}^{\otimes 2}\,\boldsymbol{\varepsilon}^{\otimes 2\mathrm{H}}] \tag{30}$$

According to [6], in the gaussian case, $\mathbf{R}_{\boldsymbol{\varepsilon}}^4$ is given as a function of $\mathbf{R}_{\boldsymbol{\varepsilon}}$ and $\mathbf{C}_{\boldsymbol{\varepsilon}} = \mathrm{E}[\,\boldsymbol{\varepsilon}\,\boldsymbol{\varepsilon}^{\mathrm{T}}]$:

$$\mathrm{R}_\varepsilon^4 = \begin{bmatrix} \mathbf{R}_{1,1} & \cdots & \mathbf{R}_{1,1+2NM} \\ \vdots & \ddots & \vdots \\ \mathbf{R}_{1+2NM,1} & \cdots & \mathbf{R}_{1+2NM,1+2NM} \end{bmatrix} \tag{31}$$

with:

$$\mathrm{R}_{i,j} = \mathrm{R}_\varepsilon\,\mathrm{R}_\varepsilon(i,j) + \mathbf{r}_\varepsilon(j)\,\mathbf{r}_\varepsilon(i)^{\mathrm{H}} + \mathbf{c}_\varepsilon(i)\,\mathbf{c}_\varepsilon(j)^{\mathrm{H}} - 2\,\boldsymbol{\delta}_i\boldsymbol{\delta}_j\,\mathbf{1}\mathbf{1}^{\mathrm{H}} \tag{32}$$

where $\mathbf{1}=[1\ \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$, $\mathbf{C}_\varepsilon=[\ \mathbf{c}_\varepsilon(1)\ldots\ \mathbf{c}_\varepsilon(2NM+1)]$, $\mathbf{R}_\varepsilon=[\ \mathbf{r}_\varepsilon(1)\ldots\ \mathbf{r}_\varepsilon(2NM+1)]$, $\mathbf{C}_\varepsilon(i,j)$ and $\mathbf{R}_\varepsilon(i,j)$ are the i^{th} line and j^{th} column of the respective matrices $\mathbf{C}_\varepsilon$ and $\mathbf{R}_\varepsilon$ and $\boldsymbol{\delta}_i = 1$ for $i=1$ and 0 in the other cases.

5. SIMULATIONS

In the case of $M=2$ sources of DOAs $\theta_l=100$ degree and variable θ_2, figure 1 and figure 2 compare the estimated performances of the first source (RMS error) by simulation on 1000 achievements, with the theoretical performances found in this paper and those of reference [4]. The array of $N=5$ sensors is circular with $R/\lambda=1$. There is no additive noise ($\sigma^2=0$) on the observations. The elements of the sources covariance matrix $\mathbf{P}$ check: $\mathbf{P}(1,1)=\mathbf{P}(2,2)=1$ and $\mathbf{P}(1,2)=\mathbf{P}(2,1)=0.8$. The steering vector modeling error is Gaussian and circular with $\mathrm{E}[\mathbf{e}\,\mathbf{e}^{\mathrm{H}}] = \sigma_e^2\,\mathrm{I}_{NM}$. More precisely in figure 1, the comparison is carried out as a function of the modeling error level $\boldsymbol{\sigma}_e$ when $\theta_2=86°$ where as it is made out as a function of $|\theta_2-\theta_1|$ with $\boldsymbol{\sigma}_e=0.26$ in figure 2.

The simulations show that the proposed expression of the RMS error of this paper at the 2nd order is in adequation with the simulation results and that the first order is not representative for $\sigma_e>0.15$ in figure-1 and for $|\theta_2-\theta_1|<20°$ in figure-2.

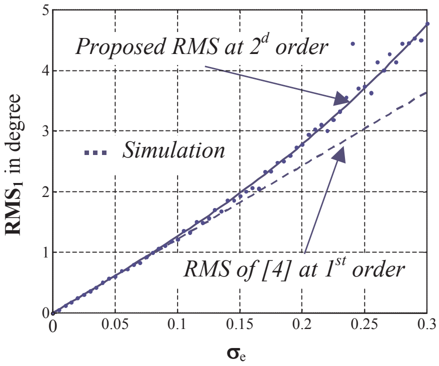

Figure 1: RMS_1 according to the level of the Gaussian modeling error : $\theta_2-\theta_1=14°$.

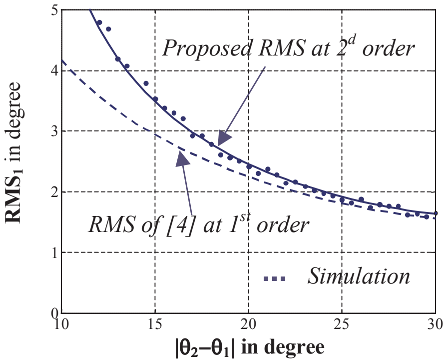

Figure 2: RMS_1 according to $|\theta_2-\theta_1|$: $\sigma_e=0.26$.

6. CONCLUSIONS

This paper has proposed a second order Taylor expansion of the analytic expression of the bias and RMS error of the DOA estimation of the ML algorithm. The DOA estimation errors have been written as hermitean forms with a stochastic vector composed by the steering vectors errors. This hermitean form allowed us to easily deduce the bias and the RMS error of the ML algorithm. This new closed form expression of performances is representative of the simulation results for larger modeling errors than in reference [4] and provides a tool for material designing.

APPENDIX-A

The purpose of this appendix is to demonstrate the expression (14) of $\Delta\underline{\theta}=f(\mathbf{e})$. The second order Taylor expansion of $f(\mathbf{e}) = -\mathbf{H}(\mathbf{e})^{-1}\,\nabla_0(\mathbf{e})$ checks around $\mathbf{e}$=0:

$$\Delta\underline{\theta} = f(\mathbf{e}) = f(\mathbf{e=0}) + df(\mathbf{e=0}) + \tfrac{1}{2}\, d^2 f(\mathbf{e=0}) + o(\|\mathbf{e}\|^2) \quad (33)$$

where $df(\mathbf{e})$ and $d^2f(\mathbf{e})$ are the first and second derivative of $f(\mathbf{e})$. According to (9) the expression of $df(\mathbf{e})$ is as following:

$$df(\mathbf{e}) = -d(\mathbf{H}(\mathbf{e})^{-1})\,\nabla_0(\mathbf{e}) - \mathbf{H}(\mathbf{e})^{-1}\, d(\nabla_0(\mathbf{e})) \quad (34)$$

where $d(\mathbf{M})$ is the first derivative of $\mathbf{M}$. Knowing that the first derivative $d(\mathbf{M}^{-1}) = -\mathbf{M}^{-1}\, d(\mathbf{M})\, \mathbf{M}^{-1}$, $df(\mathbf{e})$ becomes:

$$df(\mathbf{e}) = \mathbf{H}(\mathbf{e})^{-1}\, d(\mathbf{H}(\mathbf{e}))\, \mathbf{H}(\mathbf{e})^{-1}\nabla_0(\mathbf{e}) - \mathbf{H}(\mathbf{e})^{-1}\, d(\nabla_0(\mathbf{e})) \quad (35)$$

Using that according to (13) $\mathbf{H}(\mathbf{e=0})=\mathbf{H_0}$ and that $\nabla_0(\mathbf{e=0})=0$ according to (12), $df(\mathbf{e=0})$ can be rewritten as following :

$$df(\mathbf{e=0}) = -\mathbf{H_0}^{-1}\, d(\nabla_0(\mathbf{e}))_{\mathbf{e}=0} \quad (36)$$

where $d(\nabla_0(\mathbf{e}))_{\mathbf{e}=0}$ is the first differential of $\nabla_0(\mathbf{e})$ around $\mathbf{e}$=0. Appendix-B shows that :

$$d(\nabla_0(\mathbf{e}))_{\mathbf{e}=0} = \nabla_1(\mathbf{e}) = -2\Re(diag(\mathbf{P}\,\mathbf{E}^H\,\mathbf{\Pi}_0\,\dot{\mathbf{A}}_0)) \quad (37)$$

According to (35) and using that $d(\mathbf{M}^{-1}) = -\mathbf{M}^{-1}\, d(\mathbf{M})\, \mathbf{M}^{-1}$, the expression of $d^2f(\mathbf{e})$ is as following:

$$\begin{aligned} d^2f(\mathbf{e}) = & -\mathbf{H}(\mathbf{e})^{-1}\, d(\mathbf{H}(\mathbf{e}))\, \mathbf{H}(\mathbf{e})^{-1}\, d(\mathbf{H}(\mathbf{e}))\, \mathbf{H}(\mathbf{e})^{-1}\nabla_0(\mathbf{e}) \\ & + \mathbf{H}(\mathbf{e})^{-1}\, d^2(\mathbf{H}(\mathbf{e}))\, \mathbf{H}(\mathbf{e})^{-1}\,\nabla_0(\mathbf{e}) \\ & - \mathbf{H}(\mathbf{e})^{-1}\, d(\mathbf{H}(\mathbf{e}))\, \mathbf{H}(\mathbf{e})^{-1}\, d(\mathbf{H}(\mathbf{e}))\, \mathbf{H}(\mathbf{e})^{-1}\,\nabla_0(\mathbf{e}) \\ & + \mathbf{H}(\mathbf{e})^{-1}\, d(\mathbf{H}(\mathbf{e}))\, \mathbf{H}(\mathbf{e})^{-1}\, d(\nabla_0(\mathbf{e})) \\ & + \mathbf{H}(\mathbf{e})^{-1}\, d(\mathbf{H}(\mathbf{e}))\, \mathbf{H}(\mathbf{e})^{-1}\, d(\nabla_0(\mathbf{e})) \\ & - \mathbf{H}(\mathbf{e})^{-1}\, d^2(\nabla_0(\mathbf{e})) \end{aligned} \quad (38)$$

where $d^2(\mathbf{M})$ is the second derivative of $\mathbf{M}$. Using that $\mathbf{H}(\mathbf{e=0})=\mathbf{H_0}$ according to (13) and that $\nabla_0(\mathbf{e=0})=0$ according to (12), $d^2f(\mathbf{e=0})$ can be rewritten as following :

$$\begin{aligned} d^2f(\mathbf{e=0}) = & \; 2\,\mathbf{H_0}^{-1}\, d(\mathbf{H}(\mathbf{e}))_{\mathbf{e}=0}\, \mathbf{H_0}^{-1}\, d(\nabla_0(\mathbf{e}))_{\mathbf{e}=0} \\ & - \mathbf{H_0}^{-1}\, d^2(\nabla_0(\mathbf{e}))_{\mathbf{e}=0} \end{aligned} \quad (39)$$

where $d^2(\nabla_0(\mathbf{e}))_{\mathbf{e}=0}$ is the second differential of $\nabla_0(\mathbf{e})$ around $\mathbf{e}$=0. Appendix-C shows that :

$$d^2(\nabla_0(\mathbf{e}))_{\mathbf{e}=0} = -4\Re(diag(\mathbf{A_0}^{\#}\,\mathbf{E}\,\mathbf{P}\,\mathbf{E}^H\,\mathbf{\Pi}_0\,\dot{\mathbf{A}}_0)). \quad (40)$$

Appendix-D shows that the first differential $d(\mathbf{H}(\mathbf{e}))_{\mathbf{e}=0}$ of $\mathbf{H}(\mathbf{e})$ around $\mathbf{e}$=0 in equation (39) is as following:

$$d(\mathbf{H}(\mathbf{e}))_{\mathbf{e}=0} = \Delta\mathbf{H}(\mathbf{e}) = 4\Re((\dot{\mathbf{A}}_0^{\,H}\,\mathbf{\Pi}_0\,\dot{\mathbf{A}}_0) \odot (\mathbf{A_0}^{\#}\,\mathbf{E}\,\mathbf{P})^T) \quad (41)$$

Noticing that $\nabla_0(\mathbf{e}) = d(\nabla_0(\mathbf{e}))_{\mathbf{e}=0} + d^2(\nabla_0(\mathbf{e}))_{\mathbf{e}=0}$ and using (36)(37) (39) and (41), the expression (33) becomes:

$$f(\mathbf{e}) = -\mathbf{H_0}^{-1}\,\nabla_0(\mathbf{e}) + \mathbf{H_0}^{-1}\,\Delta\mathbf{H}(\mathbf{e})\,\mathbf{H_0}^{-1}\,\nabla_0(\mathbf{e}) + o(\|\mathbf{e}\|^2) \quad (42)$$

and equation (14) is proven.

APPENDIX-B

The purpose of this appendix is to show that relation (37) is verified around $\mathbf{e}$=0. Using expression (12), $d(\nabla_0(\mathbf{e}))$ becomes:

$$\begin{aligned} d(\nabla_0(\mathbf{e})) = & -2\Re(diag(\mathbf{P}\, d(\mathbf{E})^H\,\mathbf{\Pi}_0\,\dot{\mathbf{A}}_0)) \\ & - 2\Re(diag(\mathbf{A_0}^{\#}\, d(\mathbf{E})\,\mathbf{P}\,\mathbf{E}^H\,\mathbf{\Pi}_0\,\dot{\mathbf{A}}_0)) \\ & - 2\Re(diag(\mathbf{A_0}^{\#}\,\mathbf{E}\,\mathbf{P}\, d(\mathbf{E})^H\,\mathbf{\Pi}_0\,\dot{\mathbf{A}}_0)) \end{aligned} \quad (43)$$

According to (43), the expression (37) of $d(\nabla_0(\mathbf{e}))_{\mathbf{e}=0}$ is proven because around $\mathbf{e}$=0, $\mathbf{E}_0=\tilde{\mathbf{A}}(\underline{\theta}_0,\mathbf{e=0})-\mathbf{A}_0=\mathbf{0}$ and $d(\mathbf{E})=\mathbf{E}-\mathbf{E}_0=\mathbf{E}$.

APPENDIX-C

The purpose of this appendix is to show that relation (40) is verified around $\mathbf{e}$=0. According to equation (43), $d^2(\nabla_0(\mathbf{e}))$ is:

$$d^2(\nabla_0(\mathbf{e})) = -4\Re(diag(\mathbf{A_0}^{\#}\, d(\mathbf{E})\,\mathbf{P}\, d(\mathbf{E})^H\,\mathbf{\Pi}_0\,\dot{\mathbf{A}}_0)) \quad (44)$$

According to (44), the expression (40) of $d^2(\nabla_0(\mathbf{e}))_{\mathbf{e}=0}$ is proven because around $\mathbf{e}$=0, $\mathbf{E}_0=\tilde{\mathbf{A}}(\underline{\theta}_0,\mathbf{e=0})-\mathbf{A}_0=\mathbf{0}$ and $d(\mathbf{E})=\mathbf{E}-\mathbf{E}_0=\mathbf{E}$.

APPENDIX-D

The purpose of this appendix is to show that relation (41) is verified around $\mathbf{e}$=0. Using expression (13), $d(\mathbf{H}(\mathbf{e}))$ becomes:

$$d(\mathbf{H}(\mathbf{e})) = 4\Re((\dot{\mathbf{A}}_0^{\,H}\,\mathbf{\Pi}_0\,\dot{\mathbf{A}}_0) \odot (\mathbf{dU}(\mathbf{e})\,\mathbf{P}\,\mathbf{U}(\mathbf{e})^H)^T) \quad (45)$$

where $\mathbf{dU}(\mathbf{e}) = \mathbf{A_0}^{\#} d(\mathbf{E})$ and $\mathbf{U}(\mathbf{e}) = \mathbf{I}_M + \mathbf{A_0}^{\#}\,\mathbf{E}$. Equation (41) of $d(\mathbf{H}(\mathbf{e}))_{\mathbf{e}=0}$ is then proven because around $\mathbf{e}$=0, $\mathbf{E}_0=\tilde{\mathbf{A}}(\underline{\theta}_0,\mathbf{e=0})-\mathbf{A}_0=\mathbf{0}$, $d(\mathbf{E})=\mathbf{E}-\mathbf{E}_0=\mathbf{E}$ and $\mathbf{U}(\mathbf{e=0})=\mathbf{I}_M$.

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