ON PERFECT RECONSTRUCTION WITH LOST CHANNEL DATA IN LAPPED PSEUDO-ORTHOGONAL TRANSFORM

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ABSTRACT

We address a problem to reconstruct the original signal with lost data by using FIR synthesis filters in a class of linear-phase perfect reconstruction (PR) oversampled filter banks (FB) called lapped pseudo-orthogonal transform (LPOT), which belongs to a class of overcomplete linear-phase paraunitary FB's, in which the synthesis filters are given as the paraconjugate of the analysis filters. When some subband coefficients are lost, the use of the Moore-Penrose pseudo-inverse of the analysis FB can achieve PR, but the corresponding synthesis filters can have IIR. We firstly discuss the possibility of PR with FIR filters of the same length as the analysis filters, and show the synthesis filters which can optimally suppress additive noise in some sense. We then show an example of the resulting synthesis filters. Finally, we suggest some open problems.

1. INTRODUCTION

Linear-phase perfect reconstruction filter banks (LPPRFB's) or lapped transforms (LT's) are powerful and extensively used tools for signal and image processing [1,2]. The analysis part of the LPPRFB consists of a bank of M linear-phase (LP) filters followed by N-fold downsamplers (or decimators). The synthesis part is composed of upsamplers followed by synthesis filters. The LPPRFB guarantees perfect reconstruction (PR), or that the output of the synthesis part is the same as the original input signal. The LT is equivalent to the LPPRFB. The only difference lies in that the analysis and the synthesis filters are called the basis functions in the context of LT's.

For design and implementation of LPPRFB's, several kinds of lattice factorization/parameterization have been investigated [1–6]. A filter bank (FB) in which the number of channels is equal to the decimation factor is referred to as *maximally decimated* or *critically sampled*. On the other hand, a FB with a smaller decimation factor M than the number of channels N, i.e. $N > M$, is called *oversampled*. The oversampled FB's have some advantages such as their improved design freedom and noise immunity [7]. Several efficient lattice structures for oversampled LPPRFB's have been proposed [3,5,6,8]. Recently, Liang *et al* [8] suggested a possibility for the application to the OFDM as well as a reduced lattice structure.

In this reduced structure [8], the oversampled LPPRFB is obtained by applying left-invertible matrices to any critically sampled LPPRFB. Suppose that some channel data are lost as illustrated in Fig. 1. Then, if the number of lost channel is less than or equal to $N-M$, the original data can be recovered by applying the associated left-inverses. This is a fundamental fact of linear algebra. However, for an oversampled LPPRFB which cannot be represented by this structure based on the postprocessing, the perfect reconstruction problem with lost channel data is not trivial and has not been solved yet. If we permit the corresponding synthesis filters to have infinite impulse responses (IIR), PR is achieved by using the so-called *para-pseudoinverse* [7] as the synthesis part. From a practical point of view, however, this is ineffective, because infinite iteration is needed for implementation of the para-pseudoinverse; therefore, a method to achieve perfect reconstruction by the synthesis filters with finite impulse responses (FIR) is desirable. In the literature, no results of this problem has been reported. In this paper, therefore, we address the reconstruction problem in an oversampled LPPRFB where $K = 2$, that is, we consider the case where the length of the analysis filters is twice the decimation factor M and that of the synthesis filters to be found is also $2M$. We clarify how many channels can be lost for PR, and derive the synthesis filters, which are not uniquely determined in many cases. Therefore, we moreover find a special solution which consider the presence of noise.

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1.1 Notation

The following conventions are adopted in terms of notation: Boldfaced characters are used to denote vectors and matrices. $\|\boldsymbol{f}\|$, $\boldsymbol{I}_n$, $\boldsymbol{J}_n$, and $\boldsymbol{0}$ denote the Euclidean norm of $\boldsymbol{f}$, the $n \times n$ identity matrix, the $n \times n$ reversal matrix, and the null matrix, respectively We sometimes omit the subscript of these matrices if the size is obvious. $\boldsymbol{A}^T$, $\mathrm{tr}[\boldsymbol{A}]$, $R(\boldsymbol{A})$, $N(\boldsymbol{A})$, $\mathrm{rank}(\boldsymbol{A})$ denote the transposition, the trace, the range, the null space, and the rank of $\boldsymbol{A}$, respectively. Let $\dim S$ be the dimension of a subspace S.

2. LAPPED PSEUDO-ORTHOGONAL TRANSFORM: A CLASS OF OVERSAMPLED LPPRFB'S

A simple way to construct an oversampled LPPRFB is to put a left-inverse matrix before an N-channel LPPRFB. Generally, uniform FB's are described by polyphase matrices [1]. Let $\boldsymbol{E}_g(z)$ be an $N \times N$ order-K polyphase matrix of the analysis part of an LPPRFB. An oversampled LPPRFB can be obtained as follows:

$$\boldsymbol{E}(z) = \boldsymbol{E}_g(z)\boldsymbol{\Phi}_0, \tag{1}$$

where $\boldsymbol{\Phi}_0$ is an appropriately chosen left-invertible matrix of size $N \times M$. (for details, see [3, 5, 6], for example). This represents N-channel oversampled LPPRFB in which the decimation factor is M and the length of all filters is KM. The corresponding synthesis polyphase $\boldsymbol{R}$ is described as $\boldsymbol{R}(z) = \boldsymbol{\Phi}_0^{-}\boldsymbol{R}_g(z)$, where $\boldsymbol{R}_g(z)$ is the synthesis polyphase matrix with respect to $\boldsymbol{E}_g(z)$ and $\boldsymbol{\Phi}_0^{-}$ is a left-inverse of $\boldsymbol{\Phi}_0$. This is very general structure which can represents a very wide range of oversampled LPPRFB [6]. An alternative reduced structure has been proposed by Liang *et al* [8], and is of the form

$$\boldsymbol{E}(z) = \boldsymbol{\Psi}_0\boldsymbol{E}_r(z), \tag{2}$$

where $\boldsymbol{E}_r(z)$ is an $M \times M$ polyphase matrix of order K, and $\boldsymbol{\Psi}_0$ is an appropriately chosen left-invertible matrix of size $N \times M$. It has been proven in [8] that this structure can be always converted to the form (1).

Assume that an oversampled LPPRFB is paraunitary (pseudo-orthogonal) and consists of the filters of length $2M$, i.e., $K = 2$ Then, the corresponding analysis polyphase matrix $\boldsymbol{E} = \boldsymbol{E}_0 + z^{-1}\boldsymbol{E}_1$ is specifically represented as

$$\boldsymbol{E}_1 = \begin{bmatrix} \boldsymbol{U}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_0 - \boldsymbol{V}_0 & (\boldsymbol{U}_0 + \boldsymbol{V}_0)\boldsymbol{J} \\ \boldsymbol{U}_0 - \boldsymbol{V}_0 & (\boldsymbol{U}_0 + \boldsymbol{V}_0)\boldsymbol{J} \end{bmatrix},$$

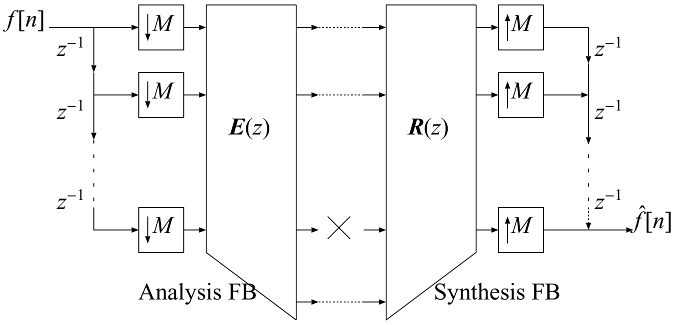

Figure 1: A polyphase representation for a pair of the analysis and the synthesis parts of an oversampled FB's: One channel is lost in this example.

$$\boldsymbol{E}_0 = \begin{bmatrix} \boldsymbol{U}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_0 + \boldsymbol{V}_0 & (\boldsymbol{U}_0 - \boldsymbol{V}_0)\boldsymbol{J} \\ -(\boldsymbol{U}_0 - \boldsymbol{V}_0) & -(\boldsymbol{U}_0 - \boldsymbol{V}_0)\boldsymbol{J} \end{bmatrix}, \quad (3)$$

where $\boldsymbol{U}_i$ and $\boldsymbol{V}_i$ $(i = 0, 1)$ are as follows:

General Structure $\boldsymbol{U}_0$ and $\boldsymbol{V}_0$ are $N/2 \times M/2$ column-orthogonal matrices (columns are orthonormal to each other). $\boldsymbol{U}_1 = \boldsymbol{I}_{N/2}$. $\boldsymbol{V}_1$ is an $N/2 \times N/2$ orthogonal matrix.

Reduced Structure $\boldsymbol{U}_0 = \boldsymbol{I}_{M/2}$. $\boldsymbol{V}_0$ is an $M/2 \times M/2$ orthogonal matrix. $\boldsymbol{U}_1$ and $\boldsymbol{V}_1$ are $N/2 \times M/2$ column-orthogonal matrices.

From the context of lapped transforms, we would like to call this transform *lapped pseudo-orthogonal transform* (LPOT), because this is a special case of the generalized lapped pseudo-biorthogonal transform developed by the authors [6]. In the rest of this paper, we deal only with the LPOT which can be represented by only the general structure, since the solution of the PR problem in the reduced structure is trivial.

A transformed vector is obtained as

$$\boldsymbol{g}_i = \boldsymbol{E}_1 \boldsymbol{f}_{i-1} + \boldsymbol{E}_0 \boldsymbol{f}_i. \quad (4)$$

By this transformation, the sequence $\{\boldsymbol{g}_i\}$ is generated from the sequence of input vectors $\{\boldsymbol{f}_i\}$ by $\boldsymbol{E}$.

3. PROBLEM FORMULATION

An oversampled representation enables us to recover the original signal, even though some transform coefficients are lost. To make the problem more specific, we introduce the index sequence for lost channels $\mathcal{L}_\beta = \{l_n\}_{n=0}^{\beta-1}$ such that $0 \le l_n \le N-1$ and $l_m < l_n$ if $m < n$ The corresponding complement sequence $\mathcal{L}_\beta^c$ can be also defined as $\mathcal{L}_\beta^c = \{l_n^c\}_{n=0}^{N-\beta-1} = \{0, \ldots, N-1\} - \mathcal{L}_\beta$. For example, when $N = 8$ and 0th and 4th channel data (or elements in $\boldsymbol{g}$) are lost, the index sets are written as $\mathcal{L}_2 = \{0, 4\}$ and $\mathcal{L}_2^c = \{1, 2, 3, 5, 6, 7\}$. We then define the "losing operator" $\boldsymbol{P}_{\mathcal{L}_\beta}$ of size $(N-\beta) \times N$ with respect to $\mathcal{L}_\beta$ as

$$\boldsymbol{P}_{\mathcal{L}_\beta} = [\boldsymbol{e}_{l_0^c}, \ldots, \boldsymbol{e}_{l_{N-\beta-1}^c}]^T, \quad (5)$$

where $\boldsymbol{e}_i$ is a natural basis vector in $\mathbb{R}^N$. For simplicity, we use the notation $\boldsymbol{P}$ instead of $\boldsymbol{P}_{\mathcal{L}_\beta}$ hereafter. When $\beta = 0$, $\boldsymbol{P}$ equals to the identity matrix.

Consider the case where some coefficients of the channel data generated by the LPOT is lost. The best linear unbiased estimator (BLUE) in this case is given by the so-called *para-pseudoinverse* [7] of $\boldsymbol{PE}(z)$ that is defined as

$$\boldsymbol{R}(z) = (\boldsymbol{PE}(z))^+ = (\tilde{\boldsymbol{E}}(z)\boldsymbol{P}^T\boldsymbol{PE}(z))^{-1}\tilde{\boldsymbol{E}}(z)\boldsymbol{P}^T, \quad (6)$$

where $\tilde{\boldsymbol{E}}(z)$ denotes the para-conjugate of $\boldsymbol{E}(z)$. The para-pseudoinverse corresponds to the Moore-Penrose pseudoinverse when the matrix has no delay.

It should be, however, noted that unless the determinant of $\tilde{\boldsymbol{E}}(z)\boldsymbol{P}^T\boldsymbol{PE}(z)$ is a monomial, $\boldsymbol{R}(z)$ does not represent an FIR FB. Consider the reduced structure. If $\beta \le N - M$, it follows from (2) that the para-pseudoinverse as in (6) yields

$$\begin{aligned} (\boldsymbol{PE}(z))^+ &= (\tilde{\boldsymbol{E}}(z)\boldsymbol{P}^T\boldsymbol{PE}(z))^{-1}\tilde{\boldsymbol{E}}(z)\boldsymbol{P}^T \\ &= \boldsymbol{E}_r^{-1}(z)(\boldsymbol{\Psi}_0^T\boldsymbol{P}^T\boldsymbol{P}\boldsymbol{\Psi}_0)^{-1}\tilde{\boldsymbol{E}}^{-1}(z)\tilde{\boldsymbol{E}}_r(z)\boldsymbol{\Psi}_0^T\boldsymbol{P}^T \\ &= \boldsymbol{R}_r(z)(\boldsymbol{P}\boldsymbol{\Psi}_0)^+, \end{aligned} \quad (7)$$

where $\boldsymbol{R}_r(z)$ is the corresponding synthesis polyphase of $\boldsymbol{E}_r(z)$. As we have seen, the reconstruction problem for the reduced structure is very easy. However, the problem would become difficult in more general settings, that is, when we consider the LPOT that is not represented by the reduced structure. The para-pseudoinverse for the general case becomes

$$(\boldsymbol{PE}(z))^+ = (\boldsymbol{\Phi}_0^T\tilde{\boldsymbol{E}}_g(z)\boldsymbol{P}^T\boldsymbol{PE}_g(z)\boldsymbol{\Phi}_0)^{-1}\boldsymbol{\Phi}_0^T\tilde{\boldsymbol{E}}_g(z)\boldsymbol{P}^T. \quad (8)$$

In order to obtain the FIR solution, the determinant of the term in the brackets $\boldsymbol{\Phi}_0^T\tilde{\boldsymbol{E}}_g(z)\boldsymbol{P}^T\boldsymbol{PE}_g(z)\boldsymbol{\Phi}_0$ must be a monomial. However, this is not usually guaranteed. This argument implies that although the para-pseudoinverse can achieve PR and give the optimal inverse, the synthesis filters generally have IIR, and then its implementation is difficult. Thus, in this paper, we clarify the possibility for perfect reconstruction and establish a method to construct an inverse of $\boldsymbol{PE}(z)$ with FIR filters. The synthesis filters of the same length as the analysis LPOT are of particular interest.

Let $\boldsymbol{R}(z) = \boldsymbol{R}_0 + z^{-1}\boldsymbol{R}_1$ be the synthesis LPOT polyphase matrix of size $M \times (N-\beta)$ associated with $\boldsymbol{E}(z)$. Then, we obtain the estimation of the input signals from the "incomplete" channel signals $\tilde{\boldsymbol{g}}_i = \boldsymbol{Pg}_i$ and $\tilde{\boldsymbol{g}}_{i+1} = \boldsymbol{Pg}_{i+1}$ as follows:

$$\hat{\boldsymbol{f}}_i = \boldsymbol{R}_1\tilde{\boldsymbol{g}}_i + \boldsymbol{R}_0\tilde{\boldsymbol{g}}_{i-1}. \quad (9)$$

For PR $\hat{\boldsymbol{f}}_i = \boldsymbol{f}_{i-1}$, it is needed that

$$\boldsymbol{R}_0\boldsymbol{PE}_0 + \boldsymbol{R}_1\boldsymbol{PE}_1 = \boldsymbol{I}, \quad (10)$$

$$\boldsymbol{R}_1\boldsymbol{PE}_0 = \boldsymbol{R}_0\boldsymbol{PE}_1 = \boldsymbol{0}. \quad (11)$$

If $\boldsymbol{P} = \boldsymbol{I}$, that is, there is no loss of information, then the BLUE can be obtained as $\boldsymbol{R}_0 = \boldsymbol{E}_0^T$ and $\boldsymbol{R}_1 = \boldsymbol{E}_1^T$, which is the ordinary synthesis matrix of the oversampled paraunitary FB's [3].

In the rest of the paper, we would like to address the problems as the following:

1. Clarify the possible number of coefficients to be lost that permits PR;
2. Find the synthesis matrices $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ which enable us to obtain the original signal without noise.

4. RECONSTRUCTION ABILITY AND METHOD

Firstly, we would like to show the main result regarding the ability of PR with lost data.

Theorem 1 *Let β be the number of lost coefficients. For perfect reconstruction, it at least holds that*

$$M \le 2(N-\beta) - (\mathrm{rank}(\boldsymbol{PE}_0) + \mathrm{rank}(\boldsymbol{PE}_1)). \quad (12)$$

Proof. The condition (11) implies that

$$R(\boldsymbol{PE}_1) \subset N(\boldsymbol{R}_0),\ R(\boldsymbol{PE}_0) \subset N(\boldsymbol{R}_1), \quad (13)$$

which can be rewritten with the subspaces $\tilde{S}_0$ and $\tilde{S}_1$ as such that

$$N(\boldsymbol{R}_0) = \tilde{S}_0 \oplus R(\boldsymbol{PE}_1),\ N(\boldsymbol{R}_1) = R(\boldsymbol{PE}_0) \oplus \tilde{S}_1. \quad (14)$$

We then have the decomposition of the whole space $\mathbb{R}^{N-\beta}$ as follows:

$$\mathbb{R}^{N-\beta} = S_0 \oplus (\tilde{S}_0 \oplus R(\boldsymbol{PE}_1)) = (R(\boldsymbol{PE}_0) \oplus \tilde{S}_1) \oplus S_1, \quad (15)$$

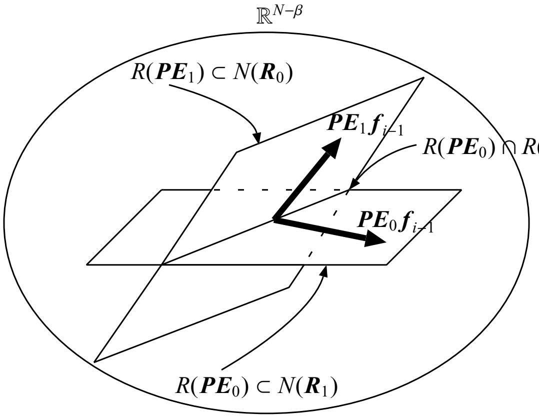

Figure 2: Geometric illustration of the lost channel space.

where S_0 and S_1 are the compliments of $N(\boldsymbol{R}_0)$ and $N(\boldsymbol{R}_1)$, respectively. If we take $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ such that for all non-zero $\boldsymbol{y}_0 \in S_0$ and $\boldsymbol{y}_1 \in S_1$, $\boldsymbol{R}_0\boldsymbol{y}_0 \neq \boldsymbol{0}$ and $\boldsymbol{R}_1\boldsymbol{y}_1 \neq \boldsymbol{0}$, respectively. Then, we have

$$\dim S_0 = \dim R(\boldsymbol{R}_0),\ \dim S_1 = \dim R(\boldsymbol{R}_1). \tag{16}$$

Now, we can use the subspaces S_0 and S_1 for reconstruction. Perfect reconstruction is therefore possible if and only if

$$\mathbb{R}^M = R(\boldsymbol{R}_0) + R(\boldsymbol{R}_1). \tag{17}$$

To satisfy the above condition, it is necessary that

$$\dim S_0 + \dim S_1 \geq M. \tag{18}$$

Moreover, (15) yields that

$$\dim S_0 \leq N - \beta - \dim R(\boldsymbol{PE}_0),\ \dim S_1 \leq N - \beta - \dim R(\boldsymbol{PE}_1). \tag{19}$$

Together with (18) and (19), we finally have

$$M \leq \dim S_0 + \dim S_1 \leq 2(N-\beta) - (\mathrm{rank}(\boldsymbol{PE}_0) + \mathrm{rank}(\boldsymbol{PE}_1)). \tag{20}$$

This completes proof. □

We next show how to construct the synthesis matrices $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$. First of all, the subspaces S_0 and S_1, which are used for reconstruction, should be taken as "largely" as possible. In (15), therefore, we set that

$$\tilde{S}_0 = \tilde{S}_1 = \{\boldsymbol{0}\}, \tag{21}$$

which implies that

$$N(\boldsymbol{R}_0) = R(\boldsymbol{PE}_1),\ N(\boldsymbol{R}_1) = R(\boldsymbol{PE}_0). \tag{22}$$

From (15), (21), and Lemma 1 in Appendix, we obtain the following fact:

Proposition 1 *We have the decomposition of* $\mathbb{R}^{N-\beta}$ *as*

$$\mathbb{R}^{N-\beta} = (S_0 \oplus S_1) \oplus S, \tag{23}$$

where $S = R(\boldsymbol{PE}_0) \cap R(\boldsymbol{PE}_1)$.

Proof is straightforward. We also have the variation of the above proposition:

Proposition 2 *Assuming that*

$$\mathbb{R}^{N-\beta} = R(\boldsymbol{PE}_0) + R(\boldsymbol{PE}_1), \tag{24}$$

we have

$$R(\boldsymbol{PE}_0) = S \oplus S_0,\ R(\boldsymbol{PE}_1) = S \oplus S_1. \tag{25}$$

Keep in mind that we have the freedom to choose S_0 and S_1 in the above decomposition. For simplicity, we will decompose the subspaces $R(\boldsymbol{PE}_0)$ and $R(\boldsymbol{PE}_1)$ into the orthogonal direct sums given as $R(\boldsymbol{PE}_0) = S \dot{\oplus} S_0$ and $R(\boldsymbol{PE}_1) = S \dot{\oplus} S_1$. In this case, for $i = 0, 1$, writing the orthogonal projectors onto $R(\boldsymbol{PE}_i)$, S, and S_i by $\boldsymbol{P}_i$, $\boldsymbol{P}_S$, and $\boldsymbol{P}_{S_i}$, respectively, we have the relation

$$\boldsymbol{P}_{S_i} = \boldsymbol{P}_i - \boldsymbol{P}_S. \tag{26}$$

Then, the orthogonal projectors $\boldsymbol{P}_{S_0\cdot 1}$ and $\boldsymbol{P}_{S_1\cdot 0}$ onto S_0 and S_1 along $R(\boldsymbol{PE}_1)$ and $R(\boldsymbol{PE}_0)$ are respectively given as

$$\begin{aligned} \boldsymbol{P}_{S_0\cdot 1} &= \boldsymbol{P}_{S_0}[\boldsymbol{P}_{S_0} + (\boldsymbol{PE}_1)(\boldsymbol{PE}_1)^T]^{-1}, \\ \boldsymbol{P}_{S_1\cdot 0} &= \boldsymbol{P}_{S_1}[\boldsymbol{P}_{S_1} + (\boldsymbol{PE}_0)(\boldsymbol{PE}_0)^T]^{-1}. \end{aligned} \tag{27}$$

At this moment, we have obtained the useful orthogonal projectors: $\boldsymbol{P}_{S_0\cdot 1}$ and $\boldsymbol{P}_{S_1\cdot 0}$ vanish elements in $R(\boldsymbol{PE}_1)$ and $R(\boldsymbol{PE}_0)$, respectively. We can find the decomposition of $\boldsymbol{P}_{S_0}$ and $\boldsymbol{P}_{S_1}$ as

$$\boldsymbol{P}_{S_0} = \boldsymbol{U}_0\boldsymbol{U}_0^T,\ \boldsymbol{P}_{S_1} = \boldsymbol{U}_1\boldsymbol{U}_1^T, \tag{28}$$

where $\boldsymbol{U}_0$ and $\boldsymbol{U}_1$ are matrices of respective size $(N-\beta) \times \dim S_0$ and $(N-\beta) \times \dim S_1$ whose columns are orthogonal to each other. Finally, we obtain the following theorem:

Theorem 2 *If* $\dim(S_0 + S_1) \geq M$, *there exist matrices* $\boldsymbol{Z}_0$ *and* $\boldsymbol{Z}_1$ *of respective size* $M \times \dim S_0$ *and* $M \times \dim S_1$ *which satisfy that*

$$\boldsymbol{Z}_0\boldsymbol{U}_0^T\boldsymbol{P}_{S_0\cdot 1}\boldsymbol{PE}_0 + \boldsymbol{Z}_1\boldsymbol{U}_1^T\boldsymbol{P}_{S_1\cdot 0}\boldsymbol{PE}_1 = \boldsymbol{I}. \tag{29}$$

Proof is omitted here. This theorem implies that we obtain synthesis matrices which we would like to find by letting

$$\begin{aligned} \boldsymbol{R}_0 &= \boldsymbol{Z}_0\boldsymbol{U}_0^T\boldsymbol{P}_{S_0\cdot 1} = \boldsymbol{Z}_0\boldsymbol{U}_0^T[\boldsymbol{P}_{S_0} + (\boldsymbol{PE}_1)(\boldsymbol{PE}_1)^T]^{-1}, \\ \boldsymbol{R}_1 &= \boldsymbol{Z}_1\boldsymbol{U}_1^T\boldsymbol{P}_{S_1\cdot 0} = \boldsymbol{Z}_1\boldsymbol{U}_1^T[\boldsymbol{P}_{S_1} + (\boldsymbol{PE}_0)(\boldsymbol{PE}_0)^T]^{-1}. \end{aligned} \tag{30}$$

The remained problem is to find the optimal $\boldsymbol{Z}_0$ and $\boldsymbol{Z}_1$ in some sense. Consider the case where the incomplete channel data is corrupted by the additive random noise. As shown in (9), the reconstruction can be written as

$$\begin{aligned} \hat{\boldsymbol{f}}_i &= \boldsymbol{R}_0(\tilde{\boldsymbol{g}}_{i-1} + \boldsymbol{n}_{i-1}) + \boldsymbol{R}_1(\tilde{\boldsymbol{g}}_i + \boldsymbol{n}_i) \\ &= \boldsymbol{R}_0(\boldsymbol{PE}_1\boldsymbol{f}_{i-2} + \boldsymbol{PE}_0\boldsymbol{f}_{i-1} + \boldsymbol{n}_{i-1}) + \boldsymbol{R}_1(\boldsymbol{PE}_1\boldsymbol{f}_{i-1} + \boldsymbol{PE}_0\boldsymbol{f}_i + \boldsymbol{n}_i) \\ &= \boldsymbol{f}_{i-1} + \boldsymbol{R}_0\boldsymbol{n}_{i-1} + \boldsymbol{R}_1\boldsymbol{n}_i. \end{aligned} \tag{31}$$

In a manner similar to the BLUE, we introduce the cost function defined as

$$J[\boldsymbol{Z}_0, \boldsymbol{Z}_1] = E_{\boldsymbol{n}_{i-1},\boldsymbol{n}_i}\|\boldsymbol{R}_0\boldsymbol{n}_{i-1} + \boldsymbol{R}_1\boldsymbol{n}_i\|^2, \tag{32}$$

which should be minimized subject to (29). Note that (29) becomes

$$\boldsymbol{ZA} = \boldsymbol{I}, \tag{33}$$

where

$$\boldsymbol{Z} = [\boldsymbol{Z}_0\ \boldsymbol{Z}_1],\ \boldsymbol{A} = \begin{bmatrix} \boldsymbol{U}_0^T\boldsymbol{P}_{S_0\cdot 1}\boldsymbol{PE}_0 \\ \boldsymbol{U}_1^T\boldsymbol{P}_{S_1\cdot 0}\boldsymbol{PE}_1 \end{bmatrix}. \tag{34}$$

Then, the reconstruction process as in (31) can be also modified to

$$\hat{\boldsymbol{f}}_i = \boldsymbol{Z}(\boldsymbol{A}\boldsymbol{f}_{i-1} + \boldsymbol{\epsilon}_{i-1}), \tag{35}$$

where

$$\boldsymbol{\epsilon}_{i-1} = \underbrace{\begin{bmatrix} \boldsymbol{U}_0^T\boldsymbol{P}_{S_0\cdot 1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_1^T\boldsymbol{P}_{S_1\cdot 0} \end{bmatrix}}_{\boldsymbol{B}} \underbrace{\begin{bmatrix} \boldsymbol{n}_{i-1} \\ \boldsymbol{n}_i \end{bmatrix}}_{\boldsymbol{\nu}_{i-1}}. \tag{36}$$

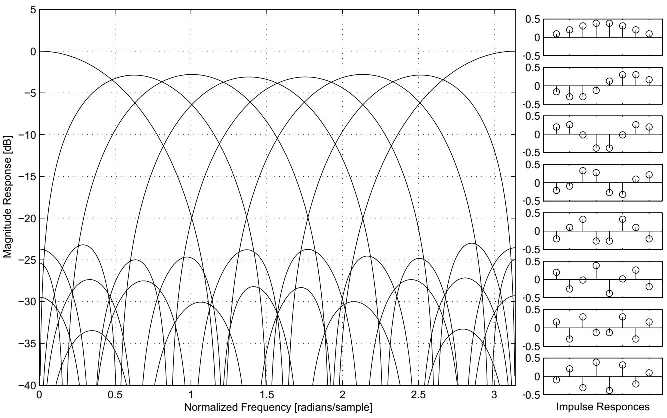

Figure 3: The analysis filters of the LPOT ($N = 8$ and $M = 4$)

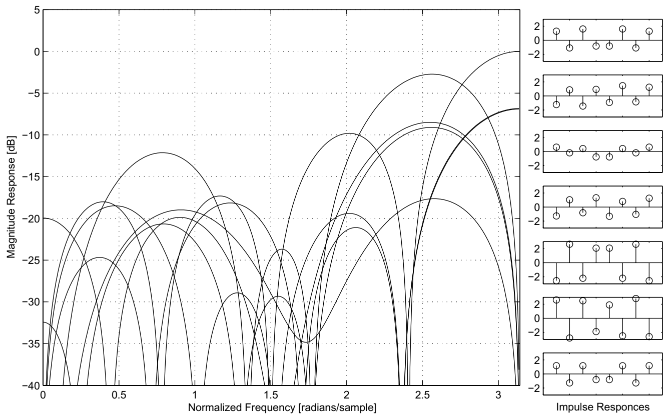

Figure 4: The seven synthesis filters when the eighth channel is lost.

Surprisingly, the problem to find the synthesis matrices reduces to a basic estimation problem in a linear degradation model as in (35).

Assume that the additive noise $\boldsymbol{\nu}_i = [\boldsymbol{n}_i^T, \boldsymbol{n}_{i+1}^T]^T$ is stationary, that is, $\boldsymbol{Q}_{\boldsymbol{\nu}} = E_{\boldsymbol{\nu}_i}[\boldsymbol{\nu}_i\boldsymbol{\nu}_i^T] = E_{\boldsymbol{\nu}_j}[\boldsymbol{\nu}_j\boldsymbol{\nu}_j^T]$ for all $i \neq j$. Then, let

$$\boldsymbol{Q}_{\boldsymbol{\epsilon}} = E_{\boldsymbol{\nu}}[\boldsymbol{\epsilon}_{i-1}\boldsymbol{\epsilon}_{i-1}^T] = \boldsymbol{B}\boldsymbol{Q}_{\boldsymbol{\nu}}\boldsymbol{B}^T, \tag{37}$$

where $\boldsymbol{Q}_{\boldsymbol{\epsilon}}$ is also independent of the index of $\boldsymbol{\epsilon}_{i-1}$. It follows from (30) and (36) that the cost function (32) is reduced to

$$J[\boldsymbol{Z}_0, \boldsymbol{Z}_1] = J[\boldsymbol{Z}] = E_{\boldsymbol{\nu}}\|\boldsymbol{Z}\boldsymbol{\epsilon}_{i-1}\|^2 = \mathrm{tr}[\boldsymbol{Z}\boldsymbol{Q}_{\boldsymbol{\epsilon}}\boldsymbol{Z}^T]. \tag{38}$$

Then, we obtain the synthesis filters which achieve PR and noise suppression simultaneously as follows:

Theorem 3 *The cost function as in (32) or (38) is minimized subject to (31) (equivalently (33)) when*

$$\boldsymbol{Z} = (\boldsymbol{A}^T\boldsymbol{Q}_{\boldsymbol{\epsilon}}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{Q}_{\boldsymbol{\epsilon}}^{-1}. \tag{39}$$

If there is no lost channel, the filters obtained in the above theorem are equivalent to the ordinal paraunitary filters, that is, $\boldsymbol{R}_0 = \boldsymbol{E}_0^T$ and $\boldsymbol{R}_1 = \boldsymbol{E}_1^T$.

5. EXAMPLE

An example of the synthesis filters that lead to perfect reconstruction as well as noise suppression is shown in this section. In Fig. 3, the frequency and the impulse responses of the analysis filters of the eight-channel ($N = 8$) LPOT with the decimation factor $M = 4$ are illustrated.

It is obvious that for PR, a possible number of channels to be lost is less than or equal to $N - M$. Therefore, we only consider the loosing operator such that the number of rows is up to $N - M$. In this example, the number of channel is twice the decimation factor, i.e., $M = N/2$. From (3) and the fact that $\boldsymbol{U}_1 = \boldsymbol{I}$ in the general structure, it follows that for $i = 0, 1$, $\mathrm{rank}(\boldsymbol{P}\boldsymbol{E}_i) = M = N/2$, unless the angles of Givens rotations in $\boldsymbol{V}_1$ are exactly zero or the multiple of π. Then, Theorem 1 tells us that the relation $\beta \leq M/2$ holds. Therefore, the number of channel to be lost is up to two in this example. We show the impulse and frequency responses of the synthesis filters derived by (39) in the case where the seventh channel is lost, and all channels are corrupted by the white noise. As we have seen, PR is possible in this case.

6. CONCLUSIONS AND OPEN PROBLEMS

We have shown a solution of a reconstruction problem from lost channel data for a class of oversampled LPPRFB's called LPOT's. We have discussed the possibility of PR with FIR filters of the same length as the analysis filters, and shown the reconstruction method with these FIR filters.

We now have two open problems. We have assumed that the subspaces S_0 and S_1 are chosen such that the decompositions as in (25) are the orthogonal direct sums. However, the optimal choice has not been clarified. The second problem is to address the PR problem for oversampled LPPRFB's in the case $K > 2$.

A. APPENDIX

Lemma 1 ([9]) *Let H_N be an N-dimensional vector space. When there exist two kinds of decomposition as $H_N = V_1 \oplus W_1 = V_2 \oplus W_2$, if $V_1 \subset W_2$ or $V_2 \subset W_1$, then it holds that $H_N = (V_1 \oplus V_2) \oplus (W_1 \cap W_2)$.*

REFERENCES

[1] G. Strang and T. Nguyen, *Wavelets and Filter Banks*. Wellesley MA: Wellesley-Cambridge Press, 1996.

[2] T. D. Tran, R. L. de Queiroz, and T. Q. Nguyen, “Linear-phase perfect reconstruction filter bank: Lattice structure, design, and application in image coding,” *IEEE Trans. Signal Processing*, vol. 48, pp. 133–147, Jan. 2000.

[3] F. Labeau, L. Vandendorpe, and B. Macq, “Structures, factorizations, and design criteria for oversampled paraunitary filterbanks yielding linear-phase filters,” *IEEE Trans. Signal Processing*, vol. 48, pp. 3062–3071, Nov. 2000.

[4] X. Gao, T. Q. Nguyen, and G. Strang, “On factorization of *m*-channel paraunitary filterbanks,” *IEEE Trans. Signal Processing*, vol. 49, pp. 1433–1446, July 2001.

[5] L. Gan and K.-K. Ma, “Oversampled linear-phase perfect reconstruction filterbanks: theory, lattice structure and parameterization,” *IEEE Trans. Signal Proc.*, vol. 51, pp. 744–759, Mar. 2003.

[6] T. Tanaka and Y. Yamashita, “The generalized lapped pseudo-biorthogonal transform: Oversampled linear-phase perfect reconstruction filter banks with lattice structures,” *IEEE Trans. Signal Processing*, vol. 52, Feb. 2004.

[7] H. Bölcskei, F. Hlawatsch, and H. G. Feichtinger, “Frame-theoretic analysis of oversampled filter banks,” *IEEE Trans. Signal Processing*, vol. 46, pp. 3256–3268, Dec. 1998.

[8] J. Liang, L. Gan, C. Tu, T. D. Tran, and K.-K. Ma, “On efficient implementation of oversampled linear phase perfect reconstruction filter banks,” in *ICASSP 2003*, vol. VI, pp. 501–504, 2003.

[9] H. Yanai and K. Takeuchi, *Projection Matrix, Generalized Inverse Matrix, and Singular Value Decomposition*. Tokyo, Japan: Univ. of Tokyo Press, 1983. in Japanese.