MEAN-SQUARE CONSISTENCY OF STATISTICAL-FUNCTION ESTIMATORS FOR GENERALIZED ALMOST-CYCLOSTATIONARY PROCESSES

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ABSTRACT

In this paper, the problem of estimating second-order statistical functions for generalized almost-cyclostationary (GACS) processes is addressed. The class of such nonstationary processes includes, as a special case, the almost-cyclostationary (ACS) processes. ACS processes filtered by some linear time-variant channels are further examples. It is shown that, for GACS processes, the cyclic correlogram is a mean-square consistent estimator of the cyclic autocorrelation function. Moreover, well-known consistency results for ACS processes can be obtained by specializing the results of this paper.

1. INTRODUCTION

In the last two decades, a big effort was devoted to the analysis and exploitation of the properties of the almost-cyclostationary (ACS) processes. In fact, almost-all modulated signals adopted in communications can be modelled as ACS (see e.g., [4] and [13] for a comprehensive treatment in terms of higher-order statistics). For an ACS process, multivariate statistical functions are almost-periodic functions of time and can be expressed by (generalized) Fourier series expansions whose frequencies, referred to as cycle frequencies, do not depend on the lag shifts of the processes.

More recently, wider classes of nonstationary processes have been considered in [7], [8], [9], [10], and [11]. In particular, in [7], the class of the generalized almost-cyclostationary (GACS) processes has been introduced and characterized. Processes belonging to this class exhibit multivariate statistical functions that are almost-periodic functions of time whose Fourier series expansions have coefficients and frequencies, referred to as lag-dependent cycle frequencies, that can depend on the lag shifts of the processes. The class of the GACS processes includes, as a special case, the class of the ACS processes. Moreover, chirp signals and several angle-modulated and time-warped communication signals are GACS processes.

In [8] and [9], it is shown that several time variant channels of interest in communications transform a transmitted ACS signal into a GACS one. In particular, in [9] it is shown that the GACS model is appropriate to describe the output signal of some mobile communication channels when the input signal is ACS and the product transmitted-signal-bandwidth times data-record-length is not too small. Thus, the GACS model turns out to be useful in modern mobile communication systems where wider and wider bandwidths are required to get higher and higher bit rates and, moreover, large data-record lengths are necessary for blind channel identification techniques or detection algorithms in highly noise- and interference-corrupted environments. Therefore, to properly equalize such time variant channels, statistical functions of the output GACS process need to known or estimated.

In the present paper, mean-square consistent estimators for second-order statistical functions of GACS processes are proposed. In particular, it is shown that for a GACS stochastic process satisfying some mixing conditions expressed in terms of the summability of its second- and fourth-order cumulants, the cyclic correlogram is a mean-square consistent estimator of the cyclic autocorrelation function. It is shown that well-known consistency results derived for ACS processes (see e.g., [2], [3], [5], [6]) can be obtained as a special case of the results of this paper.

2. SECOND-ORDER GACS STOCHASTIC PROCESSES

In this section, the second-order characterization of GACS stochastic processes is briefly reviewed. See [7] and [8] for a more comprehensive treatment in the nonstochastic approach.

A finite-power complex-valued continuous-time stochastic process $x(t)$ is said *second-order GACS in the wide sense* if its autocorrelation function

$$\mathcal{R}_{xx^*}(t,\tau) \triangleq \mathrm{E}\{x(t+\tau)\,x^*(t)\} \tag{1}$$

with $\mathrm{E}\{\cdot\}$ denoting statistical expectation, is an almost-periodic function of time in the sense of Bohr (or, equivalently, uniformly almost periodic in t in sense of Besicovitch [1]). That is, for each fixed τ, $\mathcal{R}_{xx^*}(t,\tau)$ is the limit of an uniformly convergent sequence of trigonometric polynomials in t:

$$\mathcal{R}_{xx^*}(t,\tau) = \sum_{\alpha\in A_\tau} R_{xx^*}(\alpha,\tau)\,e^{j2\pi\alpha t}\,. \tag{2}$$

In (2), the real numbers α and the complex-valued functions $R_{xx^*}(\alpha,\tau)$, referred to as cycle frequencies and cyclic autocorrelation functions, are the frequencies and coefficients, respectively, of the (generalized) Fourier series expansion of $\mathcal{R}_{xx^*}(t,\tau)$, that is,

$$R_{xx^*}(\alpha,\tau) \triangleq \lim_{T\to\infty}\frac{1}{T}\int_{t_0-T/2}^{t_0+T/2}\mathcal{R}_{xx^*}(t,\tau)\,e^{-j2\pi\alpha t}\,\mathrm{d}t \tag{3}$$

with the limit independent of t_0. Moreover,

$$A_\tau \triangleq \{\alpha\in\mathbb{R} : R_{xx^*}(\alpha,\tau)\neq 0\} \tag{4}$$

is a countable set which, in general, depends on τ.

Note that, even if the set A_τ is always countable, the set

$$A \triangleq \bigcup_{\tau\in\mathbb{R}} A_\tau \tag{5}$$

is not necessarily countable. Thus, the class of the second-order wide-sense GACS processes extends that of the wide-sense ACS which are obtained as a special case of GACS processes when the set A is countable [3], [6].

A useful characterization of wide sense GACS processes can be obtained by observing that the set A_τ can be expressed as

$$A_\tau = \bigcup_{n\in\mathbb{I}}\{\alpha\in\mathbb{R} : \alpha = \alpha_n(\tau)\} \tag{6}$$

where $\mathbb{I}$ is a countable set and the functions $\alpha_n(\tau)$, referred to as *lag-dependent cycle frequencies*, are such that, for each α and τ, there exists at most one $n\in\mathbb{I}$ such that $\alpha=\alpha_n(\tau)$. Thus, accounting for the countability of A_τ for each τ, the support in the (α,τ) plane of the cyclic autocorrelation function $R_{xx^*}(\alpha,\tau)$ is constituted by the closure of the set of curves defined by the explicit equations $\alpha=\alpha_n(\tau)$, $n\in\mathbb{I}$. Furthermore, we have the following result.

Theorem 2.1 The autocorrelation function $\mathcal{R}_{xx^*}(t,\tau)$ of a second-order wide-sense GACS process can be expressed as

$$\mathcal{R}_{xx^*}(t,\tau) = \sum_{n\in\mathbb{I}} R_{xx^*}^{(n)}(\tau)\, e^{j2\pi\alpha_n(\tau)t} \tag{7}$$

where the functions $R_{xx^*}^{(n)}(\tau)$, referred to as *generalized cyclic autocorrelation functions*, are defined as

$$R_{xx^*}^{(n)}(\tau) \triangleq \begin{cases} \lim\limits_{T\to\infty} \dfrac{1}{T}\displaystyle\int_{t_0-T/2}^{t_0+T/2} \mathcal{R}_{xx^*}(t,\tau)\, e^{-j2\pi\alpha_n(\tau)t}\,\mathrm{d}t\,, \\ \tau\in\mathcal{T}^{(n)} \\ 0, \quad \tau\in\mathbb{R}-\mathcal{T}^{(n)} \end{cases} \tag{8}$$

with

$$\mathcal{T}^{(n)} \triangleq \{\tau\in\mathbb{R} : \alpha_n(\tau) \text{ is defined}\} \tag{9}$$

and the limit in (8) independent of t_0. □

Note that, in (7) the sum ranges over a set not depending on τ as, on the contrary, it occurs in (2). Moreover, unlike the case of second-order ACS processes, both coefficients and frequencies of the Fourier series in (7) depend on the lag parameter τ. Thus, the wide-sense ACS processes are obtained as a special case of GACS processes when the lag-dependent cycle frequencies are constant with respect to τ and, hence, are coincident with the cycle frequencies [7].

The functions $\alpha_n(\tau)$ in (6)–(9) are such that, for each τ, $\alpha_n(\tau)\neq\alpha_m(\tau)$ for $n\neq m$. However, if more functions $\alpha_{n_1}(\tau),\cdots,\alpha_{n_K}(\tau)$ are defined in K (not necessarily coincident) neighborhoods of the same point τ_0, all have the same limit, say α_0, for $\tau\to\tau_0$, and only one of them is defined in τ_0, then it is convenient to assume all the functions $\alpha_{n_1}(\tau),\cdots,\alpha_{n_K}(\tau)$ defined in τ_0 with $\alpha_{n_1}(\tau_0)=\cdots=\alpha_{n_K}(\tau_0)=\alpha_0$ and, consequently, define

$$R_{xx^*}^{(n_i)}(\tau_0) \triangleq \lim_{\tau\to\tau_0} R_{xx^*}^{(n_i)}(\tau) \quad i=1,\dots,K \tag{10}$$

where, for each i, the limit is made with τ ranging in the neighborhood where the function $\alpha_{n_i}(\tau)$ is defined. With convention (10), by taking the coefficient of the complex sinewave at frequency α (see (3)) in both sides of (7), it follows that the cyclic autocorrelation function and the generalized cyclic autocorrelation functions are linked by the relationship

$$R_{xx^*}(\alpha,\tau) = \sum_{n\in\mathbb{I}} R_{xx^*}^{(n)}(\tau)\,\delta_{\alpha-\alpha_n(\tau)} \tag{11}$$

where δ_γ denotes Kronecker delta, that is, $\delta_\gamma=1$ for $\gamma=0$ and $\delta_\gamma=0$ otherwise.

In the special case of ACS processes, the lag-dependent cycle frequencies are constant and coincident with the cycle frequencies, only one term is present in the sum in (11) and, consequently, the generalized cyclic autocorrelation functions are coincident with the cyclic autocorrelation functions. Moreover, the autocorrelation function $\mathcal{R}_{xx^*}(t,\tau)$ depends uniformly on the parameter τ and is uniformly continuous and the cyclic autocorrelation functions $R_{xx^*}(\alpha,\tau)$ are continuous in τ for each $\alpha\in A$ [3].

Two finite-power complex-valued continuous-time stochastic process $y(t)$ and $x(t)$ are said jointly GACS in the wide sense if

$$\begin{aligned} \mathcal{R}_{yx^{(*)}}(t,\tau) &\triangleq \mathrm{E}\left\{y(t+\tau)\,x^{(*)}(t)\right\} \\ &= \sum_{\alpha\in B_\tau} R_{yx^{(*)}}(\alpha,\tau)\,e^{j2\pi\alpha t} \\ &= \sum_{n\in\mathbb{I}_{yx^{(*)}}} R_{yx^{(*)}}^{(n)}(\tau)\,e^{j2\pi\alpha_n(\tau)t}\,. \end{aligned} \tag{12}$$

In (12), B_τ and $\mathbb{I}_{yx^{(*)}}$ are countable sets, superscript $(*)$ denotes an optional complex conjugation, the lag-dependent cycle frequencies $\alpha_n(\tau)$ depend on the choice made for $(*)$ and, in general, are not coincident with those of $x(t)$ or $y(t)$, $R_{yx^{(*)}}(\alpha,\tau)$ is the *cyclic cross-correlation function*

$$R_{yx^{(*)}}(\alpha,\tau) \triangleq \lim_{T\to\infty}\frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} \mathcal{R}_{yx^{(*)}}(t,\tau)\,e^{-j2\pi\alpha t}\,\mathrm{d}t \tag{13}$$

and $R_{yx^{(*)}}^{(n)}(\tau)$ are the *generalized cyclic cross-correlation functions* defined as

$$R_{yx^{(*)}}^{(n)}(\tau) \triangleq \begin{cases} \lim\limits_{T\to\infty}\dfrac{1}{T}\displaystyle\int_{t_0-T/2}^{t_0+T/2} \mathcal{R}_{yx^{(*)}}(t,\tau)\,e^{-j2\pi\alpha_n(\tau)t}\,\mathrm{d}t\,, \\ \tau\in\mathcal{T}_{yx^{(*)}}^{(n)} - D^{(n)} \\ 0, \quad \tau\in\mathbb{R}-\mathcal{T}_{yx^{(*)}}^{(n)} - D^{(n)} \end{cases} \tag{14}$$

$$R_{yx^{(*)}}^{(n)}(\tau_0) \triangleq \lim_{\tau\to\tau_0} R_{yx^{(*)}}^{(n)}(\tau) \quad \tau_0\in D^{(n)} \tag{15}$$

where

$$\mathcal{T}_{yx^{(*)}}^{(n)} \triangleq \{\tau\in\mathbb{R} : \alpha_n(\tau) \text{ is defined, } \alpha_n(\tau)\neq\alpha_m(\tau)\ \forall m\neq n\} \tag{16}$$

$$D^{(n)} \triangleq \{\tau\in\mathbb{R} : \alpha_n(\tau)=\alpha_m(\tau) \text{ for some } m\neq n\} \tag{17}$$

and the limit in (15) is made with τ ranging in $\mathcal{T}_{yx^{(*)}}^{(n)} - D^{(n)}$. Moreover, by reasoning as for a single process, it can be shown that

$$R_{yx^{(*)}}(\alpha,\tau) = \sum_{n\in\mathbb{I}_{yx^{(*)}}} R_{yx^{(*)}}^{(n)}(\tau)\,\delta_{\alpha-\alpha_n(\tau)}\,. \tag{18}$$

3. FINITE DATA-RECORD BASED ESTIMATORS

In this section, for GACS processes, the cyclic cross-correlogram, the cyclic correlogram, and the conjugate cyclic correlogram are proposed as estimators of the cyclic cross-correlation function (13), the cyclic autocorrelation function (3), and the conjugate cyclic autocorrelation function

$$R_{xx}(\alpha,\tau) \triangleq \lim_{T\to\infty}\frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} \mathrm{E}\{x(t+\tau)\,x(t)\}\,e^{-j2\pi\alpha t}\,\mathrm{d}t \tag{19}$$

respectively. Moreover, their bias and variance are determined for finite data-record length.

Definition 3.1 Given two stochastic processes $y(t)$ and $x(t)$, their *cyclic cross-correlogram* is defined as

$$R_{yx^{(*)}}(\alpha,\tau;t_0,T) \triangleq \int_{\mathbb{R}} w_T(t-t_0)\,y(t+\tau)\,x^{(*)}(t)\,e^{-j2\pi\alpha t}\,\mathrm{d}t \tag{20}$$

where $w_T(t)$ is a unit-area data-window nonzero in $(-T/2,T/2)$. □

By specializing (20) for $y(t)\equiv x(t)$ and $(*)$ present, one obtains the cyclic correlogram. Furthermore, for $y(t)\equiv x(t)$ and $(*)$ absent, one obtains the conjugate cyclic correlogram.

Assumption 3.1 ***a)*** The stochastic processes $y(t)$ and $x(t)$ are singularly and jointly (second-order) GACS in the wide sense, that is, for any choice of z_1 and z_2 in $\{x,x^*,y,y^*\}$

$$\begin{aligned} &\mathrm{E}\{z_1(t+\tau_1)\,z_2(t+\tau_2)\} \\ &\quad= \sum_{n\in\mathbb{I}_{z_1z_2}} R_{z_1z_2}^{(n)}(\tau_1-\tau_2)\,e^{j2\pi\alpha_{z_1,z_2}^{(n)}(\tau_1-\tau_2)(t+\tau_2)} \end{aligned} \tag{21}$$

(uniformly almost-periodic in t in the sense of Besicovitch [1]).

b) The fourth-order cumulant $\operatorname{cum}\{y(t+\tau_1), x^{(*)}(t+\tau_2), y^*(t+\tau_3), x^{(*)*}(t)\}$ can be expressed as

$$\begin{aligned} &\operatorname{cum}\{y(t+\tau_1), x^{(*)}(t+\tau_2), y^*(t+\tau_3), x^{(*)*}(t)\} \\ &= \sum_{n \in \mathbb{I}_4} C^{(n)}_{yx^{(*)}y^*x^{(*)*}}(\tau_1,\tau_2,\tau_3)\, e^{j2\pi\beta_n(\tau_1,\tau_2,\tau_3)t}\,. \end{aligned} \tag{22}$$

(uniformly almost-periodic in t in the sense of Besicovitch). □

Assumption 3.2 For any choice of z_1 and z_2 in $\{x, x^*, y, y^*\}$ it results

$$\sum_{n \in \mathbb{I}_{z_1,z_2}} \left\| R^{(n)}_{z_1 z_2} \right\|_\infty < \infty \tag{23}$$

where $\|R\|_\infty \triangleq \operatorname{ess\,sup}_{\tau\in\mathbb{R}} |R(\tau)|$ is the essential supremum of $R(\tau)$. □

Assumption 3.3 It results that

$$\sum_{n \in \mathbb{I}_4} \left\| C^{(n)}_{yx^{(*)}y^*x^{(*)*}} \right\|_\infty < \infty\,. \tag{24}$$

□

Assumption 3.4 The stochastic processes $x(t)$ and $y(t)$ have uniformly bounded fourth-order absolute moments. That is, for any $z \in \{x, y\}$ there exists a positive number M_4 such that

$$\mathrm{E}\left\{ |z(t)|^4 \right\} \leq M_4 < \infty\,. \tag{25}$$

□

Assumptions 3.1–3.4 regard the regularity of second and fourth-order (joint) statistical functions of $x(t)$ and $y(t)$. Specifically, from Assumption 3.1 it follows that the second-order (cross) moments of $x(t)$ and $y(t)$ and their fourth-order joint cumulant are limits of uniformly convergent sequences of trigonometric polynomials in t. Moreover, Assumptions 3.2 and 3.3 mean that, for each Fourier series in (21) and (22), the nth coefficient has amplitude approaching zero, as $n \to \infty$, sufficiently fast to assure that the infinite sums in (23) and (24) are finite. Finally, note that a sufficient condition assuring that Assumption 3.4 holds is that the fourth-order moment functions of $x(t)$ and $y(t)$ are almost-periodic in t in the sense of Besicovitch.

Assumption 3.5 $w_T(t)$ is a T-duration data-tapering window that can be expressed as

$$w_T(t) = \frac{1}{T}\, a\!\left(\frac{t}{T}\right) \tag{26}$$

with $a(t) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\int_{\mathbb{R}} a(t)\,\mathrm{d}t = 1$, and $\lim_{T\to\infty} a(t/T) = 1$. □

Assumption 3.5 is easily verified by taking $a(t)$ with finite support $[-1/2, 1/2]$ and bounded (e.g., $a(t) = \operatorname{rect}(t)$).

By taking the expected value of the cyclic cross-correlogram (20) and using (12), the following result can be proved [12], where the made assumptions allow to interchange the order of expectation, integral, and sum operations.

Theorem 3.1 *Expected Value of the Cyclic Cross-Correlogram.* Let $y(t)$ and $x(t)$ be wide-sense jointly GACS stochastic processes with cross-correlation function (12). Under Assumptions 3.1a, 3.2, and 3.5, the expected value of the cyclic cross-correlogram $R_{yx^{(*)}}(\alpha, \tau; t_0, T)$ is given by

$$\begin{aligned} &\mathrm{E}\left\{ R_{yx^{(*)}}(\alpha, \tau; t_0, T) \right\} \\ &= \sum_{n \in \mathbb{I}_{yx^{(*)}}} R^{(n)}_{yx^{(*)}}(\tau)\, W_{\frac{1}{T}}(\alpha - \alpha_n(\tau))\, e^{-j2\pi[\alpha-\alpha_n(\tau)]t_0} \end{aligned} \tag{27}$$

where $W_{\frac{1}{T}}(f)$ is the Fourier transform of $w_T(t)$. □

By expressing the covariance of the second-order lag-product $y(t+\tau)\, x^{(*)}(t)$ in terms of second-order cross-moments and a fourth-order cumulant, the following result can be proved [12], where the made assumptions allow to interchange the order of expectation, integral, and sum operations.

Theorem 3.2 *Covariance of the Cyclic Cross-Correlogram.* Let $y(t)$ and $x(t)$ be zero-mean wide-sense jointly GACS stochastic processes with cross-correlation function (12). Under Assumptions 3.1–3.5, the covariance of the cyclic cross-correlogram $R_{yx^{(*)}}(\alpha, \tau; t_0, T)$ is given by

$$\operatorname{cov}\left\{ R_{yx^{(*)}}(\alpha_1, \tau_1; t_1, T), R_{yx^{(*)}}(\alpha_2, \tau_2; t_2, T) \right\} = \mathscr{T}_1 + \mathscr{T}_2 + \mathscr{T}_3 \tag{28}$$

where

$$\begin{aligned} \mathscr{T}_1 \triangleq\; & \sum_{n'} \sum_{n''} \int_{\mathbb{R}} R^{(n')}_{yy^*}(\tau_1 - \tau_2 + s)\, R^{(n'')}_{x^{(*)}x^{(*)*}}(s) \\ & e^{j2\pi\alpha'_{n'}(\tau_1-\tau_2+s)(t_2+\tau_2)}\, e^{j2\pi\alpha''_{n''}(s)t_2}\, e^{-j2\pi\alpha_1\cdot(t_2+s)}\, e^{j2\pi\alpha_2 t_2} \\ & \frac{1}{T}\, r_{aa^*}\Big(-[\alpha'_{n'}(\tau_1 - \tau_2 + s) + \alpha''_{n''}(s) - \alpha_1 + \alpha_2]T; \gamma_{s,T} \Big)\, \mathrm{d}s \end{aligned} \tag{29}$$

$$\begin{aligned} \mathscr{T}_2 \triangleq\; & \sum_{n'''} \sum_{n^{\prime v}} \int_{\mathbb{R}} R^{(n''')}_{yx^{(*)*}}(\tau_1 + s)\, R^{(n^{\prime v})}_{x^{(*)}y^*}(s - \tau_2) \\ & e^{j2\pi\alpha'''_{n'''}(\tau_1+s)t_2}\, e^{j2\pi\alpha'^{v}_{n^{\prime v}}(s-\tau_2)(t_2+\tau_2)}\, e^{-j2\pi\alpha_1\cdot(t_2+s)}\, e^{j2\pi\alpha_2 t_2} \\ & \frac{1}{T}\, r_{aa^*}\Big(-[\alpha'''_{n'''}(\tau_1 + s) + \alpha'^{v}_{n^{\prime v}}(s - \tau_2) - \alpha_1 + \alpha_2]T; \gamma_{s,T} \Big)\, \mathrm{d}s \end{aligned} \tag{30}$$

$$\begin{aligned} \mathscr{T}_3 \triangleq\; & \sum_{n} \int_{\mathbb{R}} C^{(n)}_{yx^{(*)}y^*x^{(*)*}}(\tau_1 + s, s, \tau_2)\, e^{j2\pi[\beta_n(\tau_1+s,s,\tau_2)-\alpha_1+\alpha_2]t_2} \\ & e^{-j2\pi\alpha_1 s}\, \frac{1}{T}\, r_{aa^*}\Big(-[\beta_n(\tau_1 + s, s, \tau_2) - \alpha_1 + \alpha_2]T; \gamma_{s,T} \Big)\, \mathrm{d}s \end{aligned} \tag{31}$$

with $\gamma_{s,T} \triangleq (t_2 - t_1 + s)/T$ and

$$r_{aa^*}(\beta; s) \triangleq \int_{\mathbb{R}} a(t+s)\, a^*(t)\, e^{-j2\pi\beta t}\, \mathrm{d}t\,. \tag{32}$$

In (29)–(31), for notational simplicity, $\alpha'_{n'}(\cdot) \equiv \alpha^{(n')}_{yy^*}(\cdot)$, $\alpha''_{n''}(\cdot) \equiv \alpha^{(n'')}_{x^{(*)}x^{(*)*}}(\cdot)$, $\alpha'''_{n'''}(\cdot) \equiv \alpha^{(n''')}_{yx^{(*)*}}(\cdot)$, and $\alpha'^{v}_{n^{\prime v}}(\cdot) \equiv \alpha^{(n^{\prime v})}_{x^{(*)}y^*}(\cdot)$. □

In the special case of ACS processes, Theorems 3.1 and 3.2 reduce to the well known results of [2], [3], [5], [6].

4. MEAN-SQUARE CONSISTENCY

In this section, the cyclic cross-correlogram is shown to be a mean-square consistent estimator of the cyclic cross-correlation function.

Assumption 4.1 For any choice of z_1 and z_2 in $\{x, x^*, y, y^*\}$ it results

$$\sum_{n \in \mathbb{I}_{z_1,z_2}} \int_{\mathbb{R}} \left| R^{(n)}_{z_1 z_2}(s) \right| \mathrm{d}s < \infty\,. \tag{33}$$

□

Assumption 4.2 $\forall \tau_1, \tau_2 \in \mathbb{R}$ it results

$$\sum_{n \in \mathbb{I}_4} \int_{\mathbb{R}} \left| C^{(n)}_{yx^{(*)}y^*x^{(*)*}}(s+\tau_1, s, \tau_2) \right| \mathrm{d}s < \infty . \tag{34}$$

□

Assumptions 4.1 and 4.2 are referred to as mixing conditions and are generally satisfied if the involved stochastic processes have finite or practically finite memory, i.e., if $z_1(t)$ and $z_2(t+s)$, are asymptotically ($s \to \infty$) independent.

By taking the limit of the expected value of the cyclic-cross correlogram (27) as the data-record length T approaches infinite, the following result can be proved [12], where the made assumptions allow to interchange the order of limit and sum operations.

Theorem 4.1 *Asymptotic Expected Value of the Cyclic Cross-Correlogram.* Let $y(t)$ and $x(t)$ be wide-sense jointly GACS stochastic processes with cross-correlation function (12). Under Assumptions 3.1a, 3.2, and 3.5, the asymptotic expected value of the cyclic cross-correlogram $R_{yx^{(*)}}(\alpha, \tau; t_0, T)$ is given by

$$\lim_{T \to \infty} \mathrm{E}\left\{ R_{yx^{(*)}}(\alpha, \tau; t_0, T) \right\} = R_{yx^{(*)}}(\alpha, \tau) . \tag{35}$$

□

Starting from Theorem 3.1, the following result can be proved [12], where the made assumptions allow to interchange the order of limit and sum operations.

Theorem 4.2 *Asymptotic Covariance of the Cyclic Cross-Correlogram.* Let $y(t)$ and $x(t)$ be zero-mean wide-sense jointly GACS stochastic processes with cross-correlation function (12). Under Assumptions 3.1–3.5, 4.1, and 4.2, the asymptotic covariance of the cyclic cross-correlogram $R_{yx^{(*)}}(\alpha, \tau; t_0, T)$ is given by

$$\lim_{T \to \infty} T \operatorname{cov}\left\{ R_{yx^{(*)}}(\alpha_1, \tau_1; t_1, T), R_{yx^{(*)}}(\alpha_2, \tau_2; t_2, T) \right\} = \mathscr{T}_1' + \mathscr{T}_2' + \mathscr{T}_3' \tag{36}$$

where

$$\mathscr{T}_1' \triangleq \mathscr{E}_a \sum_{n'} \sum_{n''} \int_{\mathbb{R}} R^{(n')}_{yy^*}(\tau_1 - \tau_2 + s)\, R^{(n'')}_{x^{(*)}x^{(*)*}}(s)\, e^{j2\pi\alpha'_{n'}(\tau_1-\tau_2+s)\tau_2}\, e^{-j2\pi\alpha_1 s}\, \delta_{[\alpha'_{n'}(\tau_1-\tau_2+s)+\alpha''_{n''}(s)-\alpha_1+\alpha_2]}\, \mathrm{d}s \tag{37}$$

$$\mathscr{T}_2' \triangleq \mathscr{E}_a \sum_{n'''} \sum_{n^{\prime v}} \int_{\mathbb{R}} R^{(n''')}_{yx^{(*)*}}(\tau_1 + s)\, R^{(n^{\prime v})}_{x^{(*)}y^*}(s - \tau_2)\, e^{j2\pi\alpha'^{v}_{n^{\prime v}}(s-\tau_2)\tau_2}\, e^{-j2\pi\alpha_1 s}\, \delta_{[\alpha'''_{n'''}(\tau_1+s)+\alpha'^{v}_{n^{\prime v}}(s-\tau_2)-\alpha_1+\alpha_2]}\, \mathrm{d}s \tag{38}$$

$$\mathscr{T}_3' \triangleq \mathscr{E}_a \sum_{n} \int_{\mathbb{R}} C^{(n)}_{yx^{(*)}y^*x^{(*)*}}(\tau_1 + s, s, \tau_2)\, e^{-j2\pi\alpha_1 s}\, \delta_{[\beta_n(\tau_1+s,s,\tau_2)-\alpha_1+\alpha_2]}\, \mathrm{d}s \tag{39}$$

with $\mathscr{E}_a \triangleq \int_{\mathbb{R}} |a(u)|^2 \, \mathrm{d}u$. □

From Theorem 4.1 it follows that the cyclic cross-correlogram (20), as a function of $(\alpha, \tau) \in \mathbb{R} \times \mathbb{R}$, is an asymptotically unbiased estimator of the cyclic cross-correlation function (13). Moreover, from Theorem 4.2 it follows that its asymptotic variance is $\mathcal{O}(T^{-1})$, where $\mathcal{O}$ is the "big oh" Landau symbol. Therefore, it is mean-square consistent. Consequently, for ACS processes, we obtain as a special case the well known results of [2], [3], [5], [6].

Note that, as it is well known, for (jointly) ACS processes, if the estimation of the cyclic cross-correlation function is performed at a fixed cycle frequency, say α_0, then the not exact knowledge of the value of α_0 leads to a biased estimate. Moreover, an analogous result can be found for GACS processes if the estimation is performed along a fixed lag-dependent cycle frequency curve $\alpha = \alpha_n(\tau)$. However, if the estimation of the cyclic cross-correlogram $R_{yx^{(*)}}(\alpha, \tau; t_0, T)$ as a function of the two variables (α, τ) is performed, then, in the limit for $T \to \infty$, the regions of the (α, τ) plane where $R_{yx^{(*)}}(\alpha, \tau; t_0, T)$ is significantly different from zero tend to the support curves of the cyclic cross-correlation function, that is, the curves $\alpha = \alpha_n(\tau)$, $n \in \mathbb{I}$ (see (18)). Therefore, the unknown lag-dependent cycle frequencies can potentially be estimated.

REFERENCES

[1] A. S. Besicovitch, *Almost Periodic Functions*. London: Cambridge University Press, 1932 and New York: Dover Publications, Inc., 1954.

[2] A. V. Dandawaté and G. B. Giannakis, "Asymptotic theory of mixed time averages and *k*th-order cyclic-moment and cumulant statistics," *IEEE Transactions on Information Theory*, vol. 41, pp. 216-232, January 1995.

[3] D. Dehay and H. L. Hurd, "Representation and estimation for peridically and almost periodically random processes," in *Cyclostationarity in Communications and Signal Processing,* (W.A. Gardner, Ed.). IEEE Press, New York, 1994.

[4] W.A. Gardner and C.M. Spooner, "The cumulant theory of cyclostationary time-series, Part I: Foundation," *IEEE Trans. Signal Processing*, vol. 42, pp. 3387-3408, December 1994.

[5] M. J. Genossar, H. Lev-Ari, and T. Kailath, "Consistent estimation of the cyclic autocorrelation," *IEEE Transactions on Signal Processing*, vol. 42, pp. 595-603, March 1994.

[6] H. L. Hurd, "Correlation theory of almost periodically correlated processes," *Journal of Multivariate Analysis*, vol. 31, pp. 24-45, April 1991.

[7] L. Izzo and A. Napolitano, "The higher-order theory of generalized almost-cyclostationary time-series," *IEEE Trans. Signal Processing*, vol. 46, pp. 2975-2989, November 1998.

[8] L. Izzo and A. Napolitano, "Linear time-variant transformations of generalized almost-cyclostationary signals, Part I: Theory and method", *IEEE Transactions on Signal Processing*, vol. 50, pp. 2947-2961, December 2002.

[9] L. Izzo and A. Napolitano, "Linear time-variant transformations of generalized almost-cyclostationary signals, Part II: Development and applications", *IEEE Transactions on Signal Processing*, vol. 50, pp. 2962-2975, December 2002.

[10] L. Izzo and A. Napolitano, "Sampling of generalized almost-cyclostationary signals", *IEEE Transactions on Signal Processing*, vol. 51, pp. 1546-1556, June 2003.

[11] A. Napolitano, "Uncertainty in measurements on spectrally correlated stochastic processes," *IEEE Transactions on Information Theory*, vol. 49, pp. 2172-2191, September 2003.

[12] A. Napolitano, "Asymptotic properties of statistical function estimators for generalized almost-cyclostationary signals," in preparation.

[13] C. M. Spooner and W. A. Gardner, "The cumulant theory of cyclostationary time-series, Part II: Development and aplications," *IEEE Trans. Signal Processing*, vol. 42, pp. 3409-3429, December 1994.