AN ITERATIVE REWEIGHTED LEAST-SQUARES ALGORITHM FOR THE DESIGN OF 2D IIR FILTERS

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ABSTRACT

We present a new method for optimizing two-dimensional IIR filters with respect to a p-norm error criterion, especially for large p, when quasi-equiripple filters are obtained. We combine a Gauss-Newton convexification of the criterion with the iterative reweighted least-squares (IRLS) algorithm. Stability is obtained by using 1-D or 2-D convex stability domains based on a positive realness description. A final optimization of the numerator, keeping the denominator fixed, allows more freedom in choosing the value of a key parameter.

1. INTRODUCTION

A quarter plane 2-D IIR filter has the transfer function

$$H(z_1,z_2) = \frac{B(z_1,z_2)}{A(z_1,z_2)} = \frac{\sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2} b_{k_1,k_2} z_1^{-k_1} z_2^{-k_2}}{\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} a_{k_1,k_2} z_1^{-k_1} z_2^{-k_2}}, \quad (1)$$

with $a_{00} = 1$. Optimization of 2-D IIR filters is a challenging problem due to at least two factors. The first is its complexity, which becomes significant even for moderate degrees. The second is the difficulty in guaranteeing the stability of the filter without imposing too restrictive constraints that might make unreachable many near-optimum filters. If the factorization $A(z_1,z_2) = A_1(z_1)A_2(z_2)$ is possible, then the denominator is separable; in this case, complexity is reduced and stability easier to enforce.

Given a complex desired frequency response $D(\omega_1,\omega_2)$, with $\omega_1,\omega_2 \in [-\pi,\pi]$, whose values are D_{ℓ_1,ℓ_2} on a grid of frequencies defined by $\omega_1(\ell_1)$, $\omega_2(\ell_2)$, $\ell_1 = 1 : L_1$, $\ell_2 = 1 : L_2$, the design problem is to find a filter (1) whose coefficients minimize the p-norm error

$$J(A,B,p) = \sum_{\ell_1=1}^{L_1}\sum_{\ell_2=1}^{L_2} \lambda_{\ell_1,\ell_2} \left| D_{\ell_1,\ell_2} - \frac{B(\omega_1(\ell_1),\omega_2(\ell_2))}{A(\omega_1(\ell_1),\omega_2(\ell_2))} \right|^p, \quad (2)$$

where $A(\omega_1,\omega_2)$ and $B(\omega_1,\omega_2)$ are the frequency responses of the denominator and numerator of (1). The numbers $\lambda_{\ell_1,\ell_2} \geq 0$ represent weights. For further use, we denote $A_{\ell_1,\ell_2} = A(\omega_1(\ell_1),\omega_2(\ell_2))$ and $B_{\ell_1,\ell_2} = B(\omega_1(\ell_1),\omega_2(\ell_2))$.

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We are interested mainly in the optimization of (2) for relatively large values of p, when the resulting filter is almost equiripple and thus (2) can replace a Chebyshev criterion.

Although the research on optimization methods for 2-D IIR filter design is more than 30 years old (see e.g. [2]), there are relatively few results. Due to the easier approach and to the fact that they cover the class of quadrantally symmetric responses, separable filters received more attention; among the recent papers, good results are obtained in [7, 8]. The trend over the years has been to go from the use of general nonlinear optimization methods to specific methods; in the latest years, several algorithms appealed to advanced convex optimization techniques like semidefinite programming (SDP) [11].

The contributions of this paper are

- to adapt the iterative reweighted least squares (IRLS) algorithm of Burrus *et al* [1], for the p-norm optimization of 2-D IIR filters with *fixed* denominator;
- to combine i) the IRLS algorithm, ii) the Gauss-Newton optimization principle [5, 7, 4], and iii) convex stability domains in 1-D [4] or 2-D [3], into a single algorithm for 2-D IIR filter (with separable or nonseparable denominator) optimization; the algorithm is iterative and in each iteration an SDP problem is solved.

The two above algorithms are presented in sections 2 and 3, respectively. We run them in reversed order, i.e. the second algorithm to obtain a full filter, then the first to further optimize the numerator. Experimental results are presented in section 4.

2. OPTIMIZATION OF 2-D IIR FILTERS WITH FIXED DENOMINATOR

We assume in this section that the denominator $A(z_1,z_2)$ of the IIR filter (1) is given. Our purpose is to find the numerator $B(z_1,z_2)$ optimizing (2) for the given denominator. A first remark is that in the least squares (LS) problem (when $p = 2$), the optimal numerator can be found directly by solving (in LS sense) an overdetermined system of linear equations; we denote $B = B_{LS}(A,\lambda)$ such an optimal numerator. To extend the solution to an arbitrary $p > 2$, we adapt the IRLS algorithm [1]. The resulting algorithm is presented in figure 1. The main idea of IRLS is to solve successive LS problems (step 5 in figure 1) where the weights are computed as in (4) such to include the "more-than-square" part of the p-norm criterion (2). The values of p start from 2 and grow in geometric progression with ratio $\gamma > 1$ as in step 3. The update (5) of the numerator uses a fixed convex combination of the current numerator $B^{(i)}$ and the cur-

Algorithm IRLS_fixedA

Input: denominator $A(z_1,z_2)$ and degrees m_1, m_2 of (1);
desired response D and the weights λ from (2),
on a grid of frequencies with $L_1 \times L_2$ points;
value p for the chosen norm;
ratio γ;
a tolerance ε.

1. Set $p_0 = 2$ and compute $B^{(1)} = B_{LS}(A,\lambda)$.
2. Set $i = 1$.
3. Put $p_i = \min(\gamma p_{i-1}, p)$.
4. Compute new LS weights (for $\ell_1 = 1:L_1$, $\ell_2 = 1:L_2$)

$$\tilde{\lambda}_{\ell_1,\ell_2} = \lambda_{\ell_1,\ell_2} \left| D_{\ell_1,\ell_2} - \frac{B^{(i)}_{\ell_1,\ell_2}}{A_{\ell_1,\ell_2}} \right|^{p_i-2}. \tag{4}$$

5. Compute $\tilde{B} = B_{LS}(A,\tilde{\lambda})$.
6. Compute new numerator

$$B^{(i+1)} = \frac{1}{p_i-1}\tilde{B} + \frac{p_i-2}{p_i-1}B^{(i)}. \tag{5}$$

7. If

$$\frac{J(A,B^{(i)},p) - J(A,B^{(i+1)},p)}{J(A,B^{(i)},p)} < \varepsilon, \tag{6}$$

stop. Otherwise, put $i = i+1$ and go back to 3.

Output: optimal numerator $B = B^{(i+1)}$.

Figure 1: IRLS algorithm for the p-norm optimization of IIR filters with fixed denominator.

rent LS solution $\tilde{B}$. The stopping decision may take different forms; we have shown only a decision based on the relative improvement of the criterion (2); in a practical implementation we should also impose a maximum number of iterations; moreover, we should provide a treatment of the (rare) situation when the value of the criterion is not improved (i.e. $J(A,B^{(i+1)},p) \geq J(A,B^{(i)},p)$); we chose to stop the algorithm and put $B = B^{(i)}$.

If we want an (almost) equiripple filter, then we set p to a value sufficiently large to give good approximation of the minimax norm, but such that numerical errors do not occur in the computation of (2); we consistently used $p = 120$; larger values like e.g. 150 do not bring significant improvement and may lead occasionally to numerical overflow. We also use the criterion

$$J(A,B,\infty) = \max_{\ell_1=1:L_1} \max_{\ell_2=1:L_2} \left| D_{\ell_1,\ell_2} - \frac{B_{\ell_1,\ell_2}}{A_{\ell_1,\ell_2}} \right|, \tag{3}$$

in the stopping decision (6).

An important practical issue is the choice of the ratio γ. As reported by Burrus *et al* [1] for FIR filters, we have found that there is a sufficiently large range of values that ensure convergence; too small γ means slow decrease of the criterion (and large number of iterations), while too large γ may lead to premature stop; we have used $\gamma = 1.15$ with good results.

The main problem in using the algorithm in figure 1 is to choose the denominator A. We will discuss this matter in the following section.

We have also applied the iterative reweighting algorithm used succesfully in [9] in the 1-D case with fixed denominator. The performance of the generalization to the 2-D case was inferior to the algorithm in figure 1 in terms of values of the criterion (2), convergence and execution time. We conclude that this approach needs further study and can be now described with few words.

The underlying idea was to modify the weights of a LS optimization criterion so that the final design becomes almost equiripple, as proposed by Lim *et al* [6]. The weights corresponding to large values of the error $|D_{\ell_1,\ell_2} - B_{\ell_1,\ell_2}/A_{\ell_1,\ell_2}|$ are increased, such that at the next iteration these errors decrease. Typically, the error surface has a fast varying shape; thus, an envelope function is used instead of the exact error. In 2-D, there are many options in computing the envelope function; for example, we first compute the envelope along the ω_1 axis, then the envelope of this envelope along the ω_2 axis. Since the number of parameters to be optimized equals the number of grid points, the number of parameters to be optimized is very high, typically thousands. It is possible to impose symmetry according to the shape of the desired response; for example, when $|D(\omega_1,\omega_2)|$ was circularly symmetric, we forced the weights with equal distance from origin to be equal. However, neither of these approaches gave results worth reporting now.

3. ITERATIVE REWEIGHTING FOR IIR FILTERS

An algorithm for LS optimization of 2-D IIR filters. Several recent methods for the LS optimization of 1-D IIR filters [5, 7, 4] work upon the following general idea. We use the notations for 2-D IIR filters, since they fit naturally in this optimization framework. Suppose that, at iteration i, the current filter parameters are $A^{(i)}$ and $B^{(i)}$; the filter is stable, i.e. the zeros of $A^{(i)}$ are inside the unit circle. Let $\mathcal{D}_i$ be a convex domain including $A^{(i)}$ and containing only stable polynomials of degree (n_1,n_2). The cited methods seek polynomials $\Delta A^{(i)}$ and $\Delta B^{(i)}$ such that

$$\begin{aligned} &J(A^{(i)}+\Delta A^{(i)}, B^{(i)}+\Delta B^{(i)}, p) < J(A^{(i)},B^{(i)},p), \\ &A^{(i)}+\Delta A^{(i)} \in \mathcal{D}_i. \end{aligned} \tag{7}$$

If the search is successful, they put $A^{(i+1)} = A^{(i)} + \Delta A^{(i)}$, $B^{(i+1)} = B^{(i)} + \Delta B^{(i)}$ and continue similarly. A method is characterized by the way in which a good descent direction $\Delta A^{(i)}$, $\Delta B^{(i)}$ is found and by the description of the stability domain $\mathcal{D}_i$. The common feature of the methods is that (7) is implemented via a convex optimization problem.

We propose the extension of the 1-D method from [4] for the LS optimization of 2-D IIR filters. The stability domain $\mathcal{D}_i$ is built using a positive realness condition which may be expressed as a linear matrix inequality (LMI) function of the coefficients of the variable $\Delta A^{(i)}$. If the denominator is chosen to be separable, then the stability domain is exactly that presented in [4] (actually we have a domain for each of the two monovariable factors of the denominator). For nonseparable denominator, a 2-D positive realness stability domain was presented in [3], also in the form of an LMI. In both cases, we can enforce robust stability, in the sense that the poles of filter are inside a circle of radius $\rho < 1$.

A good descent direction is found using the Gauss-Newton (GN) method, based on the first order approximation of the filter (1) viewed as a function of its coefficients. We denote $\Delta = [\text{vec}(\Delta A^{(i)})^T \ \text{vec}(\Delta B^{(i)})^T]^T$ the vector of coefficients of the variable polynomials $\Delta A^{(i)}$, $\Delta B^{(i)}$ (a 2-D polynomial has a matrix of coefficients, which is vectorized in Δ). Also, we denote $\nabla H^{(i)}_{\ell_1,\ell_2}$ the gradient of (1) with respect to A, B, computed in $A^{(i)}$, $B^{(i)}$, for the frequencies $\omega_1(\ell_1)$, $\omega_2(\ell_2)$. An iteration of the GN method consists of solving

$$\begin{array}{ll} \min\limits_{\Delta} & \sum\limits_{\ell_1=1}^{L_1}\sum\limits_{\ell_2=1}^{L_2} \lambda_{\ell_1,\ell_2} \left| D_{\ell_1,\ell_2} - \dfrac{B^{(i)}_{\ell_1,\ell_2}}{A^{(i)}_{\ell_1,\ell_2}} - \nabla^T H^{(i)}_{\ell_1,\ell_2} \cdot \Delta \right|^2 \\ \text{s.t.} & A^{(i)} + \Delta A^{(i)} \in \mathcal{D}_i \end{array} \quad (8)$$

Moreover, in the GN method, the resulting $\Delta A^{(i)}$, $\Delta B^{(i)}$ are used as maximal steps in the descent direction and the optimal steps are computed through line search.

With the ingredients described above, we have built an algorithm for the LS design of 2-D IIR filters. In particular, the problem (8) has an SDP form. A more detailed description and numerical results will be reported elsewhere. We concentrate now on using the ideas exposed above for p-norm optimization and especially for finding almost equiripple filters.

The GN_IRLS algorithm. The new idea we propose here is to insert the Gauss-Newton iterations into the structure of the IRLS method. In each iteration we perform a basic GN step, computing updates of the current numerator *and* denominator such that an LS criterion is improved. However, the weights of the LS problem are updated like in the IRLS method, with gradually increasing p. The new algorithm, named *GN_IRLS*, is presented in figure 2.

Comparing it with the IRLS algorithm with fixed denominator (figure 1) we remark that differences in steps 5, 6 hide a similar principle. In *IRLS_fixedA* the optimal solution of the LS problem gives actually the descent direction $\tilde{B} - B^{(i)}$; then, the update (5) uses a fixed step of length $1/(p_i - 1)$. In *GN_IRLS* the descent direction is computed using a more complex optimization problem, due to the presence of a variable denominator (which imposes an approximate optimization via a convex approximation of the original problem); also, it is probably difficult to set a fixed step length, and so the line search (9) must be performed; anyway, the complexity of line search is not significant with respect to the complexity of (8), thus a hypothetical fixed step would reduce only slightly the complexity of an iteration of *GN_IRLS*, but would probably increase the number of iterations. Finally, let us remark that the initialization of *GN_IRLS* is trivial.

Continuation of GN_IRLS. We remark that, in each iteration of *GN_IRLS*, we could optimize (in terms of a p_i-norm error) the numerator $B^{(i+1)}$ keeping fixed the denominator $A^{(i+1)}$, using e.g. *IRLS_fixedA*. This would be a way to make the algorithm more flexible; otherwise, only tandem updates (9) with the same step length of the numerator and denominator are made. This may speed-up convergence, but it may be costly in terms of complexity. We found more useful to reoptimize only the final numerator. In other terms, after running *GN_IRLS*, we input the obtained denominator to *IRLS_fixedA* and end up with a better numerator; we name *GN_IRLS+* this two-stage algorithm.

Algorithm GN_IRLS

Input: degrees m_1, m_2, n_1, n_2 of (1);
$D, \lambda, L_1, L_2, p, \gamma, \varepsilon$ as in *IRLS_fixedA*

1. Set $A^{(1)}(z_1,z_2) = 1$ and compute $B^{(1)} = B_{LS}(A^{(1)}, \lambda)$.
2. Set $p_0 = 2$, $i = 1$.
3. Put $p_i = \min(\gamma p_{i-1}, p)$.
4. For $\ell_1 = 1 : L_1$, $\ell_2 = 1 : L_2$
   compute new LS weights $\tilde{\lambda}_{\ell_1,\ell_2}$ using (4), with $A^{(i)}_{\ell_1,\ell_2}$ instead of A_{ℓ_1,ℓ_2}.
5. Compute $\Delta A^{(i)}$, $\Delta B^{(i)}$ by solving the GN optimization problem (8), with weights $\tilde{\lambda}_{\ell_1,\ell_2}$.
6. Compute optimal step $\alpha^\star$ by solving the line search problem

$$\begin{array}{ll} \min\limits_{\alpha} & J(A^{(i)} + \alpha\Delta A^{(i)}, B^{(i)} + \alpha\Delta B^{(i)}, p) \\ \text{s.t.} & 0 \le \alpha \le 1 \end{array} \quad (9)$$

7. Compute new filter

$$A^{(i+1)} = A^{(i)} + \alpha^\star \Delta A^{(i)}, \ \ B^{(i+1)} = B^{(i)} + \alpha^\star \Delta B^{(i)}.$$

8. If

$$\frac{J(A^{(i)}, B^{(i)}, p) - J(A^{(i+1)}, B^{(i+1)}, p)}{J(A^{(i)}, B^{(i)}, p)} < \varepsilon,$$

stop. Otherwise, put $i = i + 1$ and go back to 3.

Output: IIR filter with $A = A^{(i+1)}$, $B = B^{(i+1)}$.

Figure 2: Gauss-Newton IRLS algorithm for the p-norm optimization of IIR filters.

4. EXPERIMENTS

We implemented our algorithms in Matlab, using the SDP library SeDuMi [10]. We present here a single example of results, for a problem proposed in [8] for 2-D IIR filters with separable denominator. The desired response is circularly lowpass, with linear phase:

$$D(\omega_1, \omega_2) = \begin{cases} e^{-j(\tau_1\omega_1 + \tau_2\omega_2)}, & \text{if } \sqrt{\omega_1^2 + \omega_2^2} \le \omega_p, \\ 0, & \text{if } \sqrt{\omega_1^2 + \omega_2^2} \ge \omega_s. \end{cases}$$

The group delays are $\tau_1 = \tau_2 = 8$, the passband and stopband radii are $\omega_p = 0.5\pi$, $\omega_s = 0.7\pi$. The weights in the criterion (2) are 1 in the passband and stopband and zero in the transition band. The stability constraints are chosen such that the poles are inside a circle with radius $\rho = \sqrt{0.8}$, like in [8]. The degrees of the filter (1) are $m_1 = m_2 = 12$, $n_1 = n_2 = 8$. We took $L_1 = L_2 = 80$ and a uniform grid of frequency points and also $p = 120$, $\varepsilon = 10^{-5}$.

The main question is how the value of γ affects the results. In *IRLS_fixedA* we always use $\gamma = 1.15$. In figure 3 we present the values of the Chebyshev criterion (3) obtained after running *GN_IRLS* and *GN_IRLS+*. We remark that for *GN_IRLS* it would be difficult to forecast the optimal value of γ; the final optimization of the numerator (with fixed denominator) has a regularization effect; for *GN_IRLS+* there is a

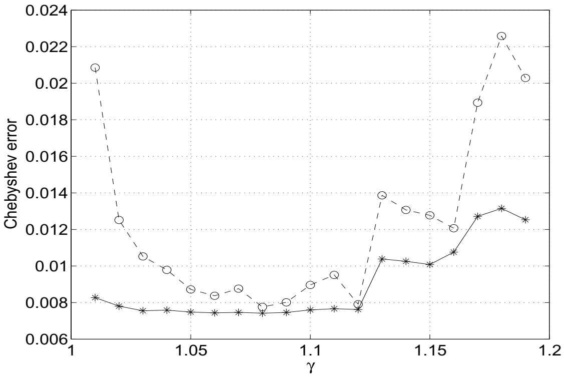

Figure 3: Maximum ripple of 2-D IIR filters obtained by running *GN_IRLS* (circles) and *GN_IRLS+* (stars).

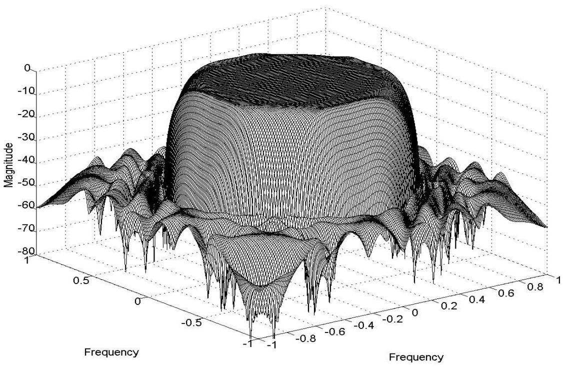

Figure 4: Magnitude of the frequency response.

range of values of γ, from 1.02 to 1.12, in which the Chebyshev error has similar values, all below 0.008. We obtained similar figures for other examples; as a general rule, γ should be given a higher value for smaller degrees of the filters. In the same time, the values of γ should be correlated with the final value of the variable p_i; a small final p_i and convergence of *GN_IRLS* indicate that a larger γ might be better; on the contrary, if the final p_i is small and the method was stopped due to an increase of the criterion (lack of convergence), then a smaller γ should be tried. We conclude that there is a fairly large range of values γ for which good filters are obtained and that experimental tuning of γ is simple.

The frequency response of the best filter obtained in the considered example is shown in figures 4 (magnitude response) and 5 (group delay). Our filter has better performances than the design reported in [8]: 42.5dB vs. 39.4 stopband attenuation; 0.0074 vs. 0.0081 maximum amplitude deviation in passband. The maximmum group delay error in passband is 0.526 (not given in [8]). The execution time of our method is about 6 minutes (5 of them for *GN_IRLS*) on a Pentium III PC at 1GHz, which compares favorably with the 27 minutes reported in [8] on a slightly slower computer.

5. CONCLUSIONS

We have presented a method for designing 2-D IIR filters using a p-norm criterion, with equiripple filters as main target. We combined the Gauss-Newton and the iterative reweighted LS [1] algorithms, using also LMI stability constraints based on positive realness [4]. The new method produces good filters with convenient execution time.

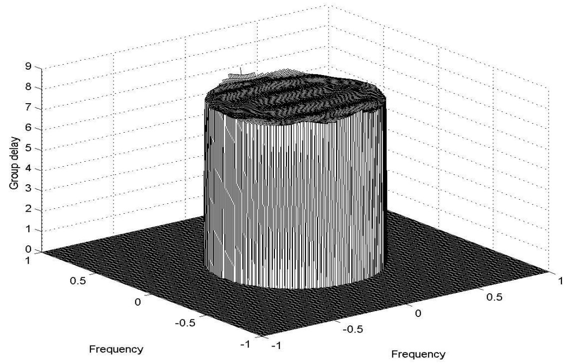

Figure 5: Group delay.

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