SEQUENCES ORDERING IN MINIMUM-PHASE SEQUENCES

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ABSTRACT

The goal of this paper is to determine the situations when a given finite set of complex numbers can be reordered such that the novel corresponding complex sequence is a minimum-phase one.

1. MOTIVATION AND PROBLEM STATEMENT

Many signal and image processing applications deal with signal reconstruction based on modulus of the Fourier transform. For instance Fourier descriptors (FD) magnitude has some invariant properties very useful in detecting shapes regardless of their size and orientation [1]. However, the FD magnitude alone is generally inadequate for reconstruction of the original shape. Indeed, FD as the discrete Fourier transform of the sampled boundary, can be obtained by sampling the Fourier transform. Thus the reconstruction of a complex sequence from its FD magnitude can be possible only when its corresponding z-transform is a minimum-phase function [2], i.e. all zeros and poles of its z-transform are inside the unit circle (such sequence is referred as minimum-phase sequence). On the other hand, we can find applications where there is no preliminary request to pick the boundary samples set in clockwise or trigonometric order. This means that one may select the succession of the phasors, and the resulting sequence is a minimum-phase one (Fig. 1). Nevertheless, this may lead to loosing some invariant properties. In the following we are interested to examine whether any

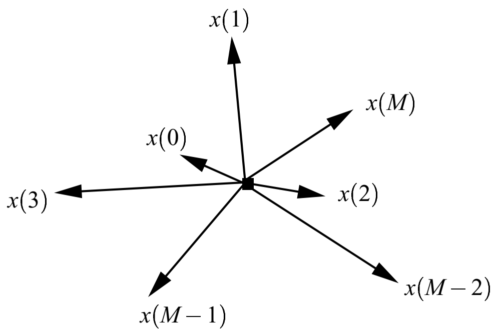

Figure 1: The sampled boundary reordered.

finite set of complex numbers can be reordered such that the corresponding complex sequence is a minimum-phase one. We shall see that this happens only for special cases and we shall try to characterize this type of sets. In this work we shall avoid zero samples, since by using another selection of coordinates the sampled boundary can skip the origin. Moreover, to simplify our analysis, we shall discuss only the case, when the samples modulus differs, i.e. they are not located on an arc of a circle.

2. FRAMEWORK

We shall focus our study on finite length complex valued sequences $x(n)$, $n = \overline{0,M}$. The z-transform of $x(n)$ is:

$$X(z) = x(0) + x(1)z^{-1} + \cdots + x(M)z^{-M}. \tag{1}$$

If we restrict our interest only to transfer functions $X(z)$, of the form given by (1), then $x(n)$ and $X(z)$ can be considered as the impulse response and the transfer function of an FIR filter. The Fourier transform of $x(n)$ is given by: $X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$. For $N = M+1$ the discrete Fourier transform of the given sequence $x(n)$ is: $\widetilde{X}(k) = X(z)|_{z=e^{j\frac{2\pi k}{N}}}$, where $k = \overline{0,N-1}$. Because the length of the sequence is finite and $M+1 = N$, $\widetilde{X}(k)$ are exactly the samples of the Fourier transform $X(e^{j\omega})$:

$$\widetilde{X}(k) = X(e^{j\omega})|_{\omega=\frac{2\pi k}{N}},\ k = 0,1,\ldots,N-1,$$

and no frequency aliasing occurs when we reconstruct $X(e^{j\omega})$ from spectrum samples $\widetilde{X}(k)$ [3]. It follows that z-transform can be found from the DFT samples:

$$X(z) = \sum_{n=0}^{N-1} x(n)z^{-n} = \frac{1}{N}\sum_{n=0}^{N-1}\left[\widetilde{X}(k)e^{\frac{j2\pi kn}{N}}\right]z^{-n}.$$

Thus it is suggesting that a change in the way we pick the samples $x(n)$ (by modifying the succession) may affect the $X(z)$'s pole-zeros configuration, and consequently if $X(z)$ is minimum phase or non-minimum phase function. Nevertheless, our interest is to identify the minimum-phase sequences, since, in their case, from:

$$|\widetilde{X}(k)| = X(z)X(z^{-1})|_{z=e^{\frac{j2\pi k}{N}}},$$

the ambiguity of zero allocation is not anymore present [4].

3. REORDERING THE SEQUENCE

One of the properties of minimum-phase systems which may start our discussion is the following [5]:

Theorem 1 (Energy concentration) *If the systems $H(z)$ (non-minimum phase) and $H_m(z)$ (minimum-phase) have the same magnitude response and their response to the same input are $g(n)$ and $y(n)$, respectively, then for any n_0,*

$$\sum_{n=0}^{n_0} |y(n)|^2 \geq \sum_{n=0}^{n_0} |g(n)|^2. \tag{2}$$

From this property, one may suppose that a condition for a certain sequence $x(n)$ to be a minimum-phase is to have its energy concentrated around origin such that:

$$|x(0)| > |x(1)| > \cdots > |x(M)| > 0. \tag{3}$$

This happens for real positive sequences, as a consequence of Eneström-Kakeya theorem [6].

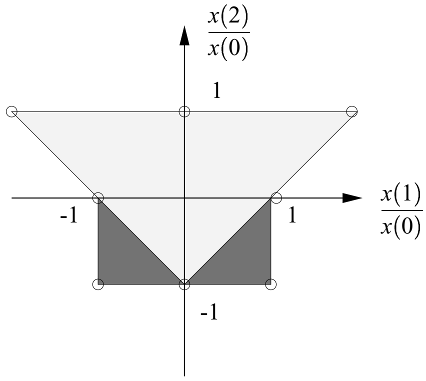

Figure 2: The stability triangle and the area where the system is unstable.

Theorem 2 (Eneström-Kakeya Theorem) Suppose that $0 < a_0 < a_1 < \cdots < a_M$. Then all the zeros of the polynomial $p(z) = a_0 + a_1 z + \cdots + a_M z^M$ lie in the disc $\{z : |z| < 1\}$.

Indeed, if the set $\{x(n)|n = 0, 1, \ldots, M\}$ respects the condition (3), by taking $a_k = x(M-k)$, we retrieve the situation described in Eneström-Kakeya theorem. We obtain:

Proposition 1 *Suppose that $\{x(n)|n = 0, 1, \ldots, M\}$ satisfies the condition (3), then $X(z)$ as given by (1) is a minimum-phase function.*

Unfortunately, this is not anymore valid if we skip to real sequences, with both positive and negative samples. To show this, it is enough to consider the case $M = 2$, and to compare the condition (3) rewritten here as:

$$|x(0)| > |x(1)| > |x(2)| > 0, \quad (4)$$

with the well known stability condition (9) for a quadratic polynomial (Appendix A.2):

$$\left|\frac{x(2)}{x(0)}\right| < 1; \quad \left|\frac{x(1)}{x(0) + x(2)}\right| < 1. \quad (5)$$

We can easily see that (4) and (5) are different. Furthermore Fig. 2 presents the area (two small rectangular triangles) which satisfies (4), but does not belong to the stability triangle.

However, we shall show that

Proposition 2 *For any set of real numbers $\{x(i)|i = \overline{0,2}\}$, there is a choice of ordering them such that the corresponding new sequence is a minimum-phase one.*

Proof: We begin with a lemma which is proven in Appendix B.

Lemma 1 *If A, B and C are real numbers such that $|A| > |B| > |C| > 0$, at least one of the following inequalities are satisfied:*
1. $|B| < |A + C|$;
2. $|C| < |A + B|$;
3. $|A| < |B + C|$.

To prove Proposition 2, we just make the following corresponding substitutions:
1. $A = x(0)$; $B = x(1)$; $C = x(2)$;
2. $A = x(0)$; $C = x(1)$; $B = x(2)$;
3. $B = x(0)$; $A = x(1)$; $C = x(2)$.

which will satisfy the conditions (5). □

One may ask if a similar statement as Proposition 2 can be found for other values of $M > 2$. For $M = 3$ it can be shown that the answer is positive only in special cases. Actually we have:

Proposition 3 *For any set of real numbers $\{x(i)|i = \overline{0,3}\}$, which differ in modulus and satisfying both $x(0)x(1)x(2)x(3) > 0$ and $x(0) + x(1) + x(2) + x(3) \neq 0$, there is a choice of ordering them such that the corresponding new sequence is a minimum-phase one.*

Proof: Again we begin with a lemma which is proven in Appendix C.

Lemma 2 *Let A, B, C and D be real positive numbers such that $A > B > C > D > 0$.*

1. *Then all the zeros of the following polynomials:*
   - $P_1(z) = Az^3 + Bz^2 + Cz + D$;
   - $P_2(z) = Az^3 - Bz^2 + Cz - D$;
   - $P_3(z) = Az^3 - Cz^2 + Bz - D$;
   - $P_4(z) = Az^3 + Cz^2 + Bz + D$.

   lie inside the unit circle.
2. *If $A + D > B + C$, then all the zeros of the polynomial $P_5(z) = Az^3 - Cz^2 - Bz + D$ lie inside the unit circle.*
3. *If $A + D < B + C$, then all the zeros of the polynomial $P_6(z) = Bz^3 - Az^2 + Cz - D$ lie inside the unit circle.*

To prove Proposition 3, we start by assuming that the set has been ordered such that:

$$|x(0)| > |x(1)| > |x(2)| > |x(3)| > 0,$$

and consequently let be $A = |x(0)|$, $B = |x(1)|$, $C = |x(2)|$ and $D = |x(3)|$.
Case 1. All the numbers have the same sign. Then we can choose $X(z) = \text{sign}[x(0)]P_1(z)$ or $X(z) = \text{sign}[x(0)]P_4(z)$.
Case 2. $x(0)$ and $x(1)$ have the same sign, but not the same as $x(2)$ and $x(3)$. Then we can select $X(z) = \text{sign}[x(0)]P_3(z)$.
Case 3. $x(0)$ and $x(1)$ have the different sign, and the sign of $x(0)$ is the same as $x(2)$ and the sign of $x(1)$ is the same as of $x(3)$. Then we can pick $X(z) = \text{sign}[x(0)]P_2(z)$.
Case 4. $x(0)$ and $x(1)$ have the different sign, and the sign of $x(0)$ is the same as $x(3)$ and the sign of $x(1)$ is the same as of $x(2)$. In this situation, to make a choice we need also the sign of the diference:

$$|x(0)| + |x(3)| - |x(1)| - |x(2)| = \text{sign}[x(0)][x(0) + x(1) + x(2) + x(3)].$$

Case 4a. If $|x(0)| + |x(3)| - |x(1)| - |x(2)| > 0$, then we can choose $X(z) = \text{sign}[x(0)]P_5(z)$.
Case 4b. If $|x(0)| + |x(3)| - |x(1)| - |x(2)| < 0$, then we can select $X(z) = \text{sign}[x(0)]P_6(z)$. □

Simulations show that for $M = 3$ we can find situations that no ordering of real sequences will produce a minimum-phase sequence. One example is presented in Table 1. Our simulations also verify that the sequences where any kind of ordering fails consists of samples with an odd number of plus and minus signs. In such situation, a special case appears when the sum of the numbers is zero, when the zeros may lie outside the unit disk or on the unit circle. It follows also that for $M \geq 3$ and for the case complex sequences one can find sequences that no ordering will provide a minimum-phase sequence.

Finally, we shall analyze whether a similar situation as mentioned for real sequences in Proposition 2, happens also for complex sequences. i.e. for any set of complex numbers $\{x(0), x(1), x(2)\}$, there is a choice of ordering them such that the corresponding new sequence is a minimum-phase one. The Schur-Cohn conditions are the following (Appendix A.2):

$$\left|\frac{x(2)}{x(0)}\right| < 1; \quad \left|\frac{\frac{x(1)}{x(0)} - \frac{x(2)}{x(0)} \cdot \frac{x^*(1)}{x^*(0)}}{1 - \left|\frac{x(2)}{x(0)}\right|^2}\right| < 1, \quad (6)$$

$x(0)$	$x(1)$	$x(2)$	$x(3)$	z_1	z_2	z_3	$\max_{i=1,2,3}\{\lvert z_i\rvert\}$
-1	-2	-3	5	-1.4473 + 1.8694j	-1.4473 - 1.8694j	0.8946	2.3642
-1	-2	5	-3	0.8067 + 0.4236j	0.8067 - 0.4236j	-3.6135	3.6135
-1	-3	-2	5	-1.9521 + 1.3112j	-1.9521 - 1.3112j	0.9042	2.3516
-1	-3	5	-2	-4.2780	0.6390 + 0.2432j	0.6390 - 0.2432j	4.2780
-1	5	-2	-3	4.3885	1.1873	-0.5758	4.3885
-1	5	-3	-2	4.1642	1.2271	-0.3914	4.1642
-2	-1	-3	5	-0.7017 - 1.5084j	-0.7017 + 1.5084j	0.9033	1.6636
-2	-1	5	-3	-2.0637	0.7818 - 0.3400j	0.7818 + 0.3400j	2.0637
-2	-3	-1	5	-1.2093 - 1.1222j	-1.2093 + 1.1222j	0.9186	1.6497
-2	-3	5	-1	-2.5550	0.8149	0.2401	2.5550
-2	5	-1	-3	1.6180	1.5000	-0.6180	1.6180
-2	5	-3	-1	1.3669 - 0.5203j	1.3669 + 0.5203j	-0.2338	1.4625
-3	-1	-2	5	-0.6257 - 1.1933j	-0.6257 + 1.1933j	0.9180	1.3474
-3	-1	5	-2	-1.6180	0.6667	0.6180	1.6180
-3	-2	-1	5	-0.7954 - 1.0820j	-0.7954 + 1.0820j	0.9241	1.3429
-3	-2	5	-1	-1.7368	0.8423	0.2279	1.7368
-3	5	-1	-2	1.0756 - 0.4678j	1.0756 + 0.4678j	-0.4846	1.1729
-3	5	-2	-1	0.9717 - 0.5102j	0.9717 + 0.5102j	-0.2767	1.0975
5	-1	-2	-3	1.0821	-0.4410 - 0.6000j	-0.4410 + 0.6000j	1.0821
5	-1	-3	-2	1.0886	-0.4443 - 0.4123j	-0.4443 + 0.4123j	1.0886
5	-2	-1	-3	1.0893	-0.3446 - 0.6573j	-0.3446 + 0.6573j	1.0893
5	-2	-3	-1	1.1060	-0.3530 - 0.2371j	-0.3530 + 0.2371j	1.1060
5	-3	-1	-2	1.1070	-0.2535 - 0.5450j	-0.2535 + 0.5450j	1.1070
5	-3	-2	-1	1.1179	-0.2589 - 0.3345j	-0.2589 + 0.3345j	1.1179

Table 1: Example: $\{x(0),x(1),x(2),x(3)\} = \{-1,-2,-3,5\}$, when no ordering will provide a minimum-phase sequence.

To simplify our analysis, let us consider $x(1) = x(0)\cdot\rho_1 e^{j\theta_1}$ and $x(2) = x(0)\cdot\rho_2 e^{j\theta_2}$, where $\rho_i \geq 0,\ i = 1,2$. The previous condition (6) can be written as follows:

$$\rho_2 < 1; \quad \left|\rho_1 e^{j\theta_1} - \rho_1\rho_2 e^{\theta_2-\theta_1}\right| < 1-\rho_2^2. \tag{7}$$

The last inequality is valid whenever:

$$\cos(2\theta_1-\theta_2) > \frac{1+\rho_1^2-\left(\frac{1-\rho_2^2}{\rho_1}\right)^2}{2\rho_1}. \tag{8}$$

Note that the right-hand side of (8) is greater than 1, if $\rho_1 > 1+\rho_2$ and smaller than -1 , if $\rho_1 < 1-\rho_2$. It follows:

Proposition 4 *For any set of complex numbers $\{x(i)|i=\overline{0,2}\}$ which may be phasors of a triangle, there is a choice of ordering them such that the corresponding new sequence is a minimum-phase one.*

A. SCHUR-COHN STABILITY TEST

A.1 Schur-Cohn recursion

Let $A_M(z)$ be a complex polynomial of order M in z^{-1}:

$$A_M(z) = \alpha_M(0)+\alpha_M(1)z^{-1}+\cdots+\alpha_M(M)z^{-M},\ \alpha_M(0)=1.$$

All the zeros of $A_M(z)$ lie inside the unit circle if and only if $|k_m| < 1$, for $m = M, M-1,\cdots,1$, where [7]:

$$k_m = \alpha_m(m);\ B_m(z) = z^{-m}A_m^*(z^{-1}),$$

$$A_{m-1}(z) = \frac{A_m(z)-k_m B_m(z)}{1-|k_m|^2}.$$

A.2 Second order polynomial

For a second order complex polynomial in z^{-1}, the Schur-Cohn recursion is the following one:

$$k_2 = \alpha_2(2);\ B_2(z) = \alpha_2^*(2)+\alpha_2^*(1)z^{-1}+\alpha_2(0)z^{-2},$$

$$A_1(z) = \frac{A_2(z)-k_2B_2(z)}{1-|k_2|^2} = 1-\frac{\alpha_2(1)-\alpha_2(2)\alpha_2^*(1)}{1-|\alpha_2(2)|^2}z^{-1},$$

and all the zeros lie inside the closed unit disk if and only if

$$|\alpha_2(2)| < 1; \quad \left|\frac{\alpha_2(1)-\alpha_2(2)\alpha_2^*(1)}{1-|\alpha_2(2)|^2}\right| < 1. \tag{9}$$

A.3 Third order polynomial

For a third order real polynomial in z^{-1}, the Schur-Cohn recursions are the following:

$$k_3 = \alpha_3(3); B_3(z) = \alpha_3(3)+\alpha_3(2)z^{-1}+\alpha_3(1)z^{-2}+z^{-3},$$

$$A_2(z) = 1+\frac{\alpha_3(1)-\alpha_3(2)\alpha_3(3)}{1-\alpha_3^2(3)}z^{-1}+\frac{\alpha_3(2)-\alpha_3(1)\alpha_3(3)}{1-\alpha_3^2(3)}z^{-2},$$

$$k_2 = \frac{\alpha_3(2)-\alpha_3(1)\alpha_3(3)}{1-\alpha_3(3)^2};$$

$$B_2(z) = \frac{\alpha_3(2)-\alpha_3(1)\alpha_3(3)}{1-\alpha_3^2(3)}+\frac{\alpha_3(1)-\alpha_3(2)\alpha_3(3)}{1-\alpha_3^2(3)}z^{-1}+z^{-2},$$

$$A_1(z) = 1+\frac{\alpha_3(1)-\alpha_3(2)\alpha_3(3)}{1-\alpha_3^2(3)+\alpha_3(2)-\alpha_3(1)\alpha_3(3)}z^{-1},$$

and all the zeros lie inside the closed unit disk if and only if

$$\begin{aligned} &|\alpha_3(3)| < 1; \left|\frac{\alpha_3(2)-\alpha_3(1)\alpha_3(3)}{1-\alpha_3(3)^2}\right| < 1; \\ &\left|\frac{\alpha_3(1)-\alpha_3(2)\alpha_3(3)}{1-\alpha_3^2(3)+\alpha_3(2)-\alpha_3(1)\alpha_3(3)}\right| < 1. \end{aligned} \tag{10}$$

B. PROOF OF LEMMA 1

At least one of the numbers has the same sign as the sum $A+B+C$, otherwise the sum of all three will have a different sign than of the every composing number.

Certainly, this number is A or B. Indeed, if A and B have both the same sign, but opposite sign with $A+B+C$, it results that

$$(A+B)(A+B+C) < 0. \tag{11}$$

By substituting $A = |A|\text{sign}A$, $B = |B|\text{sign}A$, $C = |C|\text{sign}C$, inequality (11) becomes

$$\left(|A|+|B|\right)\left(|A|+|B|+|C|\frac{\text{signC}}{\text{signA}}\right) < 0,$$

which cannot be true as we have $|A| > |B| > |C| > 0$.

Now, suppose that no one of inequalities from the statement are satisfied; it follows that we have

1. $|B| \geq |A+C|$;
2. $|C| \geq |A+B|$;
3. $|A| \geq |B+C|$.

Taking into account that for real numbers we have $|x|^2 = x^2$, we find:

1. $B^2 \geq (A+C)^2$;
2. $C^2 \geq (A+B)^2$;
3. $A^2 \geq (B+C)^2$.

By adding the first two inequalities, and the last two inequalities, respectively, we get:

$$A(A+B+C) \leq 0; \quad B(A+B+C) \leq 0,$$

which cannot be true as we have shown before.

C. PROOF OF LEMMA 2

We start with the first part of the Lemma. It can be easily seen that $P_2(-z) = -P_1(z)$ and $P_4(-z) = -P_3(z)$. Moreover, $P_1(z)$ satisfies the Eneström-Kakeya theorem. It remains to show that $P_4(z)$ has all the zeros inside the unit circle. This is equivalent to prove that the polynomial:

$$A_3(z) = 1 + \alpha_3(1)z^{-1} + \alpha_3(2)z^{-2} + \alpha_3(3)z^{-3},$$

satisfies the conditions of Schur-Cohn, when

$$1 > \alpha_3(2) = \frac{B}{A} > \alpha_3(1) = \frac{C}{A} > \alpha_3(3) = \frac{D}{A} > 0.$$

From Appendix A.3 we need to prove the inequalities:

$$\left|\frac{D}{A}\right| < 1; \left|\frac{\frac{B}{A} - \frac{C}{A}\cdot\frac{D}{A}}{1-\left(\frac{D}{A}\right)^2}\right| < 1; \left|\frac{\frac{C}{A} - \frac{B}{A}\cdot\frac{D}{A}}{1-\left(\frac{D}{A}\right)^2 + \frac{B}{A} - \frac{C}{A}\cdot\frac{D}{A}}\right| < 1. \tag{12}$$

Note that the terms

$$\frac{D}{A}; \quad \frac{B}{A} - \frac{C}{A}\cdot\frac{D}{A}, \quad 1-\frac{D}{A}, \quad 1-\left(\frac{D}{A}\right)^2 + \frac{B}{A} - \frac{C}{A}\cdot\frac{D}{A}$$

are all positive. Furthermore,

$$1-\left(\frac{D}{A}\right)^2 > \frac{B}{A} - \frac{C}{A}\cdot\frac{D}{A};$$
$$1-\left(\frac{D}{A}\right)^2 + \frac{B}{A} - \frac{C}{A}\frac{D}{A} > \frac{B}{A} - \frac{C}{A}\cdot\frac{D}{A},$$

which ends the prove of the first part of our Lemma.

For the second part of this Lemma, we proceed as before. For $P_5(z)$ the stability conditions are:

$$\left|\frac{D}{A}\right| < 1; \quad \left|\frac{-\frac{B}{A} + \frac{C}{A}\cdot\frac{D}{A}}{1-\left(\frac{D}{A}\right)^2}\right| < 1;$$

$$\left|\frac{-\frac{C}{A} + \frac{B}{A}\cdot\frac{D}{A}}{1-\left(\frac{D}{A}\right)^2 - \frac{B}{A} + \frac{C}{A}\cdot\frac{D}{A}}\right| < 1.$$

Note that the terms

$$\frac{D}{A}; \quad \frac{C}{A} - \frac{B}{A}\cdot\frac{D}{A}, \quad 1-\frac{D}{A}, \quad 1-\left(\frac{D}{A}\right)^2 - \frac{B}{A} + \frac{C}{A}\cdot\frac{D}{A}$$

are all positive. Furthermore,

$$1-\left(\frac{D}{A}\right)^2 > \frac{B}{A} - \frac{C}{A}\cdot\frac{D}{A}.$$

However, the last stability condition:

$$1-\left(\frac{D}{A}\right)^2 - \frac{B}{A} + \frac{C}{A}\frac{D}{A} > \frac{C}{A} - \frac{B}{A}\cdot\frac{D}{A},$$

is equivalent with $(A-D)(A+D-B-C) > 0$ or $A+D-B-C > 0$ as $A > D$.

Now, suppose that the reverse is happened $A+D-B-C < 0$ and let analyze the stability conditions for $P_6(z)$:

$$\left|\frac{D}{B}\right| < 1; \quad \left|\frac{\frac{C}{B} - \frac{A}{B}\cdot\frac{D}{B}}{1-\left(\frac{D}{B}\right)^2}\right| < 1;$$

$$\left|\frac{-\frac{A}{B} + \frac{C}{B}\cdot\frac{D}{B}}{1-\left(\frac{D}{B}\right)^2 - \frac{C}{B} - \frac{A}{B}\cdot\frac{D}{B}}\right| < 1.$$

or in an equivalent form:

$$D < B;$$
$$D^2 - B^2 < BC - AD < B^2 - D^2;$$
$$-B^2 + D^2 - BC + AD < AB - CD < B^2 - D^2 + BC - AD,$$

which are all true if $B+C > A+D$.

This ends the proof of Lemma 2.

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