MULTIVARIATE MIXED POISSON DISTRIBUTIONS

A. Ferrari[‡], *G. Letac*[†] *and J.-Y. Tourneret*[*]

[‡] UMR6525 Astrophysique, Université de Nice-Sophia-Antipolis, 06108 Nice cedex 2, France
[†] LSP-UPS, 118 route de Narbonne, 31062 Toulouse cedex 4, France
[*] IRIT/ENSEEIHT/TéSA, 2 rue Charles Camichel, BP 7122, 31071 Toulouse cedex 7, France
letac@cict.fr, ferrari@unice.fr, jean-yves.tourneret@tesa.prd.fr

ABSTRACT

Univariate Mixed Poisson distributions (MPDs) are commonly used to model data recorded from low flux objects or with short exposure times. They assume that the number of recorded events, conditioned on the received random intensity, is Poisson distributed. This communication focuses on the generalization of the MPDs to the multivariate case. This generalization is required to tackle new challenging problems such as exo-planet detection using direct imaging. The joint moments and the moment generating function of a multivariate mixed Poisson distribution (MMPD) are derived. These quantities allow to characterize the over-dispersion, dependency or unicity properties of the distribution. The important example of negative multinomial distributions is considered. These distributions are obtained when the mixing distribution is a multivariate Gamma distribution. Conditions ensuring that MMPDs belong to a natural exponential family (NEF) are finally investigated.

1. INTRODUCTION

Mixed Poisson distributions (MPDs) are commonly used to model data recorded from low flux objects or with short exposure times using photocounting cameras. This model arises from the semi-classical theory of statistical optics [5]. In this theory, the classical theory of propagation is used up to the camera, leading to a high flux image. Denote by λ_ℓ the intensity on pixel ℓ and by $\mu(d\lambda_\ell)$ its probability. Conditionally to this image, the number N_ℓ of photons counted on pixel ℓ is distributed according to a Poisson distribution whose mean is the high flux intensity. Consequently, the probability of detecting k photoelectrons on pixel ℓ can be expressed as:

$$\begin{aligned}\Pr(N_\ell = k) &= \int_0^{+\infty} p(N_\ell = k|\lambda_\ell)\mu(d\lambda_\ell), \\ &= \int_0^{+\infty} \frac{(\alpha\lambda_\ell)^k}{k!} \exp(-\alpha\lambda_\ell)\mu(d\lambda_\ell). \qquad (1)\end{aligned}$$

Eq. (1) is referred to as the Poisson Mandel transform in optics or the mixed Poisson distribution in statistics. This paper assumes $\alpha = 1$ without loss of generality.

MPDs can be used to address important estimation or/and detection problems in astronomy, image processing or medical imaging. For instance, in astronomy the random nature of the intensity arises from the wave-front phase distortion by the atmospheric turbulence. This produces images with speckle patterns changing continually in time. Since the work of Labeyrie [11], many authors have proposed to take advantage of speckle patterns for high resolution imaging. In order to avoid the bluring of the speckle which destroys the high resolution information, the turbulence is "freezed" by very short time exposures which leads to photo counting. In the last decade an important amount of work has been devoted to adaptive optics in order to correct the distortion of the wave-front. However, the challenging problem of exo-planet detection using direct imaging is at the origin of a renewed interest for these models. The intensity ratio between the planet and the star is very low ($\approx 10^{-3}$ after coronography). As a consequence, residues of turbulence coming from adaptive optics can reduce considerably the detection performance. The MPDs defined in (1) are also useful in many other applications. These applications include active imaging where the image is formed from a scene illuminated with laser light [6] or actuarial statistics to model the number of accidents [7].

Univariate MPDs have been extensively studied in the literature (for example see [7, 8] and references herein). However, their multivariate extensions have received less attention. These extensions are important since they allow to model statistical dependence between the observed data, which is required to achieve tasks such as estimation or detection with high performance. This paper reviews the main properties of Multivariate Mixed Poisson Distributions denoted as MMPDs.

Section 2 derives the joint moments and the moment generating function of MMPDs. The relation between the joint factorial moments of MMPDs and the mixing density moments is also provided. Other properties such as over-dispersion, unicity of the mixing density for a given MMPD, and necessary and sufficient conditions for independence are also studied. Section 3 focuses on MMPD examples with a particular interest for intensities distributed according to multivariate Gamma distributions. Section 4 studies conditions ensuring that MMPDs belong to a natural exponential family (NEF). This result is important since the computational complexity of most estimation or detection methods is usually reduced when applied to distributions belonging to an NEF.

2. PROPERTIES OF MMPDS

2.1 Definitions

An MMPD is defined by assuming that the random variables $N_i, i = 1, \ldots, d$ are independent and distributed according to Poisson distributions with means $(\lambda_1, \ldots, \lambda_d)$, conditioned upon the vector of intensities $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$. In this case, the probability masses of $\boldsymbol{N} = (N_1, \ldots, N_d)$ are defined as

$$\Pr(\boldsymbol{N} = \boldsymbol{k}) = \int \cdots \int_{(\mathbb{R}^+)^d} \prod_\ell \frac{(\lambda_\ell)^{k_\ell}}{k_\ell!} \exp(-\lambda_\ell)\mu(d\boldsymbol{\lambda}),$$

where $\mu(d\boldsymbol{\lambda})$ is the probability of $\boldsymbol{\lambda}$ defined on $(\mathbb{R}^+)^d$. The MMPD defined above is fully characterized by the measure $\mu(d\lambda)$ and will be denoted by MPPD(μ) in this paper.

2.2 Joint moments

Multivariate factorial moments yield much simpler expressions than classical joint moments as in the univariate case.

By denoting $N^{[r]} = N(N-1)(N-r+1)$, the following results can be obtained:

$$\mathsf{E}\left(\prod_{k=1}^{d} N_k^{[r_k]}\right) = \mathsf{E}\left(\prod_{k=1}^{d} \mathsf{E}(N_k^{[r_k]}|\lambda_k)\right) = \mathsf{E}\left(\prod_{k=1}^{d} \lambda_k^{r_k}\right). \quad (2)$$

The last equality has been obtained from the factorial moments of a Poisson distribution. Joint moments can be derived by substituting in $\mathsf{E}(\prod_{k=1}^{d} N_k^{r_k})$ each $N_k^{r_k}$ by its expression as a function $N_k^{[r]}$, $r \le r_k$, (see for example [9, p. 44]). Expanding the products and using (2) yields:

$$\mathsf{E}\left(\prod_{k=1}^{d} N_k^{r_k}\right) = \sum_{j_1=0}^{r_1} \cdots \sum_{j_d=0}^{r_d} \prod_{k=1}^{d} S(r_k, j_k)\mathsf{E}\left(\prod_{k=1}^{d} \lambda_k^{j_k}\right),$$

where $S(j,k)$ are the Stirling numbers of the second kind [9].

2.3 Covariance matrix

Eq. (2) allows to obtain the following relation between the covariance matrices of $\boldsymbol{N}$ and $\boldsymbol{\lambda}$:

$$\begin{aligned} \mathrm{Cov}(\boldsymbol{N}) &= \mathrm{Cov}(\boldsymbol{\lambda}) + \mathrm{Diag}(\mathsf{E}(\lambda_1), \ldots, \mathsf{E}(\lambda_d)), \quad (3)\\ &= \mathrm{Cov}(\boldsymbol{\lambda}) + \mathrm{Diag}(\mathsf{E}(N_1), \ldots, \mathsf{E}(N_d)). \quad (4) \end{aligned}$$

Note that (3) has been used in [4] to study an approximate autoregressive model for high flux images.

2.4 Over-dispersion

For $d = 1$, eq. (4) shows that an univariate MPD is an over-dispersed distribution, i.e. a distribution with mean m and variance σ^2such that $m < \sigma^2$ [7, p. 3]. The definition of over-dispersion for multivariate distributions is not standard. In this paper, we propose the following definition: the distribution of $\boldsymbol{N} = (N_1, \ldots, N_d)$ on $\mathbb{N}^d$ is over-dispersed if $\mathrm{Cov}(\boldsymbol{N}) - \mathrm{Diag}(\mathsf{E}(N_1), \ldots, \mathsf{E}(N_d))$ is a semi positive definite matrix. Thus, (4) shows that MMPDs are over-dispersed.

Checking that $\mathrm{Cov}(\boldsymbol{N}) - \mathrm{Diag}(\mathsf{E}(N_1), \ldots, \mathsf{E}(N_d))$ is not semi positive definite allows to reject the hypothesis that $\boldsymbol{N}$ has a MMPD. For example, consider multivariate correlated Poisson distributions $\boldsymbol{N}$ defined on $\mathbb{N}^d$ by the following generating function [12]:

$$\mathsf{E}\left(\prod_{k=1}^{d} z_k^{N_k}\right) = \exp\left(\sum_{T} a_T(z^T - 1)\right) \quad (5)$$

where the sum $\sum_T$ covers all non empty subsets T of $\{1, \ldots, d\}$, $a_T \ge 0$ and $z^T = \prod_{i \in T} z_i$. Eq. (5) shows that each random variable N_k has a Poisson distribution (such that $\mathsf{E}(N_k) = \mathrm{var}(N_k)$, $\forall k = 1, \ldots, d$). This implies that the matrix $\mathrm{Cov}(\boldsymbol{N}) - \mathrm{Diag}(\mathsf{E}(N_1), \ldots, \mathsf{E}(N_d))$ has zeros on its main diagonal. Consequently, this matrix is semi positive definite if it is zero, or equivalently if the random variables $(N_1, \ldots, N_d)$ are independent.

2.5 Moment generating function

Denote by $\psi_\mu(\boldsymbol{z})$, $\boldsymbol{z} \in \mathbb{C}^d$ the Laplace transform of μ:

$$\psi_\mu(\boldsymbol{z}) = \int_{\mathbb{R}^d} e^{\langle \boldsymbol{z}, \lambda\rangle} \mu(d\lambda) = \mathsf{E}\left(e^{\sum_{\ell=1}^{L} z_\ell \lambda_\ell}\right). \quad (6)$$

The moment generating function of $\boldsymbol{N}$ expresses as:

$$\begin{aligned} \mathsf{E}\left(\prod_{k=1}^{d} z_k^{N_k}\right) &= \mathsf{E}\left(\prod_{k=1}^{d} \mathsf{E}(z_k^{N_k}|\lambda_k)\right) = \mathsf{E}\left(\prod_{k=1}^{d} e^{\lambda_k(z_k - 1)}\right)\\ &= \psi_\mu(z_1 - 1, \ldots, z_d - 1) = \psi_\mu(\boldsymbol{z} - \boldsymbol{1}). \quad (7) \end{aligned}$$

2.6 Unicity

The relation between ψ_μ and the moment generating function of $\boldsymbol{N}$ allows to prove the unicity of the mixing density for a given MMPD:

$$\mathrm{MMPD}(\mu_1) = \mathrm{MMPD}(\mu_2) \Leftrightarrow \mu_1 = \mu_2.$$

Consider the univariate case ($d = 1$) for simplicity. Eq. (1) can be written:

$$\Pr(N = k)k! = \int_0^\infty \lambda^k e^{-\lambda} \mu(d\lambda)$$

Thus, the knowledge of $\mathrm{MMPD}(\mu)$ allows to determine the moments of the measure $e^{-\lambda}\mu(d\lambda)$. However the knowledge of the moments is generally not sufficient to determine the measure itself [3]. This section shows that this is the case in the context of MMPDs.

Proof ($d = 1$). The knowledge of $\mathrm{MPPD}(\mu)$ allows to determine the function $z \mapsto \mathsf{E}(z^N)$ defined in the open unit disk $D = \{z \in \mathbb{C};\ \|z\| < 1\}$. Thus from (7) the function $z \mapsto \psi_\mu(z-1)$ is known in D. However, $\psi_\mu(z)$ is analyic in the half plane $\{z \in \mathbb{C};\ \Re z < 0\}$ (where $\Re z$ is the real part of z). Thus $z \mapsto \psi_\mu(z-1)$ is also analytic in the half plane $\{z \in \mathbb{C};\ \Re z < 1\}$. From analytic continuation it is known on $\{z \in \mathbb{C};\ \Re z < 1\}$. The unicity of Laplace transform allows to determine μ, which concludes the proof.

Note. The proof is similar for $d > 1$. However, it requires some uncommon material about several complex variables and we skip it here.

2.7 Independency

Eq. (7) shows that $(N_1, \ldots, N_d)$ are independent if and only if $(\lambda_1, \ldots, \lambda_d)$ are independent.

3. EXAMPLES OF MMPDS

3.1 Dirac Intensities

If μ is the Dirac mass at $(\lambda_1, \ldots, \lambda_d)$ then $N_1, \ldots, N_d$ are independent random variables distributed according to Poisson distributions with parameters $\lambda_1, \ldots, \lambda_d$.

3.2 Multivariate Gamma Intensities

3.2.1 Definition

A polynomial $P(\boldsymbol{z})$ with respect to $\boldsymbol{z} = (z_1, \ldots, z_d)$ is said to be affine if $\forall j = 1, \ldots, d$ the one variable polynomial $z_j \mapsto P(\boldsymbol{z})$ has the form $Az_j + B$, where A and B are polynomials with respect to the z_i's with $i \neq j$. For any $q \ge 0$ and for any affine polynomial $P(\boldsymbol{z})$, a multivariate Gamma distribution on $(\mathbb{R}^+)^d$ with shape parameter q and scale parameter $P(\boldsymbol{z})$ (denoted as $\gamma_{q,P}$) is defined by its Laplace transform [2]:

$$\psi_{\gamma_{q,P}}(\boldsymbol{z}) = [P(\boldsymbol{z})]^{-q}, \quad (8)$$

on a suitable domain of existence.

This distribution has an important practical application in optics. Indeed, the complex wave-front amplitude is generally modeled as a zero mean circular Gaussien vector with covariance matrix C. Consequently, the square modulus of the complex amplitude referred to as *intensity* is distributed as the diagonal terms of a Pseudo-Wishart distribution whose Laplace transform is:

$$\psi_\mu(\boldsymbol{z}) = \det(I_d - \mathrm{Diag}(z_1, \ldots, z_d)C)^{-1}.$$

Eq. (8) shows that μ is a multivariate Gamma distribution.

3.2.2 MMPDs generated by multivariate Gamma intensities

For any $q \geq 0$ and for any affine polynomial $P(\boldsymbol{z})$, a negative multinomial distribution $\mathrm{NM}_{q,P}$ on $\mathbb{N}^d$ is defined by its generating function [1]:

$$\mathsf{E}\left(\prod_{k=1}^{d} z_k^{N_k}\right) = [P(\boldsymbol{z})]^{-q}. \quad (9)$$

Determining necessary and sufficient conditions on the pair (q, P) such that $\gamma_{q,P}$ or $\mathrm{NM}_{q,P}$ do exist is a difficult problem. The reader is invited to look at [2, 1] for more details.

For any real numbers a_i's and b_i's, a new affine polynomial can be easily constructed as follows:

$$P_1(z_1, \ldots, z_d) = P(a_1 z_1 + b_1, \ldots, a_d z_d + b_d).$$

Equations (7,8,9) show that the MMPDs associated to the Gamma distribution $\gamma_{q,P}$ are the **negative multinomial distributions** NM_{q,P_1} with $P_1(\boldsymbol{z}) = P(\boldsymbol{z} - \mathbf{1})$.

3.2.3 Line Multivariate Gamma Distributions (LMGDs)

These distributions are a particular case of multivariate Gamma distributions where the affine polynomial is $P(\boldsymbol{z}) = 1 - a_1 z_1 - \cdots - a_d z_d$. They are the distributions of the vector $(a_1 Y, \ldots, a_d Y)$, where Y is distributed according to an univariate Gamma distribution $\gamma_{q,1}$. Note that LMGDs are concentrated on $[0, \infty)^d$ if and only if $a_k \geq 0$ for all k. The MMPD images of these distributions by $\mu \mapsto MP(\mu)$ are the negative multinomial distributions on $\mathbb{N}^d$ with generating function

$$\mathbb{E}(z_1^{N_1} \ldots z_d^{N_d}) = c^q (1 - c(a_1 z_1 + \cdots + a_d z_d))^{-q},$$

where $c = (1 + a_1 + \cdots + a_d)^{-1}$. If some of the $a_i's$ are zero (say $a_i > 0$ if and only if $i \leq m$), the line $y \mapsto (a_1 y, \ldots, a_m y)$ is concentrated on $[0, \infty)^m$ and the corresponding LMGDs are concentrated on $\mathbb{N}^m$. LMGDs will play an important role in the next section of this paper.

4. MPDS BELONGING TO A NEF

Denote by $\psi_\nu(\theta)$ the Laplace transform of a positive measure ν defined for $\theta \in \mathbb{R}^d$ (see (6). The Hölder inequality proves that the set $D(\nu)$ of $\theta \in \mathbb{R}^d$ such that $\psi_\nu(\theta) < \infty$ is a convex set and that $k = \log \psi_\nu$ is a convex function on this set. Denote by $\Theta(\nu)$ the interior of $D(\nu)$ and assume that $\Theta(\nu)$ is not empty. Then the set $F(\nu)$ of probabilities

$$\mu_\theta(d\lambda) = e^{\langle \theta, \lambda \rangle - k(\theta)} \nu(d\lambda),$$

where θ runs $\Theta(\nu)$ is called the NEF generated by ν. Note that $F(\nu) = F(\nu_1)$ does not imply $\nu = \nu_1$ but only the existence of some $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that $\nu(d\lambda) = e^{\langle a, \lambda \rangle + b} \nu_1(d\lambda)$. Thus a member μ of the NEF $F(\nu)$ can always be taken as a generating measure. However, some generating measures are not necessarily probabilities and can be even unbounded. We mention also that $\theta \mapsto e^{\langle \theta, \lambda \rangle - k(\theta)} \nu(d\lambda)$ is called a *canonical parametrization of the NEF*. Other parametrizations of the type $t \mapsto e^{\langle \alpha(t), \lambda \rangle + \beta(t)} \nu(d\lambda)$, with $\beta(t) = -k(\alpha(t))$ could be considered. The problem addressed in this section is the following; given an NEF $F(\nu)$, can we claim that $\{\mathrm{MP}(\mu_\theta);\ \theta \in \Theta\}$ is also an NEF, possibly with a non canonical parametrization? The univariate $(d = 1)$ and multivariate $(d > 1)$ cases are studied separately in the two following sections.

4.1 Univariate MPDs ($d = 1$)

The problem is simple when $F(\nu)$ is the family of Gamma distributions with fixed shape parameter q. We know that a generating measure for this family is for instance $\nu(d\lambda) = \lambda^{q-1} \mathbf{1}_{\mathbb{R}^+}(\lambda) \frac{d\lambda}{\Gamma(q)}$ [10, p. 662]. The image family $\{MP(\mu_\theta);\ \theta \in \Theta\}$ is the NEF of negative binomial distributions generated by $\sum_{k=0}^{\infty} \frac{1}{k!} p(p+1) \ldots (p+k-1) \delta_k$ [13]. However, eq. (1) does not automatically maps a distribution belonging to an NEF to an NEF. The next proposition shows that N and λ belong together to an NEF only if $F(\nu)$ is the family of Gamma distributions with fixed shape parameter.

Theorem 4.1. If the image of the NEF $F(\nu)$ on $[0, \infty)$ by $\mu \mapsto \mathrm{MP}(\mu)$ is still an NEF, then either ν is a Dirac measure or there exists $q > 0$ such that $F(\nu)$ is the family of Gamma distributions with fixed shape parameter q.

Proof. If ν is the Dirac mass at zero, the result is obvious since $MP(\nu)$ has a Poisson distribution. In the other case, denote by $\psi_\nu(\theta) = \int_0^\infty e^{\lambda\theta} \nu(d\lambda)$ for $\theta \in \Theta$, where Θ is the interior of the convergence domain of $\psi_\nu(\theta)$. Note that Θ is either $\mathbb{R}$ or some half line $(-\infty, a)$. Suppose that the image of $F(\nu)$ by $\mu \mapsto \mathrm{MP}(\mu)$ is a NEF on $\mathbb{N}$ generated by some measure $\sum_{n=0}^{\infty} p_n \delta_n$. Thus there exists two functions α and β defined on $\Theta + 1$ such that for all n

$$\int_0^\infty e^{\lambda(\theta-1)} \frac{\lambda^n}{n!} \nu(d\lambda) = p_n e^{n\alpha(\theta) + \beta(\theta)}, \quad (10)$$

which can be rewritten

$$\psi_\nu^{(n)}(\theta - 1) = n! p_n e^{n\alpha(\theta) + \beta(\theta)}. \quad (11)$$

Since ν is not the Dirac mass at zero, ψ is not a constant. Being a Laplace transform, ψ cannot be a polynomial and $\psi^{(n)}$ cannot be identically 0. This implies $p_n \neq 0$ for all n. Eq. (10) shows that α and β are real-analytic functions on the interval $\Theta + 1$. Indeed $\theta \mapsto \psi_\nu(\theta - 1)$ is analytic in the half complex plane $\Theta + 1 + i\mathbb{R}$ as well as its nth derivative $\psi_\nu^{(n)}(\theta - 1)$. Furthermore $\psi_\nu^{(n)}(\theta - 1)$ is positive on $\Theta + 1$ (since $p_n > 0$) thus its logarithm is real-analytic. Consequently $n\alpha + \beta$ and $(n+1)\alpha + \beta$ are real-analytic on $\Theta + 1$, which implies by linear combination that α and β are real-analytic on $\Theta + 1$. This proves the existence of $\alpha'(\theta)$ and $\beta'(\theta)$. By taking the logarithms of both sides of (11) and differentiating with respect to θ, the following result can be obtained

$$\frac{\psi_\nu^{(n+1)}(\theta - 1)}{\psi_\nu^{(n)}(\theta - 1)} = n\alpha'(\theta) + \beta'(\theta). \quad (12)$$

Assume first that $a = \alpha'(\theta_0) \neq 0$ and denote $\theta_0^* = \theta_0 + 1$ and $q = \beta'(\theta_0)/\alpha'(\theta_0)$. Eq. (12) can be written $\psi_\nu^{(n+1)}(\theta_0^*) = a(q+n)\psi^{(n)}(\theta_0^*)$ and thus

$$\frac{\psi_\nu^{(n)}(\theta_0^*)}{n!} = \psi_\nu(\theta_0^*) q(q+1) \ldots (q+n-1) \frac{a^n}{n!}.$$

The Taylor formula applied to the analytic function ψ (for small values of h) can be written as follows

$$\psi_\nu(\theta_0^* + h) = \psi_\nu(\theta_0^*) \sum_{n=0}^{\infty} q(q+1) \ldots (q+n-1) \frac{(ah)^n}{n!} = \psi_\nu(\theta_0^*)(1 - ah)^{-q}.$$

Since the Laplace transform is an analytic function, the result $\frac{\psi_\nu(\theta_0^* + h)}{\psi_\nu(\theta_0^*)} = (1 - ah)^{-q}$ is valid for any $h \in (-\infty, 1/a)$. For $a > 0$, the right hand side of this expression is the

Laplace transform of the Gamma distribution $\gamma_{q,1-ah}$ (usually denoted as $\gamma_{q,a}$). Moreover, the Laplace transform of $\mu_{\theta_0^*}$ is

$$\int_0^\infty e^{\lambda h}\mu_{\theta_0^*}(d\lambda) = \int_0^\infty \frac{e^{\lambda(h+\theta_0^*)}}{e^{k(\theta_0^*)}}\nu(d\lambda) = \frac{\psi(\theta_0^*+h)}{\psi(\theta_0^*)},$$

which shows that $\mu_{\theta_0^*} = \gamma_{q,a}$. In other words, the exponential family for $\{\mathrm{MP}(\mu_\theta);\ \theta \in \Theta\}$ is the family of Gamma distributions with fixed shape parameter q.

If $\alpha'(\theta_0) = 0$, (12) yields $\frac{\psi^{(n+1)}(\theta_0^*)}{\psi^{(n)}(\theta_0^*)} = \beta'(\theta_0)$ which leads to $\frac{\psi(\theta_0^*+h)}{\psi(\theta_0^*)} = e^{\beta'(\theta_0)h}$. This is the non interesting case where the exponential family for $\{\mathrm{MP}(\mu_\theta);\ \theta \in \Theta\}$ is the Dirac measure concentrated on the point $\beta'(\theta_0)$.

4.2 Multivariate MPDs ($d > 1$)

4.2.1 NEFs generated by multivariate Gamma distributions

Denote by $\Theta(\gamma_{q,P})$ the set of $\theta \in \mathbb{R}^d$ such that the Laplace transform of $\gamma_{q,P}$ converges (one can prove that $\Theta(\gamma_{q,P})$ is open). The NEF generated by $\gamma_{q,P}$ is the set of distributions whose Laplace transforms are $\psi(\boldsymbol{z}) = [P_\theta(\boldsymbol{z})]^{-q}$ where

$$P_\theta(\boldsymbol{z}) = \frac{P(\theta_1+z_1,...,\theta_d+z_d)}{P(\theta_1,...,\theta_d)},\ \theta \in \Theta(\gamma_{q,P}).$$

Since P_θ is an affine polynomial, the NEF generated by $\gamma_{q,P}$ is included in the set of multivariate Gamma distributions $\{\gamma_{q,P_\theta}; \theta \in \Theta(\gamma_{q,P})\}$. Note that for $d > 1$ and for a fixed affine polynomial P, $\{\gamma_{q,P_\theta}; \theta \in \Theta(\gamma_{q,P})\}$ does not contain all Gamma multivariate distributions of shape parameter q.

4.2.2 NEFs generated by Negative Multinomial distributions

Similarly, denote as $\Theta(\mathrm{NM}_{q,P})$, the set of $\theta \in \mathbb{R}^d$ such that the moment generating function of $\mathrm{NM}_{q,P}$ converges (here again one can prove that $\Theta(\mathrm{NM}_{q,P})$ is open). The following notation is useful

$$W(\mathrm{NM}_{q,P}) = \{w = (e^{\theta_1},...,e^{\theta_d}); \theta \in \Theta(\mathrm{NM}_{q,P})\}.$$

The NEF generated by $\mathrm{NM}_{q,P}$ is the set of distributions whose generating functions can be written $\mathsf{E}(\prod_{k=1}^d z_k^{N_k}) = [P_w(\boldsymbol{z})]^{-q}$ where

$$P_w(\boldsymbol{z}) = \frac{P(w_1z_1,...,w_dz_d)}{P(w_1,...,w_d)},\ w \in W(\mathrm{NM}_{q,P}).$$

Since P_w is an affine polynomial, the NEF generated by $\mathrm{NM}_{q,P}$ is included in the set of negative multinomial distributions $\{\mathrm{NM}_{q,P_w}; w \in W(\mathrm{NM}_{q,P})\}$. Note again that this family does not contain all negative multinomial distributions with shape parameter q.

4.2.3 MMPDs generated by the NEF of MPDs

Consider a Gamma multivariate distribution $\gamma_{q,P}$ and the corresponding NEF $\{\gamma_{q,P_\theta}, \theta \in \Theta(\gamma_{q,P})\}$. The MMPDs generated by this family are the distributions whose generating functions are

$$\mathsf{E}\left(\prod_{k=1}^d z_k^{N_k}\right) = \psi_\mu(\boldsymbol{z}-\boldsymbol{1}) = [Q_\theta(\boldsymbol{z})]^{-q},$$

where $Q_\theta(\boldsymbol{z}) = P_\theta(\boldsymbol{z}-\boldsymbol{1})$. As a consequence these distributions belong to the set of negative multinomial distributions $\{\mathrm{NM}_{q,Q_\theta}; \theta \in \Theta(\gamma_{q,P})\}$. Surprisingly, this family is NOT in general an exponential family. Indeed, it is generally not possible to find an affine polynomial Q and a map $\theta \mapsto w(\theta)$ from $\Theta(\gamma_{q,P})$ to $(\mathbb{R}^+)^d$ such that the following identity holds

$$Q_\theta(z_1,\ldots,z_d) = \frac{Q(w_1(\theta)z_1,...,w_d(\theta)z_d)}{Q(w_1(\theta),...,w_d(\theta)}.$$

The solution to the initial problem of finding NEFs on $[0,\infty)^d$ such that the MMPD image by $\mu \mapsto MP(\mu)$ is still an NEF is provided by the following theorem:

Theorem 4.2. If the image of the NEF $F(\nu)$ on $[0,\infty)^d$ by $\mu \mapsto MP(\mu)$ is still an NEF, then there exists a partition $\{T_0, T_1, \ldots, T_q\}$ of $\{1,\ldots,d\}$ (with possibly $T_0 = \emptyset$) and there exist non negative numbers $a_1,\ldots,a_d$ and positive numbers $p_1,\ldots,p_q$ such that $F(\nu)$ has a generating measure μ with Laplace transform

$$\psi_\mu(\theta) = e^{\sum_{k\in T_0} a_k\theta_k}\prod_{m=1}^q\left(1-\sum_{k\in T_m}a_k\theta_k\right)^{-p_m}.$$

Equivalently, μ_θ is the product of LMGDs or the product of LGMDs and a Dirac measure.

Proof. The proof is omitted for lack of space.

5. CONCLUSIONS AND OPEN PROBLEMS

This communication has studied important properties of MMPDs: joint moments, moment generating function, overdispersion and conditions ensuring that MMPDs belong to a natural exponential family.

The use of MMPDs to solve various imaging problems is currently under investigation. These problems include 1) *indirect imaging*, where the speckle intensity fluctuations are usually assumed to be (marginally) Gamma distributed and 2) *planet detection by using direct imaging* and assuming a perfect coronograph.

REFERENCES

[1] P. Bernardoff. Which negative multinomial distributions are infinitely divisible? *Bernoulli*, 9(5):877–893, Dec. 2003.

[2] P. Bernardoff. Which multivariate Gamma distributions are infinitely divisible? *Bernoulli*, to be published.

[3] W. Feller. *An Introduction to Probability Theory and its Applications*, volume 2. Wiley, New York, 2nd edition, 1971.

[4] A. Ferrari, J. Tourneret, and G. Alengrin. Parametric modeling of photometric signals. *Signal Processing*, 82:649–661, 2002.

[5] J. Goodman. *Statistical Optics*. Wiley, New York, 1985.

[6] F. Goudail, N. Roux, and P. Réfrégier. Performance parameters for detection in low-flux coherent images. *Optical Letters*, 28(2):81–83, Jan. 2003.

[7] J. Grandell. *Mixed Poisson Processes*. Chapman & Hall, London, 1997.

[8] N. Hengartner. Adaptive demixing in Poisson mixture models. *Annals of Statistics*, 25(3):917–928, 1997.

[9] N. Johnson, S. Kotz, and A. Kemp. *Univariate Discrete Distributions*. Wiley, New York, 1993.

[10] S. Kotz, N. Balakrishnan, and N. L. Johnson. *Continuous Multivariate Distributions*, volume 1. Wiley, New York, 2nd edition, 2000.

[11] A. Labeyrie. High resolution techniques in optical astronomy. *Progress in Optics*, 14:46–87, 1976.

[12] G. Letac. Stationary sequences with simple joint Poisson distributions. *J. Stat. Planning and Inference*, 90:1–20, 2000.

[13] G. Letac and M. Mora. Natural exponential families with cubic variance. *Annals of Statistics*, 18:1–37, 1990.