ASYMPTOTIC ANALYSIS OF REDUCED RANK DOWNLINK CDMA WIENER RECEIVERS.

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ABSTRACT

In this paper, we study the performance of reduced rank Wiener filters in the context of downlink CDMA systems corrupted by a frequency selective channel. For this, we consider the output signal to interference plus noise ratio (SINR), and study its convergence speed versus the order of the receiver. Unfortunately, this is a difficult task because the SINR expressions depend on the spreading codes allocated to the various users in a rather complicated way. In order to be able to obtain positive results, we follow the classical approach used for the first time in [12]: the code matrix is modelled as the realization of a certain random matrix, and the behavior of the SINRs is studied when the spreading factor N and the number of users K converges to $+\infty$ in such a way that $\frac{K}{N} \rightarrow +\alpha$. As the code matrices used in the downlink of CDMA systems are very often orthogonal, we model the code matrix allocated to the various users as a realization of a Haar distributed random unitary matrix. In this context, we show that the SINR of each order n reduced rank receiver converge toward a deterministic limit β_n independent of the spreading codes. In order to study the performance of the receiver versus n, we therefore study the convergence speed of β_n when $n \rightarrow +\infty$, a simpler problem. For this, we use the results of [8] based on the theory of orthogonal polynomials for the power moment problem. We obtain the convergence rate of β_n, and exhibit the parameters influencing the convergence speed.

1. INTRODUCTION

In multidimensional signal processing, it is often useful to approximate the Wiener filter by a reduced rank version of this filter. The latter acts on a projection of the received signal on a judiciously chosen small dimensional subspace. The use of a reduced rank filter can be motivated by complexity constraints or, in an adaptive setting, by fast convergence requirements. It is then of major interest to quantify the SNR loss at the output of this filter due to its non optimum character.
The *Krylov subspaces*, widely used as projection subspaces, will be considered in this paper. To fix our ideas, let us begin with the generic signal model

$$\mathbf{y}_N = \mathbf{h}_N b + \mathbf{x}_N \tag{1}$$

where $\mathbf{y}_N$ is the received $N \times 1$ signal, b is the unit-variance scalar signal to be estimated and $\mathbf{x}_N$ is a signal decorrelated with b representing interference and/or background noise . The $N \times N$ covariance matrix of $\mathbf{x}_N$ is denoted $\mathbf{R}_{N,I}$ and will be assumed invertible. Recall that the MMSE receiver is described by the equation $s_{\text{MMSE}} = \mathbf{h}_N^H \mathbf{R}_N^{-1} \mathbf{y}$ where $\mathbf{R}_N = \mathbf{h}_N \mathbf{h}_N^H + \mathbf{R}_{N,I}$ is the received signal $\mathbf{y}_N$ covariance matrix. This receiver will be called in the sequel the full rank MMSE receiver. Its output SNR that we index by the number of dimensions of the received signal is given by the standard expression

$$\beta^{(N)} = \frac{\eta^{(N)}}{1-\eta^{(N)}} \tag{2}$$

where $\eta^{(N)}$ is defined by

$$\eta^{(N)} = \mathbf{h}_N^H \mathbf{R}_N^{-1} \mathbf{h}_N \ . \tag{3}$$

The n^{th} Krylov subspace associated to the pair $(\mathbf{R}_N, \mathbf{h}_N)$ is the subspace of $\mathbb{C}^N$ spanned by the columns of $\mathbf{K}_{n,N} = [\mathbf{h}_N, \mathbf{R}_N \mathbf{h}_N, \ldots, \mathbf{R}_N^{n-1} \mathbf{h}_N]$. The n-th stage reduced rank Wiener filter considered in this paper is the MMSE estimator of b operating on the transformed signal $\tilde{\mathbf{y}}_{n,N} = \mathbf{K}_{n,N}^H \mathbf{y}_N$. The motivation behind choosing the Krylov subspaces and the implementation of the subsequent filters are discussed in a number of works (see e.g. [6] and [4]).

The output SINR $\beta_n^{(N)}$ of the n-th stage reduced rank Wiener filter is given by

$$\beta_n^{(N)} = \frac{\eta_n^{(N)}}{1-\eta_n^{(N)}} \tag{4}$$

where $\eta_n^{(N)}$ is now defined by

$$\eta_n^{(N)} = \mathbf{h}_N^H \mathbf{K}_{n,N} \left(\mathbf{K}_{n,N}^H \mathbf{R}_N \mathbf{K}_{n,N} \right)^{-1} \mathbf{K}_{n,N}^H \mathbf{h}_N \ . \tag{5}$$

The use of reduced rank Wiener filters is of course attractive if close to optimum performance can be achieved for small values n. In order to precise in which contexts this nice condition holds, the convergence speed of $\beta_n^{(N)}$ to $\beta^{(N)}$, or equivalently of $\eta_n^{(N)}$ to $\eta^{(N)}$ when n increases has to be studied. This problem has been successfully addressed in the recent work [6] (see also ([11], [10]) in the context of the following simple CDMA transmission model

$$\mathbf{y}_N = \mathbf{W}_{N,K} \mathbf{b}_K + \mathbf{v}_N \ . \tag{6}$$

$\mathbf{b}_K = [b_1, \ldots, b_K]^T$ is the $K \times 1$ symbol vector where K is the number of users, $\mathbf{W}_{N,K}$ is the $N \times K$ code matrix, and $\mathbf{v}_N$ is the classical noise with covariance matrix $\omega^2 \mathbf{I}_N$. The purpose is to estimate the symbol b_1, so this equation appears as a particular case of (1) : if we partition $\mathbf{W}_{N,K}$ and $\mathbf{b}_K$ as $\mathbf{W}_{N,K} = [\mathbf{w}_N \ \mathbf{U}_{N,K-1}]$ and $\mathbf{b}_K = [b_1 \ \mathbf{b}_I^T]^T$, then we replace $\mathbf{h}_N$ by $\mathbf{w}_N$ and $\mathbf{x}_N$ by $\mathbf{U}_{N,K-1} \mathbf{b}_I + \mathbf{v}_N$. Honig and Xiao ([6]) assumed that the code matrix $\mathbf{W}_{N,K}$ is a random matrix with centered i.i.d. elements having a variance of $1/N$, and studied the performance of the reduced rank filter in the "large system" regime where N tends to infinity in such a way that K/N converges toward a constant α. They established that $\eta_n^{(N)}$ and $\eta^{(N)}$

converge to finite limits η_n and η, and were able to show that η is a continued fraction expansion whose order n truncation coincides with η. From this, they concluded for the rapid convergence of this SNR toward the full rank SNR.

Note that partial results have been obtained in more general models than (6) (see [2] and [7]). In these works, the convergence of $\eta_n^{(N)}$ toward η_n is established. However, the convergence speed of η_n toward η is not addressed.
In [8], we also addressed the influence of n on the performance of the receiver in the asymptotic regime when $N \to +\infty$, but in the much more general context defined by model (1). Under the hypothesis that for each integer k, $s_k^{(N)} = \mathbf{h}_N^H \mathbf{R}_N^k \mathbf{h}_N$ converges when $N \to +\infty$ to a finite limit s_k, we showed that $\eta^{(N)}$ and $\eta_n^{(N)}$ also converge to certain finite limits η and η_n respectively. More importantly, the convergence speed of η_n toward η can be evaluated using properties of certain orthogonal polynomials.

The purpose of this paper is to show that the results of [8] can be used in order to study the convergence speed of reduced rank Wiener filters in the context of downlink CDMA systems corrupted by frequency selective channels. This paper is organized as follows. We first recall in section II the main results of [8]. In section III, we present the downlink CDMA system model as well as the reduced rank Wiener filters under consideration. The received data is corrupted by a frequency selective channel, and the code matrix is modelled as the realization of a orthogonal random Haar distributed matrix. In section IV, we study the performance of the above receivers in the asymptotic regime N and K converge to ∞ in such a way that $\frac{K}{N} \to \alpha$. We show that the hypotheses formulated in section II are valid, and deduce the convergence speed of the reduced rank receivers.

2. A REVIEW OF THE MAIN RESULTS OF [8]

We still consider model 1 and formulate the following assumption.

Assumption 1 *We assume that for each k, $s_k^{(N)} = \mathbf{h}_N^H \mathbf{R}_N^k \mathbf{h}_N$ converges when $N \to +\infty$ to a finite limit s_k, and that $s_0 = 1$.*

It is easily seen that $\eta_n^{(N)}$ is equal to

$$(s_0^{(N)}, \ldots, s_{n-1}^{(N)}) \begin{pmatrix} s_1^{(N)} & s_2^{(N)} & \ldots & s_n^{(N)} \\ s_2^{(N)} & s_3^{(N)} & \ldots & s_{n+1}^{(N)} \\ \vdots & \vdots & \vdots & \vdots \\ s_n^{(N)} & s_{n+1}^{(N)} & \ldots & s_{2n-1}^{(N)} \end{pmatrix}^{-1} \begin{pmatrix} s_0^{(N)} \\ \vdots \\ s_{n-1}^{(N)} \end{pmatrix} \tag{7}$$

Assumption 1 thus implies that for each n, $\eta_n^{(N)}$ converges to the quantity η_n obtained by replacing $(s_k^{(N)})_{k=1,2n-1}$ in (7) by sequence $(s_k)_{k=1,2n-1}$. Moreover, $\mathbf{K}_{n,N}^H \mathbf{K}_{n,N}$ and $\mathbf{K}_{n,N}^H \mathbf{R}_N \mathbf{K}_{n,N}$ are positive Hankel matrices converging to the Hankel matrices $(s_{k+l})_{(k,l)=0,\ldots,n-1}$ and $(s_{k+l+1})_{(k,l)=0,\ldots,n-1}$. Therefore, matrices $(s_{k+l})_{(k,l)=0,\ldots,n-1}$ and $(s_{k+l+1})_{(k,l)=0,\ldots,n-1}$ are also positive. Using well known results (see e.g. [1]), it exists a probability measure σ such that

$$s_k = \int_0^\infty \lambda^k d\sigma(\lambda). \tag{8}$$

Assumption 2 *Measure σ is carried by an interval $[\delta_1, \delta_2]$, and is thus uniquely defined by (8) (see [1]). Moreover, σ is absolutely continuous, and its density is almost surely strictly positive on $[\delta_1, \delta_2]$.*

absolutely continuous and is carried by an interval $[\delta_1, \delta_2]$

Assumption 3 *It exists $A > 0$ and $B > 0$ such that $\|\mathbf{R}_N^{-1}\| \leq A$ and $\|\mathbf{R}_N\| \leq B$ for each N.*

Under the above assumptions, $\eta^{(N)} = \mathbf{h}_N \mathbf{R}_N^{-1} \mathbf{h}_N$ can be shown to converge to $\eta = \int_{\delta_1}^{\delta_2} \frac{1}{\lambda} d\sigma(\lambda)$. Therefore, we have to evaluate the convergence speed of

$$\eta_n = (s_0, \ldots, s_{n-1}) \begin{pmatrix} s_1 & s_2 & \ldots & s_n \\ s_2 & s_3 & \ldots & s_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ s_n & s_{n+1} & \ldots & s_{2n-1} \end{pmatrix}^{-1} \begin{pmatrix} s_0 \\ \vdots \\ s_{n-1} \end{pmatrix}$$

toward $\eta = \int_{\delta_1}^{\delta_2} \frac{1}{\lambda} d\sigma(\lambda)$. The main result of [8] is the following theorem.

Theorem 1 *Let $\mu > 1$ and $\phi < 1$ be defined by $\mu = \frac{1+\frac{\delta_1}{\delta_2}}{1-\frac{\delta_1}{\delta_2}}$ and $\phi = \frac{1}{\mu+\sqrt{\mu^2-1}}$. Then, it exists 2 strictly positive constants C and D such that*

$$C\phi^{2n} \leq (\eta - \eta_n) \leq D\phi^{2n} \tag{9}$$

for n large enough.

This results implies that the convergence is locally exponential, and that its rate only depends on the ratio $\frac{\delta_1}{\delta_2}$, and not on the particular form of measure σ. In particular,if $\frac{\delta_1}{\delta_2}$ is close from 0, then μ is close to 1, and the convergence is slow. If however $\frac{\delta_1}{\delta_2}$ is close from 1, then μ is large, and the convergence is fast.

3. THE DOWNLINK CDMA MODEL.

We now show how to apply these results in order to evaluate the convergence speed of reduced rank suboptimum Wiener filters in the context of downlink CDMA systems. In this section, we first present the downlink CDMA model. We denote by N and K the spreading factor and the number of users of the cell respectively, and by $h(z) = \sum_{l=0}^{L} h_l z^{-l}$ the transfer function of the chip rate discrete-time equivalent channel between the base station and the mobile station of interest. $h(z)$ is assumed to be known at the receiver side, and is normalized in such a way that $\sum_{l=0}^{L} |h_l|^2 = 1$. $(d(m))_{m\in\mathbb{Z}}$ represents the chip sequence transmitted by the base station. Therefore, the received signal $(y(m))_{m\in\mathbb{Z}}$ sampled at the chip rate can be written as

$$y(m) = \sum_{l=0}^{L} h_l d(m-l) + v(m)$$

where v is an additive white noise of variance ω^2. We denote by $\mathbf{y}_N(n)$ the N–dimensional vector defined by $\mathbf{y}_N(n) = (y(nN), \ldots, y(nN+N-1))^T$. $\mathbf{y}_N(n)$ can be written as

$$\mathbf{y}_N(n) = \mathbf{H}_{0,N}\mathbf{W}_{N,K}(n)\mathbf{b}_K(n) + \mathbf{H}_{1,N}\mathbf{W}_{N,K}(n-1)\mathbf{b}_K(n-1) + \mathbf{v}_N(n) \tag{10}$$

$\mathbf{b}_K(n)$ represents the vector of transmitted symbols at time n, and we assume that the user of interest is user 1. $\mathbf{H}_{0,N}$ and $\mathbf{H}_{1,N}$ are 2 Toeplitz band matrices depending on sequence $(h_l)_{l=0,\ldots,L}$. Matrix $\mathbf{W}_{N,K}(n)$ represents the code matrix at time n. We denote $\mathbf{w}_N(n)$ the first column of $\mathbf{W}_{N,K}(n)$ (i.e. the code vector of the user of interest), and by $\mathbf{U}_{N,K-1}(n)$ the orthogonal $N \times (K-1)$ matrix such that $\mathbf{W}_{N,K}(n) = (\mathbf{w}_N(n), \mathbf{U}_{N,K-1}(n))$. In the following, we study the performance of reduced rank Wiener filters in the asymptotic regime N and K converge to $+\infty$ in such a way that $\frac{K}{N} \to \alpha$ where $0 < \alpha < 1$. It is important to notice that the lenght L of the impulse response of the channel is assumed to be kept constant. Therefore, the inter-symbol interference term $\mathbf{H}_{1,N}\mathbf{W}_{N,K}(n-1)$ can be shown to have no effect on the performance of our receivers. In particular, the term $\mathbf{H}_{1,N}\mathbf{W}_{N,K}(n-1)$ can be replaced by $\mathbf{H}_{1,N}\mathbf{W}_{N,K}(n)$ without changing the asymptotic behavior of the output SNRs of the receivers. We can therefore exchange (10) with

$$\mathbf{y} = \mathbf{H}_N\mathbf{W}_{N,K}\mathbf{s}_K + \mathbf{v} \tag{11}$$

Here, $\mathbf{H}_N$ is the circulant matrix $\mathbf{H}_N = \mathbf{H}_{0,N} + \mathbf{H}_{1,N}$, the first column of which is vector $\mathbf{h}_N$ defined by

$$\mathbf{h}_N = (h_0, \dots, h_L, 0, \dots, 0)^T.$$

This observation allows to simplify many further calculations. Note that we omit from now on the time index n which is irrelevant.

We now explain how the random matrix $\mathbf{W}_{N,K}$ is generated. For this purpose, some notations and definitions need to be introduced. Denote by $\mathcal{U}$ the multiplicative group of $N \times N$ unitary matrices, and by $\mathbf{Q}$ a random $N \times N$ unitary matrix. $\mathbf{Q}$ is said to be Haar distributed if the probability distribution of $\mathbf{Q}$ is invariant by left multiplication by constant unitary matrices. Since the group $\mathcal{U}$ is compact, this condition is known to be equivalent to the invariance of the probability distribution of $\mathbf{Q}$ by right multiplication by constant unitary matrices. In order to generate Haar distributed unitary random matrices, let $\mathbf{X} = [x_{i,j}]_{1\le i,j\le N}$ be a $N \times N$ random matrix with independent complex Gaussian centered unit variance entries. The unitary matrix $\mathbf{X}(\mathbf{X}^H\mathbf{X})^{-1/2}$ is Haar distributed. Unless otherwise stated, it will be assumed in the following that matrix $\mathbf{W}_{N,K}$ is generated by extracting K columns from a $N \times N$ Haar unitary random matrix $\mathbf{Q}$.

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Model (11) coincides with model (1) for $\mathbf{h}_N = \mathbf{H}_N\mathbf{w}_N$ and $\mathbf{R}_N = \mathbf{H}_N\mathbf{W}_{N,K}\mathbf{W}_{N,K}^H\mathbf{H}_N^H + \omega^2 I$. The SINRs of the plain Wiener filter and of the reduced rank Wiener filters are thus given by formulas (2) to (5). Moreover, in order to study the convergence speed of $\eta_n^{(N)}$ to $\eta^{(N)}$ in our asymptotic regime, the results of section (2) can be used provided assumptions 1 to 3 hold.

In order to check assumption 1, we observe that $s_k^{(N)}$ is given by $s_k^{(N)} = \mathbf{w}_N^H\mathbf{H}_N^H(\mathbf{H}_N\mathbf{W}_{N,K}\mathbf{W}_{N,K}^H\mathbf{H}_N^H + \omega^2 I)^k\mathbf{H}_N\mathbf{w}_N$. Using the properties of the Haar distribution, it can be shown as in [3] that $s_k^{(N)}$ has the same asymptotic behavior that the term

$$\frac{1}{K}\mathrm{Trace}(\mathbf{W}_{N,K}^H\mathbf{H}_N^H(\mathbf{H}_N\mathbf{W}_{N,K}\mathbf{W}_{N,K}^H\mathbf{H}_N^H + \omega^2 I)^k\mathbf{H}_N\mathbf{W}_{N,K}) \tag{12}$$

Denote by $(\lambda_l^{(N)})_{l=1,\dots,N}$ the eigenvalues of $\mathbf{H}_N\mathbf{W}_{N,K}\mathbf{W}_{N,K}^H\mathbf{H}_N^H$. Then, (12) is equal to $\frac{1}{K}\sum_{l=1}^N \lambda_l^{(N)}(\lambda_l^{(N)} + \omega^2)^k)$.

In order to precise the asymptotic behavior of this term when $N \to +\infty$ and $K/N \to \alpha$, we first note that the eigenvalue distributions of matrices $\mathbf{W}_{N,K}\mathbf{W}_{N,K}^H$ and $\mathbf{H_N}^H\mathbf{H}_N$ converge toward two probability distributions denoted ν and μ respectively. It is clear that $d\nu(t) = \alpha\delta(t-1) + (1-\alpha)\delta(t)$. In order to precise the behavior of μ, we remark that the eigenvalues of $\mathbf{H}_N^H\mathbf{H}_N$ coincide with $(|h(e^{2i\pi l/N})|^2)_{l=0,\dots,N-1}$. Therefore, μ is carried by the interval $[|h_{min}|^2, |h_{max}|^2]$ where $|h_{min}| = \min_f |h(e^{2i\pi f})|$ and $|h_{max}| = \max_f |h(e^{2i\pi f})|$, and is defined by $\int \phi(t)d\mu(t) = \int_0^1 \phi(|h(e^{2i\pi f})|^2)df$.

As matrices $\mathbf{W}_{N,K}\mathbf{W}_{N,K}^H$ and $\mathbf{H_N}^H\mathbf{H}_N$ are almost surely asymptotically free (see [3], [5]), the eigenvalue distribution of matrix $\mathbf{H}_N\mathbf{W}_{N,K}\mathbf{W}_{N,K}^H\mathbf{H}_N^H$ converges toward a probability measure, denoted $\mu \otimes \nu$, called the free multiplicative convolution product of μ and ν. This implies that

$$\lim_{N\to+\infty, K/N\to\alpha} s_k^{(N)} = \frac{1}{\alpha}\int t(t+\omega^2)^k d\mu\otimes\nu(t) \tag{13}$$

We note s_k the above limit. This shows that assumption 1 holds.

Assumption 3 is obviously satisfied. We now verify assumption 2. We first note that 0 is eigenvalue of matrix $\mathbf{H}_N\mathbf{W}_{N,K}\mathbf{W}_{N,K}^H\mathbf{H}_N^H$ with multiplicity $N-K$. The remaining eigenvalues are strictly positive, and coincide with the eigenvalues of matrix $\mathbf{W}_{N,K}^H\mathbf{H_N}^H\mathbf{H}_N\mathbf{W}_{N,K}$. Therefore, measure $d\mu\otimes\nu(t)$ can be written as $d\mu\otimes\nu(t) = (1-\alpha)\delta(t) + \alpha d\gamma(t)$ where $d\gamma(t)$ represents the limit eigenvalue distribution of $\mathbf{W}_{N,K}^H\mathbf{H_N}^H\mathbf{H}_N\mathbf{W}_{N,K}$. It can be checked that $d\gamma(t)$ is absolutely continuous, and that its density is almost surely strictly positive on a certain interval $[x_1, x_2]$. It is clear that the eigenvalues of $\mathbf{W}_{N,K}^H\mathbf{H_N}^H\mathbf{H}_N\mathbf{W}_{N,K}$ are contained in the interval $[|h_{min}|^2, |h_{max}|^2]$ for each N and K. Therefore, the interval $[x_1, x_2]$ is itself contained in $[|h_{min}|^2, |h_{max}|^2]$.

In order to complete the verification of assumption 2, we remark that s_k can be written as $s_k = \int_{x_1}^{x_2} t(t+\omega^2)^k d\gamma(t)$, or equivalently

$$s_k = \int_{x_1+\omega^2}^{x_2+\omega^2} (\lambda - \omega^2)\lambda^k d\gamma(\lambda - \omega^2)$$

This shows that measure σ defined by $s_k = \int \lambda^k d\sigma(\lambda)$ is given by

$$d\sigma(\lambda) = (\lambda - \omega^2)d\gamma(\lambda - \omega^2) \tag{14}$$

As $d\gamma(t)$ is compactly supported and absolutely continuous, so is σ. Moreover, the support of σ is the interval $[\delta_1, \delta_2]$ where $\delta_1 = x_1 + \omega^2$ and $\delta_2 = x_2 + \omega^2$, and its density is almost surely strictly positive on $[\delta_1, \delta_2]$.

As assumptions 1 to 3 hold, the results of [8] can be applied. It turns out that the convergence speed of η_n toward η is exponential, and depends on factor $\frac{(x_1+\omega^2)}{(x_2+\omega^2)}$: if this ratio is close from 1, the convergence is fast, while if it is close from 0, the convergence is slow. In order to discuss this point, we assume that the effect of ω^2 on the ratio is negligible. The important term is thus $\frac{x_1}{x_2}$, which depends both on α and $|h_{min}|^2$ and $|h_{max}|^2$. It is clear that the ratio $\frac{x_2-x_1}{|h_{max}|^2-|h_{min}|^2}$ increases from 0 to 1 when α increases from 0 to 1. Moreover, one can expect that the condition number $\frac{|h_{min}|^2}{|h_{max}|^2}$ also affects $\frac{x_1}{x_2}$. In order to be able to understand the influence of α and $(|h_{min}|^2, |h_{max}|^2)$ on (x_1, x_2), we mention that x_1 and x_2 can be evaluated numerically rather easily. For this, we denote by $G_\gamma(z)$ the Stieljès transform of $d\gamma(t)$ defined by $G_\gamma(z) = \int_{x_1}^{x_2} \frac{d\gamma(t)}{t-z}$. For each $z \in \mathbb{C} - [x_1, x_2]$, $G_\gamma(z)$ can be shown to satisfy the equation $\alpha(1 + zG_\gamma(z)) = T(z, G_\gamma(z))$ where $T(z, g)$ is defined by

$$T(z, g) = \int_0^1 \frac{|h(e^{2i\pi f})|^2}{|h(e^{2i\pi f})|^2 - z + \frac{1-\alpha}{\alpha g}} \tag{15}$$

Moreover, x_1 is the unique positive real number for which it exists $g_1 > 0$ satisfying

$$\begin{aligned} \alpha(1 + x_1 g_1) &= T(x_1, g_1) \\ \alpha x_1 &= \frac{\partial T}{\partial g}(x_1, g_1) \end{aligned} \qquad (16)$$

x_2 is characterized similarly, but the corresponding value g_2 is strictly negative. This result will be used more extensively in a forthcoming paper.

We now illustrate the influence of α and $(|h_{min}|^2, |h_{max}|^2)$ on the convergence speed of $\beta_n = \frac{\eta_n}{1-\eta_n}$ toward $\beta = \frac{\eta}{1-\eta}$. For this, we represent in the following figures the relative SINR defined as the ratio $\frac{\beta_n}{\beta}$. In figure 1, we first study the influence of α on the convergence speed of the relative SINR toward 1. Here, the ratio $\frac{E_b}{N_0}$ is equal to 10 dB. This figure confirms that the convergence speed of the reduced rank receivers depends crucially on the load factor.

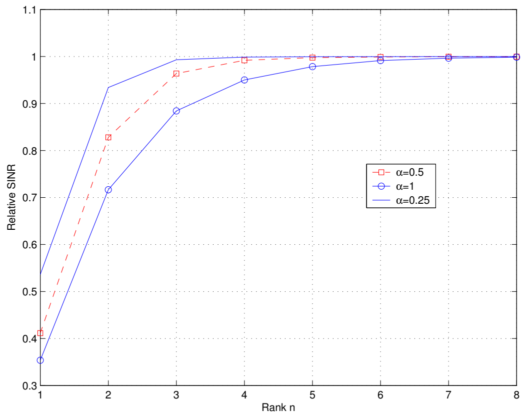

Figure 1: Influence of α

In figure 2, we study the effect of the channel on the convergence speed of β_n toward β. For this, we consider a 2 taps channel with transfer function $h(z) = h_1 + h_2 z^{-1}$. In this case, if $|h_1| = |h_2|$, $h(z)$ has a zero on the unit circle, so that $|h_{min}| = 0$. If $[h_1| = |h_2|$, the convergence speed of β_n toward β is thus expected to be minimum. This is confirmed by 4 obtained for $\alpha = \frac{1}{2}$ and $\frac{E_b}{N_0} = 17dB$.

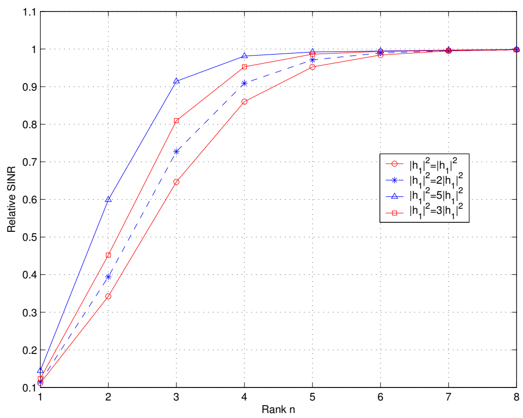

Figure 2: Influence of the channel

We finally verify that our asymptotic SINR evaluations allow to predict the empirical performance of the studied receivers. For this, we have compared the measured bit error rate with its asymptotic evaluation given by $Q(\sqrt{\beta_n})$ (we have used a QAM4 constellation). The resuts are presented in figure 3. Here, the propagation channel is the so-called Vehicular A (on each frame, a different realization of the channel is generated). The signal to noise ratio $\frac{E_b}{N_0}$ is equal to 7dB and the load factor α is equal to $\frac{1}{2}$. Figure 3 shows that our asymptotic evaluations allow to predict rather accurately the performance of the true system if $N \geq 128$. However, for smaller values of N, the asymptotic performance is too optimistic. We finally note that the receiver we implemented is based on the correct model (10), thus showing that the approximation (11) used in order to derive the asymptotic performance is justified in the context of the vehicular A channel

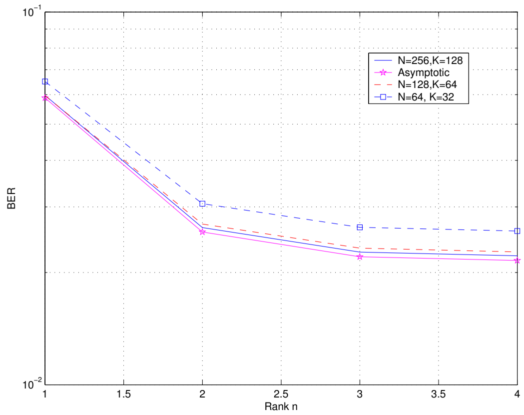

Figure 3: Comparison of empirical and theoretical BER

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