COMPLEX TIME-FREQUENCY AND DUAL-FREQUENCY SPECTRA OF HARMONIZABLE PROCESSES

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ABSTRACT

Harmonizable processes form a huge and useful class of nonstationary random processes. In this paper, we will discuss the properties of, and some consequences of employing a natural choice of complex valued densities to characterize the harmonizable class. In particular, we will discuss the Hilbert space geometry of the resulting complex time-frequency description (related to the Rihaczek distribution), and the dual-frequency description (related to the Loève spectrum). We will demonstrate that useful normalized nonstationary generalizations of coherence emerge from this picture. Finally, we extend the formalism to random fields and to higher-order cases, and we will discuss possible estimators.

1. INTRODUCTION

Nonstationary random processes are often characterized in terms of their time-frequency behavior [3, 14, 23]. Several different approaches has been taken in order to provide useful descriptors, ranging from straightforward short-time Fourier transforms, to bilinear densities among which the Wigner-Ville (WV) spectrum has a special standing [3, 9, 13].

Even though the WV spectrum is real valued, it is not everywhere non-negative [3, 9, 13]. Thus, WV spectra do not admit interpretations in terms of power as a function of time and frequency. A tremendous effort has been invested to reduce the negativeness of the WV, despite the existence of Wigner's theorem which states that *there exists no time-frequency representation that is bilinear, has correct marginals, and is non-negative everywhere* [22].

In this paper, we will seemingly remove ourselves even further from the intuitive description of power as a function of time and frequency. In fact, we will argue that certain *complex valued* distributions provide very useful insight into processes belonging to the harmonizable class. The key is to identify *other* meaningful interpretations than that of power per time and frequency, since Wigner's theorem already tells us that such an interpretation is impossible for bilinear representation.

As will be shown, a certain natural choice of time coordinates for second order correlations, leads to the Rihaczek spectrum as the corresponding time-frequency description, and the Loève spectrum as the corresponding dual-frequency description. Both the Rihaczek and the Loève spectra are complex valued, even for real data series. Powerful interpretations in terms of Hilbert space inner products exists, however, and the corresponding vector space geometry leads to time-frequency and dual-frequency coherence functions. These coherences generalize conventional coherences for stationary processes, to nonstationary processes belonging to the harmonizable class.

2. HARMONIZABLE PROCESSES

Let $X(t)$, $t \in I_t$, be a real valued stochastic process, where I_t denotes some index set for a time-like variable t. For example, time could be continuous, for which $I_t \subseteq \mathbb{R}$, or time could be discrete, for which $I_t \subseteq \mathbb{Z}$.

Assume now that the stochastic process $X(t)$ has the spectral representation [2]

$$X(t) = \int e^{j2\pi ft} d\widetilde{X}(f) \qquad (1)$$

where $d\widetilde{X}(f)$ is the complex valued increment process (or the generalized Fourier transform) of the process $X(t)$. If time is continuous, the integration limits in Eq. (1) are $\pm\infty$, and if time is discrete, the limits are $\pm 1/2\Delta t$, where Δt is the equidistant sampling interval. Since we assume $X(t) \in \mathbb{R}$, the increment process has a useful Hermitian symmetry, $d\widetilde{X}^*(f) = d\widetilde{X}(-f)$, where asterisk denotes complex conjugation.

The class of harmonizable nonstationary processes is now defined as the processes with non-orthogonal increments [12, 1, 15], i.e.,

$$Ed\widetilde{X}(f_1)d\widetilde{X}(f_2) = S_{X,L}(f_1,f_2)df_1 df_2, \qquad (2)$$

where $S_{X,L}(f_1,f_2)$ is some complex valued function of f_1 and f_2. The dual-frequency function $S_{X,L}(f_1,f_2)$ is often called the Loève spectrum of the process. The representation (1) with (2) is possible if $S_{X,L}(f_1,f_2)$ satisfies the *Loève criterion*

$$\int\int \left|S_{X,L}(f_1,f_2)\right| df_1 df_2 < \infty. \qquad (3)$$

In this case, the relation between the temporal correlation function and the Loève spectrum takes the form [12]

$$EX(t_1)X(t_2) = \iint e^{j2\pi(f_1t_1+f_2t_2)} S_{X,L}(f_1,f_2)df_1 df_2. \qquad (4)$$

From Eq. (4) we understand that in this representation, spectral correlation among different frequency components is responsible for the nonstationarities. Knowing the spectral correlation in detail is therefore useful for characterizing the nature of the nonstationarity of $X(t)$.

Note that stationary processes are also included in (1), (2) if the Loève spectrum is confined to a delta-ridge along the diagonal $f_1 = -f_2$,

$$S_{X,L}(f_1,f_2) = S_X(f_1)\delta\left(f_1+f_2\right). \qquad (5)$$

Here, $S_X(f) \geq 0$ is the conventional power spectral density. Thus, stationary processes do not possess spectral correlations, and the line $f_1+f_2 = 0$ signifies the so-called *stationary manifold* in the dual-frequency plane.

3. BASIC REPRESENTATIONS

In the following, we will assume that the process is harmonizable, i.e., the spectral representation (1) with (2) holds. Results confined to energy processes were presented in [18].

As our basic nonstationary second order moment in the time domain, we define the *correlation function* of $X(t)$ by

$$M_X(t,\tau) \equiv EX(t)X(t+\tau). \tag{6}$$

Here, t is a *global* (or absolute) time variable, and τ is a *local* (or relative) time variable.

Expressing the nonstationary correlation function (6) by means of the non-orthogonal spectral representation, we observe that

$$\begin{aligned} M_X(t,\tau) &= E\int e^{j2\pi f' t} d\widetilde{X}(f') \int e^{j2\pi f(t+\tau)} d\widetilde{X}(f) && (7)\\ &= \int\int e^{j2\pi\nu t} e^{j2\pi f\tau} S_X(\nu,f) d\nu df, && (8)\end{aligned}$$

where $\nu = f + f'$. We thus understand ν as a *frequency offset* or a *local* frequency relative to the *global* frequency f. We immediately identify the *dual-frequency spectral density* $S_X(\nu,f)$ as

$$S_X(\nu,f) d\nu df = Ed\widetilde{X}^*(f-\nu)\, d\widetilde{X}(f). \tag{9}$$

We observe that the correlation function and the dual-frequency spectrum form a 2-dimensional Fourier transform pair:

$$M_X(t,\tau) \longleftrightarrow S_X(\nu,f). \tag{10}$$

Note that we find it more convenient to deal with the frequency pair (ν,f) rather than (f_1,f_2). As will be discussed later, the local frequency ν is by definition the nonstationary frequency coordinate, while the global frequency f is the stationary coordinate. Thus, any dependence on ν is a sign of nonstationary behavior.

We understand that our dual-frequency spectrum $S_X(\nu,f)$ is related to the Loève spectrum by a coordinate transformation. Note also that $S_X(\nu,f)$ is complex valued, even for real valued processes.

We now understand that we may invoke Fourier transforms with respect to any of the variables t, ν, τ, and f, in order to exhaust all possible time and frequency representations to the n-th order. A very important quantity is now derived by means of an inverse Fourier transform of $S_X(\nu,f)d\nu df$, with respect to the local frequency ν. We obtain

$$\begin{aligned} P_X(t,f)df &\equiv df\int e^{j2\pi\nu t} S_X(\nu,f) d\nu && (11)\\ &= EX(t)d\widetilde{X}(f)e^{j2\pi ft} && (12)\end{aligned}$$

where $P_X(t,f)$ is the *time-frequency spectral density* for the process $X(t)$. Note that $P_X(t,f)$ is a function of global time t and global frequency f. It is important to note that the time-frequency and the dual-frequency spectra are a Fourier transform pair in the variables t and ν,

$$S_X(\nu,f) \longleftrightarrow P_X(t,f). \tag{13}$$

From Eq. (12) we understand that the complex valued time-frequency spectrum $P_X(t,f)$ is in fact a generalization of the deterministic Rihaczek distribution [16, 3] to random processes. It has been a widespread opinion that the Rihaczek distribution is of little value since it does not admit an interpretation as a distribution of power as a function of time and frequency. However, as will be shown in subsequent sections, the complex time-frequency representation has a powerful geometrical interpretation, which we believe is the correct way to understand this complex quantity.

$$\begin{array}{ccc} M_X(t,\tau) & \xrightarrow{\tau\longrightarrow f} & P_X(t,f) \\ \Big\downarrow t\rightarrow\nu & & \Big\downarrow t\rightarrow\nu \\ A_X(\nu,\tau) & \xrightarrow[\tau\longrightarrow f]{} & S_X(\nu,f) \end{array}$$

Figure 1: Fourier relations between the basic polyspectral densities of nonstationary harmonizable processes.

The fourth and last quantity we may construct results as an inverse Fourier transform of $S_X(\nu,f)d\nu df$, with respect to the global frequency vector f. This yields

$$\begin{aligned} A_X(\nu,\tau)d\nu &\equiv d\nu\int e^{j2\pi f\tau} S_X(\nu,f) df && (14)\\ &= E\int d\widetilde{X}(\nu-f)\, d\widetilde{X}(f)e^{j2\pi f\tau} && (15)\end{aligned}$$

The quantity $A_X(\nu,\tau)$ is the *ambiguity function*, which is a function of local frequency ν and local time τ. We observe that $S_X(\nu,f)$ and $A_X(\nu,\tau)$ constitute a Fourier transform pair in the variables f and τ,

$$S_X(\nu,f) \longleftrightarrow A_X(\nu,\tau). \tag{16}$$

We now understand that any of the four basic quantities $M_X(t,\tau)$, $P_X(t,f)$, $A_X(\nu,\tau)$, and $S_X(\nu,f)$ may be used to characterize the second order behavior of a nonstationary stochastic process.

It is very important to note that these four basic densities are interrelated by Fourier transforms, as illustrated by the four corners diagram in Fig. 1.

It is an interesting historical fact that Hagfors already in the early 1960's [6, 7] considered quantities very similar to those discussed in the present paper. His original applications were the study of fading radar backscatter from the lunar surface [6], and the characterization of nonstationary random propagation circuits [7].

4. MARGINALS

We are interested in the marginals of the global time – global frequency spectrum $P_X(t,f)$. The time marginal is readily derived as

$$\int P_X(t,f)df = M_X(t,0) = EX^2(t). \tag{17}$$

The frequency marginal is

$$\begin{aligned} df\int P_X(t,f)dt &= \int EX(t)d\widetilde{X}(f)e^{j2\pi ft}dt && (18)\\ &= E\left|d\widetilde{X}(f)\right|^2 = S_X(0,f)(df)^2. && (19)\end{aligned}$$

It is interesting and reassuring to note that the time marginal, Eq. (17) is in fact the instantaneous power of the process, and the frequency marginal, Eq. (19) is the conventional power spectrum that one would normally associate with a stationary process. Thus, even though $P_X(t,f)$ is a complex valued quantity in the (t,f)-plane, both its marginals are real and non-negative. Note that this does in no way implies that $P_X(t,f)$ could be interpreted as a power distribution in time and frequency.

5. TWO IMPORTANT COHERENCES

The concept of coherence is very important when quantifying linear relationships within random processes. Various definitions of a quantity measuring the degree of coherence can be found in the literature. The value of such a measure should preferably be bounded between zero and one. Normalized versions of $S_X(\nu,f)$ and $P_X(t,f)$ are called for since the concept of coherence is related to the *phase* of the increment process, rather than its magnitude.

5.1 Dual-frequency coherence

A meaningful way of defining a dual-frequency coherence function can be obtained by recognizing the fact that the dual-frequency spectrum $S_X(\nu,f)$ can be expressed as a Hilbert space inner product [5]

$$S_X(\nu,f)\,d\nu df = \left\langle d\widetilde{X}(f)\,,\,d\widetilde{X}(f-\nu)\right\rangle, \qquad (20)$$

where the Hilbert space inner product between any two complex valued random variables X and Y is defined by

$$\langle X\,,\,Y\rangle \equiv EXY^*. \qquad (21)$$

Now, there is an angle ψ associated with any inner product $\langle X\,,\,Y\rangle$, defined by

$$\cos\psi \equiv \frac{\langle X\,,\,Y\rangle}{\sqrt{\langle X\,,X\rangle\,\langle Y\,,\,Y\rangle}}. \qquad (22)$$

We can now define the *dual-frequency squared coherence function* [21, 5]

$$\begin{aligned}\rho_X^2(\nu,f) &\equiv \cos^2\psi_X(\nu,f)\\ &= \frac{\left|\left\langle d\widetilde{X}(f)\,,\,d\widetilde{X}(f-\nu)\right\rangle\right|^2}{\left\langle d\widetilde{X}(f)\,,\,d\widetilde{X}(f)\right\rangle\left\langle d\widetilde{X}(f-\nu)\,,\,d\widetilde{X}(f-\nu)\right\rangle}\\ &= \frac{\left|Ed\widetilde{X}^*(f-\nu)d\widetilde{X}(f)\right|^2}{E\left|d\widetilde{X}(f-\nu)\right|^2 E\left|d\widetilde{X}(f)\right|^2}\\ &= \frac{|S_X(\nu,f)|^2}{S_X(0,f-\nu)S_X(0,f)},\end{aligned} \qquad (23)$$

The Cauchy-Schwarz inequality states that

$$|\langle X\,,\,Y\rangle|^2 \le \langle X\,,X\rangle\,\langle Y\,,\,Y\rangle, \qquad (24)$$

with equality if and only if $X = \alpha Y$ for some $\alpha \in \mathbb{R}$. Applied to (20) this gives

$$0 \le \rho_X^2(\nu,f) \le 1, \qquad (25)$$

with equality if and only if

$$d\widetilde{X}(f-\nu) = \alpha d\widetilde{X}(f). \qquad (26)$$

By expressing the complex valued increment processes $d\widetilde{X}(f)$ in polar form

$$d\widetilde{X}(f) \equiv |d\widetilde{X}(f)|e^{j\phi_X(f)}, \qquad (27)$$

we arrive at the fundamental result that we have full coherence between frequency components at f and $f-\nu$ if and only if

$$\phi_X(f-\nu) = \phi_X(f) + k\pi \qquad k\in\mathbb{Z}. \qquad (28)$$

5.2 Time-frequency coherence

The complex time-frequency spectrum has a similar representation, since $P_X(t,f)$ is a Hilbert space inner product

$$P_X(t,f)df = \left\langle d\widetilde{X}(f)e^{j2\pi ft}, X(t)\right\rangle. \qquad (29)$$

The corresponding *time-frequency squared coherence function* [5] can be defined by

$$\begin{aligned}\gamma_X^2(t,f) &= \frac{\left|\left\langle d\widetilde{X}(f)e^{j2\pi ft}\,,\,X(t)\right\rangle\right|^2}{\langle X(t)\,,X(t)\rangle\left\langle d\widetilde{X}(f)\,,\,d\widetilde{X}(f)\right\rangle}\\ &= \frac{\left|EX(t)d\widetilde{X}(f)e^{j2\pi ft}\right|^2}{EX^2(t)E\left|d\widetilde{X}(f)\right|^2} = \frac{|P_X(t,f)|^2}{M_X(t,0)S_X(0,f)},\end{aligned} \qquad (30)$$

Again, the Cauchy-Schwarz inequality guarantees that $\gamma_X^2(t,f)$ is bounded by 0 and 1, and that full coherence at global time t and global frequency f is achieved if and only if

$$X(t) = \alpha d\widetilde{X}(f)e^{j2\pi ft}. \qquad (31)$$

Employing the polar form of the increment process, we can alternatively state that full coherence at time t for frequency f is achieved if and only if

$$\phi_X(f) = -2\pi ft + k\pi \qquad k\in\mathbb{Z}. \qquad (32)$$

6. GENERALIZATIONS AND EXTENSIONS

One may readily extend the formalism presented here to a number of important and relevant generalizations.

The generalization to time-frequency and dual-frequency representations of a *pair of processes*, is straightforward, as shown in [21, 10]. The resulting generalized cross-coherences are important for the study of two-channel problems, e.g., for nonstationary linear time-variant systems.

The generalization to time-varying higher-order polyspectra and higher-order time-frequency representations is non-trivial [4]. In a recent paper [5], the theoretical framework was laid, and several important applications of the higher-order theory was identified.

A generalization to random fields is mandatory for the description of temporally nonstationary and spatially inhomogeneous phenomena. Such phenomena may occcur e.g. in acoustic and electromagnetic wave propagation. In particular, a theory of nonstationary and inhomogeneous random fields may be useful for improving the results from array processors. Nonstationary and inhomogeneous random fields has been discussed in [10, 11].

7. ESTIMATORS

We can see from (1) and (9) that the spectral properties of the process $X(t)$ is contained in the (unobservable) increment process $d\widetilde{X}(f)$. With this in mind, estimation of the dual-frequency spectrum (9) and its generalizations can now be viewed as estimation of the moments of $d\widetilde{X}(f)$. Thus, estimation of the increment process itself is of great interest. This important issue was treated in detail in Thomson's seminal 1982 paper on multitaper spectral estimators [19], with further details in [20]. Interesting applications of dual-frequency spectral estimation were included in [8] and [21].

Other important classes of estimators for the time-frequency distribution were presented in [17], and a comprehensive review including numerical examples can be found in [10]. The last reference shows examples of estimators of the quantities in all four corners of Fig. 1.

8. DISCUSSION AND CONCLUSIONS

As shown in this paper, the complex valued Rihazcek time-frequency spectrum is the natural time-frequency distribution to associate with harmonizable random processes. Likewise, the complex valued Loève dual-frequency spectrum appears to be the natural dual-frequency description.

It was argued that these quantities are *not* to be understood as power as a function of time and frequency, and dual-frequency, respectively. Instead, we offered an explanation of these quantites in terms of two different Hilbert space inner products. We defined associated time-frequency and dual-frequency coherences, respectively, and argued that these were the relevant second order quantities for characterizing the nonstationarities. We thus conclude that the time-frequency squared coherence is a measure of a relevant time-frequency inner product, and the dual-frequency is a measure of a relevant dual-frequency inner product.

Armed with the Hilbert space inner product interpretations, we firmly believe that the Loève dual-frequency spectrum and the Rihaczek time-frequency spectrum will experience a revival, and become useful for the analysis of nonstationary random processes.

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