WAVE DIGITAL FILTERS WITH MULTIPLE NONLINEARITIES

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ABSTRACT

The wave digital filter (WDF) theory provides an elegant method for the discretization of linear continuous filters which is well known for possessing many desirable properties over other filter implementations. However, concerning nonlinear networks so far there are only solutions for circuits with a single nonlinearity, as the delay-free-feedback of the nonlinear element may result in delay-free-loops for multiple nonlinearities. In this paper we present a method to circumvent this problem by the introduction of a vector-nonlinearity. We show how to incorporate multiple nonlinear elements in a WDF structure and how to implement this method using the look-up-tables of the nonlinearities. This method of implementation is demonstrated for the scalar case by the well known Chua's circuit and for the vector case by a circuit with two real diodes.

1. INTRODUCTION

Wave digital filters (WDFs) can be considered to be digital models of their continuous-time reference network. The discretization is performed by the well-known bilinear transformation and the forward and backward traveling wave quantities are used instead of extrinsic and intrinsic physical variables (for example voltage and current), see [1] for details. For an electrical circuit and voltage wave variables the transformation from the Kirchhoff variables (K-variables) voltage v and current i to the wave variables (W-variables) a and b is given by (1), R is the so called port resistance.

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & R \\ 1 & -R \end{pmatrix} \begin{pmatrix} v \\ i \end{pmatrix} \tag{1}$$

WDFs include a set of advantages, like coefficient accuracy, dynamic range, stability under finite-arithmetic conditions and last but not least a structure preserving implementation of the reference network. Particularly due to their structure preserving property they are widespread used in physical modeling and for the solution of partial differential equations (see [2] for instance). Especially applications with high demands on efficiency, like sound synthesis by physical modeling [3], take advantage of WDFs.

Concerning nonlinear WDFs, methods are proposed in [4] how to incorporate a single memoryless nonlinearity in a Wave Digital (WD) system. In [5] this method was extended to nonlinearities with memory. However all these methods are restricted to single nonlinearities, as the delay free feedback of a nonlinear WDF may cause delay free loops in the entire network. In this paper we show how to circumvent this problem by the usage of a vector approach. Any arbitrary number of nonlinearities can be modeled by one single vector-nonlinearity, so that the complete network is the combination of one nonlinear vector-WDF with a delay free feedback, adapted to the linear part of the network which includes a delay.

Furthermore we show how to practically implement networks with multiple nonlinearities by using continuous piecewise linear functions to approximate the characteristics of the nonlinearities. By two examples it is demonstrated, how nonlinear systems can be implemented efficiently without any iteration method.

This paper is structured as follows: Nonlinear Wave Digital Filters are introduced in section 2. It is shown how nonlinearities have been modeled so far in a scalar fashion and how to extend this method to vector nonlinearities. In section 3 the practical implementation of multiple nonlinearities by their look-up tables is discussed. For illustration, this method is demonstrated by the scalar example of Chua's circuit in section 4.1 and by a vector example of two real diodes in section 4.2. Section 5 concludes this paper.

2. NONLINEAR WAVE DIGITAL FILTERS

In the following first a description for scalar nonlinear WDFs is given and then extended to vector nonlinear WDFs. Without loss of generality memoryless nonlinearities are used for the reason of simpler notation. Nonlinearities with memory are introduced by frequency dependent W-variables (see [5] for details).

2.1 Scalar Case

Any memoryless nonlinear network element can be described as a subset of the vector space which is spanned by its port variables. The only condition that must be fulfilled by the element to achieve a unique solution for its port variables, is that it must halve the degrees of freedom of the port variables. For the scalar case with two physical port variables, there must be a parametrization for on scalar variable α, so that the nonlinear element can be described by

$$f = \left\{ \begin{pmatrix} v \\ i \end{pmatrix} \middle| \begin{pmatrix} v \\ i \end{pmatrix} = \begin{pmatrix} f_v(\alpha) \\ f_i(\alpha) \end{pmatrix}, \alpha \in V \subseteq \mathbb{R} \right\} . \tag{2}$$

Using this graphical description of the nonlinear element, the transition from K- to W-variables can be simply interpreted as the linear transformation of the coordinate system according to equation (1). The nonlinear element is described by

$$f = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} f_a(\alpha) \\ f_b(\alpha) \end{pmatrix}, \alpha \in V \subseteq \mathbb{R} \right\} , \tag{3}$$

there the functions $f_a(\alpha)$ and $f_b(\alpha)$ are defined by

$$f_a(\alpha) = f_v(\alpha) + R f_i(\alpha) , \tag{4}$$

$$f_b(\alpha) = f_v(\alpha) - R f_i(\alpha) . \tag{5}$$

To achieve a reflected wave $b = f_b(\alpha)$ (see equation (5) as a function of the incident wave a the unique invertibility of f_a (see equation (4)) is sufficient and necessary, so that the reflected wave b can be written as

$$b = b(a) = f_b\left(f_a^{-1}(a)\right) . \tag{6}$$

Obviously f_a^{-1} exists if f_a is strictly monotonic, so if f_a is continuous and piecewise-differentiable, it has to fulfill the condition

$$f_a' > 0 \quad \vee \quad f_a' < 0 . \tag{7}$$

Note, that an explicit solution of f in the $v \times i$-domain ($i = i(v)$ for instance) is not necessarily needed for a solution of f in the wave-domain ($b = b(a)$). The special case , where i can be expressed as $i = i(v)$ is treated in more detail in [4] and is included here for $f_v(\alpha) = \alpha$ and $f_i(\alpha) = i(\alpha)$.

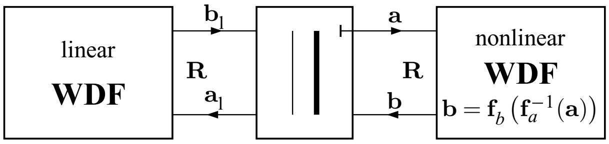

Figure 1: Principal diagram of a wave digital model of a circuit with multiple nonlinearities.

2.2 Vector Case

The extension of wave digital filtering principles to the vector case is straightforward and was already outlined in [6] and [2]. Here it is important, that we can describe any n-port by the $n \times 1$ wave vectors $\mathbf{a}$ and $\mathbf{b}$ and the $n \times n$ port resistance matrix $\mathbf{R}$. The transformation from the K-variables $\mathbf{v} = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}^T$ and $\mathbf{i} = \begin{pmatrix} i_1 & i_2 & \dots & i_n \end{pmatrix}^T$ to the W-variables $\mathbf{a}$ and $\mathbf{b}$ is done in almost the same manner as for the linear case ($\mathbf{I}$ denotes the $n \times n$ identity matrix)

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{R} \\ \mathbf{I} & -\mathbf{R} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{i} \end{pmatrix} . \tag{8}$$

In the sequel we will assume linear independent port variables, which means that the port resistance matrix $\mathbf{R}$ is of full rank. In all other cases we could reduce the number of ports without loss of information.

With these vector wave variables we can describe any nonlinear n-port element in the same way as it was done in equation (2) for the scalar case. The only difference is that we can parametrize for n independent variables $\alpha_1, \dots, \alpha_n$ and that all nonlinear functions now map one $n \times 1$ vector to another $n \times 1$ vector. With $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \end{pmatrix}^T$ we can adopt the wave variable description of the scalar nonlinear element in equation (3) to the vector case

$$\mathbf{f} = \left\{ \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \middle| \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_a(\alpha) \\ \mathbf{f}_b(\alpha) \end{pmatrix}, \alpha \in V \subseteq \mathbb{R}^n \right\} , \tag{9}$$

where the vector functions $\mathbf{f}_a(\alpha)$ and $\mathbf{f}_b(\alpha)$ are defined by

$$\mathbf{f}_a(\alpha) = \mathbf{f}_v(\alpha) + \mathbf{R}\mathbf{f}_i(\alpha) , \tag{10}$$
$$\mathbf{f}_b(\alpha) = \mathbf{f}_v(\alpha) - \mathbf{R}\mathbf{f}_i(\alpha) . \tag{11}$$

So, for an explicit solution of the nonlinear n-port element in the wave domain, $\mathbf{f}_a$ must be an one-to-one mapping with an unique inverse mapping $\mathbf{f}_a^{-1}$, so that we can write $\mathbf{b}$ as

$$\mathbf{b} = \mathbf{b}(\mathbf{a}) = \mathbf{f}_b\left(\mathbf{f}_a^{-1}(\mathbf{a})\right) . \tag{12}$$

This nonlinear vector WDF can be connected to any linear WDF by a reflection free port adaptor, as depicted in figure 1.

3. IMPLEMENTATION

So far, the definition of the nonlinear vector WDF in (12) is quite abstract and difficult to handle. Therefore in this section we will present a method to implement any nonlinearity which is given by look-up-tables by the usage of a continuous piecewise linear approximation of the function. To illustrate the method, first the scalar case is treated.

3.1 Scalar Case

Often nonlinearities are given in form of look-up tables, where a set of subsequent measurement points are archived. To extract an arbitrary point on the characteristic of the nonlinearity, one has to interpolate these measurement points. First order interpolation may be sufficient. Figure 2 depicts such an interpolation, which obviously is an continuous piecewise linear function.

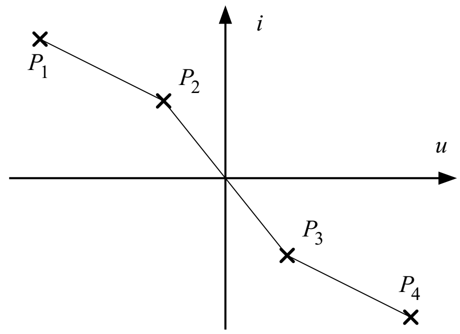

Figure 2: Nonlinear characteristic approximated by a continuous piecewise linear function.

It is straightforward to find a parameterization of this curve and write this nonlinear function in the form of equation (2). With N denoting the number of measurement points we can approximate any nonlinearity by

$$f = \bigcup_{k=1}^{N-1} \left\{ \begin{pmatrix} v \\ i \end{pmatrix} \middle| \begin{pmatrix} v \\ i \end{pmatrix} \in P_k + m \cdot \left(P_{k+1} - P_k\right) \right\} , \tag{13}$$

with $k = \lfloor \alpha \rfloor$ as the integer part and $m = \alpha - k$ as the fractional part of α and $\alpha \in [1, N]$. The transition to wave variables is very clear and just a linear transformation of the points P_k with $k \in \{1, N-1\}$ by

$$P_k^{\mathrm{w}} = \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \begin{pmatrix} 1 & R \\ 1 & -R \end{pmatrix} P_k . \tag{14}$$

The nonlinear WDF can be described by

$$f = \bigcup_{k=1}^{N-1} \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \begin{pmatrix} a \\ b \end{pmatrix} \in P_k^{\mathrm{w}} + m \cdot \left(P_{k+1}^{\mathrm{w}} - P_k^{\mathrm{w}}\right) \right\} , \tag{15}$$

there k and m are defined as above. So to achieve an explicit solution of f in the form of equation (6) we can simplify the condition given in equation (7) to the postulation, that the wave quantities a_k have to be a strictly monotonic increasing or decreasing sequence,

$$a_{k+1} > a_k \qquad \forall\ k \in \{1, \dots, N-1\} \tag{16}$$

or

$$a_{k+1} < a_k \qquad \forall\ k \in \{1, \dots, N-1\} . \tag{17}$$

For an implementation of this scalar nonlinear WDF one only has to find the appropriate k for the given incoming wave a by a suitable search algorithm and then determine m by $m = (a - a_k)/(a_{k+1} - a_k)$ to achieve the reflected wave b.

3.2 Vector Case

As in section 2.2, the extension to the vector case is straightforward. We assume, that n nonlinearities are given as continuous piecewise linear functions in the form of equation (13). We define $2n$ dimensional points

$$P_k^{(l)} = \begin{pmatrix} 0 & \dots & v_k^{(l)} & \dots & 0 & 0 & \dots & i_k^{(l)} & \dots & 0 \end{pmatrix}^T$$

analog to section 2.2, there l denotes the number of the nonlinearity and k denotes the number of the point. All coordinates with index unequal to l resp. $2l$ are zero.

With this definition of $2n$-dimensional points, the parametrization of the vector nonlinearity can be done similar to equation (13), whereas according to section 2.2 we have to find a n-dimensional parametrization. Nevertheless, due to the mutual orthogonality of the points $P_k^{(l)}$ regarding the index l, we can approximate any set of independent nonlinearities by the multidimensional continuous piecewise linear function

$$\mathbf{f} = \bigcup_{k_1=1}^{N_1-1} \cdots \bigcup_{k_n=1}^{N_n-1} \left\{ \left(\begin{array}{c} \mathbf{v} \\ \mathbf{i} \end{array} \right) \middle| \left(\begin{array}{c} \mathbf{v} \\ \mathbf{i} \end{array} \right) \in \left(P_{k_1}^{(1)} + \ldots + P_{k_n}^{(n)} \right) \right.$$
$$\left. + m_1 \left(P_{k_1+1}^{(1)} - P_{k_1}^{(1)} \right) + \ldots + m_n \left(P_{k_n+1}^{(n)} - P_{k_n}^{(n)} \right) \right\}, \quad (18)$$

where $\alpha_l \in [1, N_l]$ is the parameter of the scalar parametrization of the lth nonlinearity by equation (13) and k_l and m_l is the integer part resp. fractional part of α_l.

The transition to W-variables is performed by the definition of the vector wave variables in equation (8) analog to the scalar case in equation (14). It results in a multidimensional nonlinear WDF described by a set of points in the $2n$-dimensional space

$$\mathbf{f} = \bigcup_{k_1=1}^{N_1-1} \cdots \bigcup_{k_n=1}^{N_n-1} \left\{ \left(\begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} \right) \middle| \left(\begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} \right) \in \left(P_{k_1}^{\mathrm{w},(1)} + \ldots + P_{k_n}^{\mathrm{w},(n)} \right) \right.$$
$$\left. + m_1 \left(P_{k_1+1}^{\mathrm{w},(1)} - P_{k_1}^{\mathrm{w},(1)} \right) + \ldots + m_n \left(P_{k_n+1}^{\mathrm{w},(n)} - P_{k_n}^{\mathrm{w},(n)} \right) \right\}. \quad (19)$$

However, the proof that $\mathbf{f}$ is an unique mapping from the incoming wave $\mathbf{a}$ to the reflected wave $\mathbf{b}$ (i.e. to proof the uniqueness of equation (12)) cannot be conducted in general. One can proof local invertibility with the well known inverse function theorem by testing the Jacobian matrix for invertibility. In our case this would lead to the condition, that all the incoming wave components of the vectors in equation (19) must form an invertible matrix for all linear sub-elements of the nonlinear function $\mathbf{f}$, in more detail

$$\left| \begin{array}{ccc} \mathbf{a}_{k_1+1}^{(1)} - \mathbf{a}_{k_1}^{(1)} & \cdots & \mathbf{a}_{k_n+1}^{(n)} - \mathbf{a}_{k_n}^{(n)} \end{array} \right| \neq 0, \quad (20)$$

for all possible variations of the indices k_1 to k_n.

Another strong indication of global invertibility which holds in the majority of cases are the signs of the elements of the Jacobian matrix. Besides local invertibility, proofed by equation (20), we postulate that the signs of all elements of the Jacobian matrix are constant over the whole definition range. This condition is similar to the conditions (16) resp. (17) for the scalar case and can be checked by verifying

$$\operatorname{sign}\left(\begin{array}{ccc} \mathbf{a}_{k_1+1}^{(1)} - \mathbf{a}_{k_1}^{(1)} & \cdots & \mathbf{a}_{k_n+1}^{(n)} - \mathbf{a}_{k_n}^{(n)} \end{array} \right) = \mathrm{const} \quad (21)$$

for all $k_1 \in \{1; \ldots; N_1 - 1\}$, $k_2 \in \{1; \ldots; N_2 - 1\}$, … . However, condition (21) is an indicator not a real mathematical proof for global invertibility. In some very special cases, one may find more than one valid set of $k_1, \ldots, k_n$ that solve equation (19). An exemplification of a nonlinear vector WDF is given in section 4.2.

4. EXAMPLES

To illustrate the usage of piecewise linear functions first a scalar example of the implementation given in section 3 is shown here. Then an example with two realistic nonlinearities is given to demonstrate the capabilities of the proposed approach.

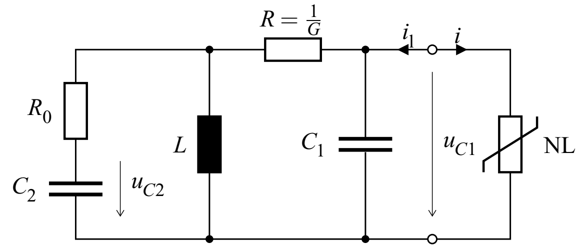

Figure 3: Chua's Circuit

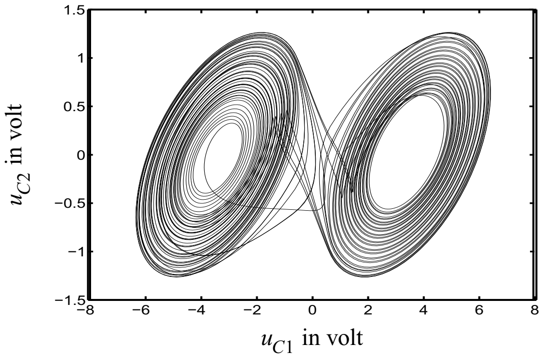

Figure 4: Behavior of Chua's circuit

4.1 Scalar Case: Chua's Circuit

The first example has already been studied by various authors (see e.g. [4], [7]) for its chaotic behavior, a property unpleasant to analyze. The so called Chua's circuit, as depicted in figure 3, contains besides several linear one-port elements one nonlinear resistance denoted by NL. The characteristic of this nonlinearity is already schematically depicted in figure 2, it is a piecewise defined negative linear resistance with two different slopes.

The network circulates around the points P_2 resp. P_3 in figure 2, as it is amplified between the points P_2 and P_3 and damped between the points P_1 and P_2 resp. P_3 and P_4. Furthermore it changes the centerpoint of this circulation in a chaotic fashion.

The network was simulated as follows. First the nonlinearity was described according to equation (13), which for this case is an exact description. Then the linear part was discretized by basic WDF principles as written in [1], which also provide the port resistance R of the linear part. With this port resistance, the nonlinear function was transformed to the WD domain by equation (14). The resulting nonlinear WDF was connected to the linear part as shown in figure 1 by a parallel port adaptor with one reflection free port.

During simulation the scalar nonlinear WDF described by equation (15) is implemented in two steps. For every incident wave a one first has to search for the correct index k which fulfills the inequality $a_k \leq a < a_{k+1}$. This can be done by efficient search-algorithms due to the subsequent definition of the a_k (see equation (16) and (17)). Then one can determine the fractional part of the parametrization m according to section 3.1 and finally one can calculate the reflected wave b. Even for large numbers of linear sub-elements N the computational effort is moderate as the actual calculation has to be performed only once and the effort for the search algorithm only increases with the logarithm of N.

A result of the simulation is shown in figure 4. Plotting the voltage of the second capacity C_2 against the voltage of the first capacity C_1 clearly demonstrates the chaotic behavior of Chua's circuit and the two attractors of the system.

4.2 Vector Case: Two Diodes

The second examples demonstrates the usage of the introduced nonlinear vector WDFs to simulate systems with multiple nonlineari-

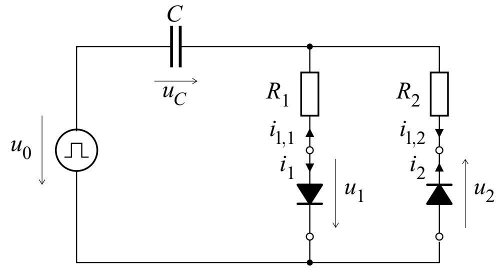

Figure 5: Electrical damper with different attack and release time constants, realized with two diodes. $R_2 = 5R_1$.

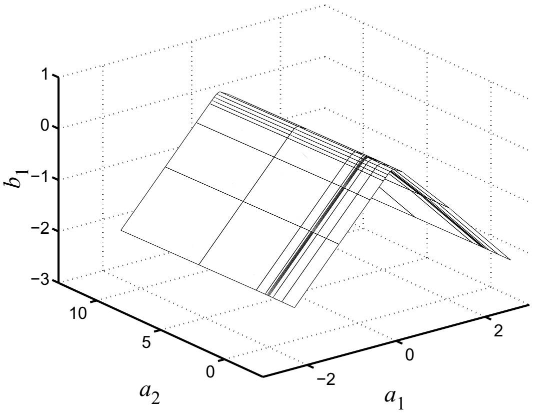

Figure 6: Reflected wave b_1 of the first diode as a function of both incident waves a_1 and a_2.

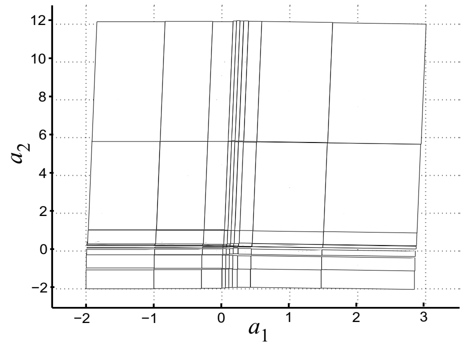

Figure 7: Projection of $b_1(a_1, a_2)$ onto the $a_1 \times a_2$ plane.

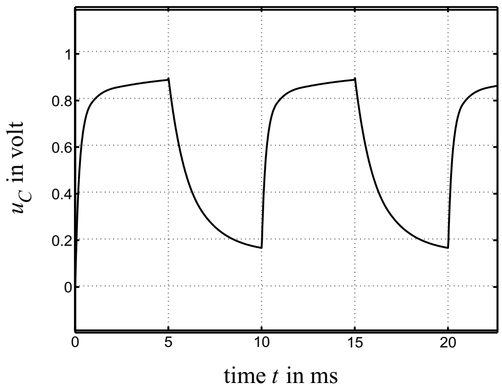

Figure 8: Voltage of the capacity from figure 5.

ties. The circuit is depicted in figure 5 and includes two diodes. Their characteristics are modeled according to W. Shockley by

$$i = i_S \left(\exp\left(\frac{e}{kT} u \right) - 1 \right) . \tag{22}$$

The approach to simulation differs only slightly from the scalar case described in section 4.1. The main differences, which are also of particular interest in this context, lie in the vector nonlinearity, which will be described in the sequel.

To achieve a description according to equation (19), the characteristics were non uniformly sampled and N_1 resp. N_2 4-dimensional points were created. Then all points were transformed by equation (8) using the port resistance matrix $\mathbf{R}$ from the linear part of the network. With these points the vector nonlinearity was defined by equation (19). Figure 6 depicts the reflected wave of the first diode b_1 as a function of the incident waves a_1 and a_2 sampled at the $N_1 \times N_2$ different points $P^{\mathrm{w},(1)}_{k_1} + P^{\mathrm{w},(2)}_{k_2}$. The sampling is more dense in interesting regions. The top view of $b_1(a_1, a_2)$ is given in figure 7. In contrast to the definition of the points $P^{(l)}_k$ in the $v \times i$-domain, here the points are neither in line with the axes a_1 and a_2 nor do they form a straight line. However, one can see that they fulfill condition (21) and that this condition is a strong one which also admits the application of suitable search algorithms to find the correct segment.

The simulation itself follows the procedure from section 4.1 for the scalar case. The absolute number of function segments now is $(N_1 - 1) \cdot (N_2 - 1)$, however the computational cost for the search algorithm only increases with $\log N_1 + \log N_2$. The result was simulated with a rectangular input function u_0 at a frequency of 100Hz and an amplitude of 1V for Germanium diodes. Figure 8 shows that the charge and discharge of the capacity follows different time constants.

5. CONCLUSIONS

In this paper we presented a new approach for the realization and efficient implementation of nonlinear WDFs, which allows multiple nonlinearities in the reference system. With some mild restrictions we are able to simulate almost any mixed linear-nonlinear network without iterative methods. An important application is the area of physical modeling, where WDFs are an object of current research.

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