NON ASYMPTOTIC EFFICIENCY OF A MAXIMUM LIKELIHOOD ESTIMATOR AT FINITE NUMBER OF SAMPLES

(1)*Alexandre Renaux,* (2)*Philippe Forster, and* (1)*Eric Boyer*

(1)SATIE UMR 8029, Ecole Normale Superieure de Cachan
61, avenue du President Wilson, 94235 Cachan Cedex, FRANCE
(2)GEA, IUT de Ville d'Avray
1, chemin Desvallieres, 92410 Ville d'Avray, FRANCE
Renaux@satie.ens-cachan.fr, Philippe.Forster@cva.u-paris10.fr, e.boyer@tele2.fr

ABSTRACT

In estimation theory, the asymptotic (in the number of samples) efficiency of the Maximum Likelihood (ML) estimator is a well known result [1]. Nevertheless, in some scenarios, the number of snapshots may be small. We recently investigated the asymptotic behavior of the Stochastic ML (SML) estimator at high Signal to Noise Ratio (SNR) and finite number of samples [2] in the array processing framework: we proved the non-Gaussiannity of the SML estimator and we obtained the analytical expression of the variance for the single source case. In this paper, we generalize these results to multiple sources, and we obtain variance expressions which demonstrate the non-efficiency of SML estimates.

1. INTRODUCTION

In array processing, the asymptotic performances of the SML estimator are well known when the number T of observations tends to infinity [3]: it is asymptotically efficient and Gaussian. This work addresses the problem of the SML behavior for a finite number of samples when the SNR tends to infinity: this is the meaning of *asymptotic* in this paper. We have recently shown that in this particular scenario, the SML estimator is non-Gaussian. We also derived its asymptotic distribution in the single source case: it is a Student distribution. Furthermore the asymptotic variance shows the non-efficiency of the SML estimator [4]. This paper extends this analysis to the multiple sources case: the two sources case is fully addressed by deriving the asymptotic variance of the SML estimates: a comparison with the Stochastic Cramer Rao Lower Bound (SCRLB) shows the asymptotic non-efficiency of the SML estimator.

This paper is organized as follows. Section 2 presents the signal model and the background of our work. The SCRLB is briefly recalled in section 3. The variance in single and two sources cases are derived and compared to the SCRLB in section 4. To confirm our results, simulations are performed in section 5. Finally, conclusions are given section 6.

2. MODEL AND BACKGROUND

2.1 Notations

The notational convention adopted is as follows: italics indicates a scalar quantity, as in A; lower case boldface indicates a vector quantity, as in $\mathbf{a}$; upper case boldface indicates a matrix quantity, as in $\mathbf{A}$. The n-th row and m-th column of matrix $\mathbf{A}$ will be denoted by $(A_{n,m})$. $\Re\{A\}$ is the real part of A. The complex conjugation of a quantity is indicated by a superscript * as in A^*. The matrix transpose is indicated by a superscript $^{\mathrm{T}}$ as in $\mathbf{A}^{\mathrm{T}}$, and the complex conjugate plus matrix transpose is indicated by a superscript $^{\mathrm{H}}$ as in $\mathbf{A}^{\mathrm{H}} = \left(\mathbf{A}^{\mathrm{T}}\right)^*$. $|A|$ indicates the absolute value of A. $Tr\{\mathbf{A}\} = \sum_{i=1}^{P} (A_{i,i})$ is the trace operator of the P order matrix $\mathbf{A}$. $Diag\{A_1, A_2, \cdots\}$ is the diagonal matrix built with $A_1, A_2, \cdots$. $\odot$ denotes the Hadamard product (element by element product). $E[\cdot]$ denotes the expectation operator. $\mathbf{I}_M$ is the identity matrix of order M. $\Gamma(\cdot)$ denotes the Gamma function[1]. ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function [5] such as:

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}. \quad (1) $$

2.2 Parametric data model

Let us consider the classical problem of localizing N sources with an array of $M \geq N$ sensors and a narrow-band far field source impinging on it. The vector $\mathbf{x}(t)$ of sensors outputs is given by the following equation [6]:

$$ \mathbf{x}(t) = \mathbf{A}(\theta_0)\mathbf{s}(t) + \mathbf{n}(t), \quad (2) $$

where $t = 1, 2, \cdots, T$ and T is the number of snapshots. $\theta = [\theta_1, \theta_2, \cdots, \theta_N]^{\mathrm{T}}$ denotes the candidate vector of the N Directions Of Arrival (DOA) whose exact value is θ_0. $\mathbf{A}(\theta) = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_N)]$ is the $M \times N$ steering vectors matrix. $\mathbf{s}(t)$ is the $N \times 1$ vector of the signals emitted by the N sources. $\mathbf{n}(t)$ is the $M \times 1$ noise vector.

2.3 Assumptions

The following assumptions will be made:

A1 The signal $\mathbf{s}(t)$ is complex, circular, Gaussian, spatially and temporally white with zero mean and covariance matrix $\mathbf{\Sigma_s} = E\left[\mathbf{s}(t)\ \mathbf{s}(t)^{\mathrm{H}}\right]$.

A2 The noise $\mathbf{n}(t)$ is complex, circular, Gaussian, spatially and temporally white with zero mean and covariance matrix $\mathbf{\Sigma_n} = \sigma^2 \mathbf{I}_M$.

A3 The signals sources are uncorrelated.

A4 $\forall \theta$, $\|\mathbf{a}(\theta)\| = 1$.

[1] $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$.
If $z \in \mathbb{N}$, $\Gamma(z) = (z-1)!$

2.4 Background

Let us denote the SML estimator by $\widehat{\theta}$ and $\widetilde{\theta} = \frac{1}{\sigma}\left(\widehat{\theta}-\theta_0\right)$. We have shown in [2] that when $\sigma \rightarrow 0$ (for fixed sources power), $\widetilde{\theta}$ is asymptotically distributed as $\mathbf{C}\,\mathbf{y}$ where $\mathbf{y}$ is a $N \times 1$ standard Gaussian vector and $\mathbf{C}$ is a $N \times N$ random matrix independent of vector $\mathbf{y}$. The matrix $\mathbf{C}$ is such that:

$$\mathbf{C}\mathbf{C}^{\mathrm{T}} = \frac{1}{2T}\left(\Re\left\{\mathbf{H}(\theta)\odot\hat{\mathbf{\Sigma}}_{\mathbf{s}}^{\mathrm{T}}\right\}\right)^{-1}, \tag{3}$$

where $\hat{\mathbf{\Sigma}}_{\mathbf{s}}$ is the sample covariance matrix of sources signals. $\hat{\mathbf{\Sigma}}_{\mathbf{s}}^{\mathrm{T}}$ is a $N \times N$ random matrix which follows a complex Wishart distribution [7] with T degrees of freedom and parameter matrix the covariance $\mathbf{\Sigma}_{\mathbf{s}}$ of sources signals $\mathbf{s}(t)$. $\mathbf{H}$ is a $N \times N$ deterministic matrix which contains the information about the DOA and the array structure:

$$\mathbf{H}(\theta) = \mathbf{D}^{\mathrm{H}}(\theta)\left[\mathbf{I}_M - \mathbf{A}(\theta)\left(\mathbf{A}^{\mathrm{H}}(\theta)\mathbf{A}(\theta)\right)^{-1}\mathbf{A}^{\mathrm{H}}(\theta)\right]\mathbf{D}(\theta), \tag{4}$$

with $\mathbf{D}(\theta) = \left[\left.\frac{\partial \mathbf{a}(\theta)}{\partial\theta}\right|_{\theta=\theta_1}, \left.\frac{\partial \mathbf{a}(\theta)}{\partial\theta}\right|_{\theta=\theta_2}, \ldots, \left.\frac{\partial \mathbf{a}(\theta)}{\partial\theta}\right|_{\theta=\theta_N}\right]$.

The asymptotic (as $\sigma \rightarrow 0$) covariance of $\widetilde{\theta}$ is given by:

$$cov\left(\widetilde{\theta}\right) = \frac{1}{2T}E\left[\left(\Re\left\{\mathbf{H}(\theta)\odot\hat{\mathbf{\Sigma}}_{\mathbf{s}}^{\mathrm{T}}\right\}\right)^{-1}\right]. \tag{5}$$

3. CRAMER RAO LOWER BOUND AT HIGH SNR

Following [6], the SCRLB can be written:

$$\mathbf{B}_{\mathrm{STO}} = \frac{\sigma^2}{2T}\left(\Re\left\{\mathbf{H}\odot\left(\mathbf{\Sigma}_{\mathbf{s}}\mathbf{A}^{\mathrm{H}}(\theta)\mathbf{\Sigma}_{\mathbf{x}}^{-1}\mathbf{A}(\theta)\mathbf{\Sigma}_{\mathbf{s}}\right)^{\mathrm{T}}\right\}\right)^{-1}, \tag{6}$$

where $\mathbf{H}$ is defined in (4).

Let us study the asymptotic behavior of (6) when the SNR tends to infinity. We will show that:

$$\lim_{\sigma\rightarrow 0}\frac{1}{\sigma^2}\mathbf{B}_{\mathrm{STO}} = \frac{1}{2T}\left(\Re\left\{\mathbf{H}\odot\mathbf{\Sigma}_{\mathbf{s}}^{\mathrm{T}}\right\}\right)^{-1}. \tag{7}$$

Proof: It is easily shown that in (6):

$$\mathbf{A}^{\mathrm{H}}(\theta)\mathbf{\Sigma}_{\mathbf{x}}^{-1}\mathbf{A}(\theta) = \left(\mathbf{\Sigma}_{\mathbf{s}} + \sigma^2\left(\mathbf{A}^{\mathrm{H}}(\theta)\mathbf{A}(\theta)\right)^{-1}\right)^{-1}, \tag{8}$$

which tends to $\mathbf{\Sigma}_{\mathbf{s}}^{-1}$ when σ^2 tends to 0. This proves (7). □

Remark: In the particular case of assumption **A3**, the sources signals covariance matrix $\mathbf{\Sigma}_{\mathbf{s}}$ is diagonal: $\mathbf{\Sigma}_{\mathbf{s}} = Diag\{\Sigma_1, \Sigma_{2,\ldots,}\Sigma_N\}$. Therefore, the SCRLB (7) can be written with the notation $H^{\mathrm{R}}_{n,m} = \Re\left\{(H)_{n,m}\right\}$:

$$\lim_{\sigma\rightarrow 0}\frac{1}{\sigma^2}\mathbf{B}_{\mathrm{STO}} = \frac{1}{2T}Diag\left\{\frac{1}{H^{\mathrm{R}}_{1,1}\Sigma_1}, \frac{1}{H^{\mathrm{R}}_{2,2}\Sigma_2}, \ldots, \frac{1}{H^{\mathrm{R}}_{N,N}\Sigma_N}\right\}. \tag{9}$$

4. VARIANCE AND NON-EFFICIENCY

We will start by recalling the analytical expression of the asymptotic SML estimator variance for the single source case together with a concise proof. Next we will address the much more involved case of two sources for which we derive an original explicit expression of the SML estimates asymptotic covariance matrix (5). It can be also shown that the SML covariance matrix in the multiple sources case can be obtained *in practice* by using the single and two sources expressions: however due to space limitation we will not address this point in this paper.

4.1 Single source case

Theorem 1: When $N = 1$, $\mathbf{\Sigma}_{\mathbf{s}} = \Sigma_1$ and matrix $\mathbf{H}$ (in equation (4)) is a scalar h_1. The asymptotic variance of $\tilde{\theta}$ is given by:

$$var(\tilde{\theta}) = \frac{1}{2h_1\Sigma_1}\frac{1}{T-1}. \tag{10}$$

This expression can be rewritten in terms of the asymptotic SCRLB (9) as:

$$var\left(\tilde{\theta}\right) = \frac{T}{T-1}k, \tag{11}$$

where $k = \lim_{\sigma\rightarrow 0}\frac{\mathrm{B_{STO}}}{\sigma^2}$.

Proof: By applying the results of section 2.4 to the single source case, we see from equation (3) that $\tilde{\theta}$ is asymptotically distributed as cy with:

$$c^2 = \frac{1}{2Th_1 v}, \tag{12}$$

and $v = \frac{1}{T}\sum_{k=1}^{T}|s(k)|^2$ is distributed as $\frac{\Sigma_1}{2T}$ times a chi-square random variable with $2T$ degrees of freedom (according to **A1**). Therefore:

$$var\left(\tilde{\theta}\right) = \frac{1}{h_1\Sigma_1}E\left[\frac{1}{\chi^2_{2T}}\right], \tag{13}$$

where χ^2_{2T} is a chi-square random variable with $2T$ degrees of freedom. According to [8], $E\left[1/\chi^2_{2T}\right] = 2T-2$: this completes the proof. □

Corollary 1: The SML estimator is not asymptotically efficient since $\frac{T}{T-1} > 1$. Furthermore, the minimal number of snapshots is $T = 2$. Indeed, if $T = 1$ the variance grows to infinity.

4.2 Two sources case

Most arrays met in practice possess two geometric properties: they have both a center and an axis of symmetry. This is for instance the case of the Uniform Linear Array (ULA) and the Uniform Circular Array. We will assume that these conditions are met. Under these assumptions, the matrix $\mathbf{H}$ of equation (4) is real. We will consider in our analysis the case of two symmetric sources $\theta_1 = -\theta_2$. In this case $\mathbf{H}$ has the following structure:

$$\mathbf{H} = \begin{pmatrix} h_1 & h_2 \\ h_2 & h_1 \end{pmatrix}. \tag{14}$$

Theorem 2: The asymptotic covariance of $\tilde{\theta}$ is finite for $T \geq 2$ and it is given

$$cov\left(\tilde{\theta}\right)=\frac{1}{2}\frac{1}{T-1}\,{}_2F_1\left(1,1;2T;\left(\frac{h_2}{h_1}\right)^2\right)Diag\left\{\frac{1}{h_1\Sigma_1},\frac{1}{h_1\Sigma_2}\right\},$$

$$cov\left(\tilde{\theta}\right)=\frac{T}{T-1}\,{}_2F_1\left(1,1;2T;\left(\frac{h_2}{h_1}\right)^2\right)\mathbf{K}\,, \tag{15}$$

with $\mathbf{K}=\lim_{\sigma\to 0}\frac{\mathbf{B}_{\mathrm{STO}}}{\sigma^2}$.

Proof: According to (5), and with the assumptions about the array geometry, we have:

$$\begin{aligned} cov\left(\tilde{\theta}\right) &= \frac{1}{2T}E\left[\left(\mathbf{H}\odot\Re\left\{\hat{\mathbf{\Sigma}}_{\mathbf{s}}^{\mathrm{T}}\right\}\right)^{-1}\right] \\ &= \frac{1}{2}E\left[\left(\mathbf{H}\odot\Re\{\mathbf{W}\}\right)^{-1}\right], \end{aligned} \tag{16}$$

where $\mathbf{W}$ is a $N\times N$ random matrix which follows a complex Wishart distribution with T degrees of freedom and parameter matrix the covariance, $\mathbf{\Sigma_s}=Diag\{\Sigma_1,\Sigma_2\}$, of emitted signals $\mathbf{s}(t)$ [7].

Therefore, with assumptions **A1** and **A3**, $\mathbf{W}_{\mathrm{R}}=\Re\{\mathbf{W}\}$ is a $N\times N$ random matrix which follows a real Wishart distribution with $2T$ degrees of freedom and parameter matrix the covariance $\frac{1}{2}\mathbf{\Sigma_s}$ of the emitted signal $\mathbf{s}(t)$. $\mathbf{W}_{\mathrm{R}}$ is a symmetric positive definite matrix. Therefore, we can use the Cholesky factorization: $\mathbf{W}_{\mathrm{R}}=\mathbf{D}\mathbf{D}^{\mathrm{T}}$, with:

$$\mathbf{D}=\left(\begin{array}{cc}\rho_1 & 0\\ \alpha & \rho_2\end{array}\right). \tag{17}$$

According to [8], the elements of $\mathbf{D}$ are distributed as follows[2]:

$$\left\{\begin{array}{l}\rho_1\sim\sqrt{\chi^2\left(2T,\frac{\Sigma_1}{2}\right)},\\ \rho_2\sim\sqrt{\chi^2\left(2T-1,\frac{\Sigma_2}{2}\right)},\\ \alpha\sim N\left(0,\frac{\Sigma_2}{2}\right),\end{array}\right. \tag{18}$$

where ρ_1, ρ_2, and α are independent.

The covariance of $\tilde{\theta}$

$$cov\left(\tilde{\theta}\right)=\left(\begin{array}{cc}var(\tilde{\theta}_1) & \Psi\\ \Psi & var(\tilde{\theta}_2)\end{array}\right), \tag{19}$$

is given by $\frac{1}{2}E\left[\left(\mathbf{H}\odot\mathbf{W}_{\mathrm{R}}\right)^{-1}\right]$ where:

$$\left(\mathbf{H}\odot\mathbf{W}_{\mathrm{R}}\right)^{-1}=\frac{1}{\Phi}\left(\begin{array}{cc}h_1\left(\rho_2^2+\alpha^2\right) & -h_2\rho_1\alpha\\ -h_2\rho_1\alpha & h_1\rho_1^2\end{array}\right), \tag{20}$$

with $\Phi=h_1^2\rho_1^2\left(\rho_2^2+\alpha^2\right)-\left(h_2\rho_1\alpha\right)^2$.

[2]The following notations are used:
$\sim$ means "distributed as".
$N(0,\varepsilon)$ is a Gaussian distribution with mean value 0 and variance ε.
Chi-square distribution: $\chi^2(t,\varepsilon)=\sum_{k=1}^{t}z_k^2$ with $z_k\sim N(0,\varepsilon)$. t is the number of degrees of freedom of the chi-square distribution.

First, we will calculate the expression of $var(\tilde{\theta}_1)$:

$$var(\tilde{\theta}_1)=\frac{1}{2h_1}E\left[\frac{1}{\rho_1^2}\frac{1}{1-\left(\frac{h_2}{h_1}\right)^2\frac{\alpha^2}{\alpha^2+\rho_2^2}}\right], \tag{21}$$

where $\alpha^2\sim\chi^2\left(1,\frac{\Sigma_2}{2}\right)$, and the ratio $\frac{\alpha^2}{\alpha^2+\rho_2^2}=Z$ follows a standard Beta distribution with 1 and $2T-1$ degrees of freedom [5] which is independent of $Y=\rho_1^2$. Therefore, (21) becomes:

$$var(\tilde{\theta}_1)=\frac{1}{2h_1}\underbrace{E\left[\frac{1}{Y}\right]}_{I_1}\underbrace{E\left[\frac{1}{1-\left(\frac{h_2}{h_1}\right)^2Z}\right]}_{I_2}=\frac{I_1\,I_2}{2h_1} \tag{22}$$

Both following integrals I_1 and I_2 must be calculated:

$$\left\{\begin{array}{l}I_1=\int_0^{\infty}\frac{1}{y}\Pi_{\chi^2}(y)\,dy,\\ I_2=\int_0^{1}\frac{1}{1-\left(\frac{h_2}{h_1}\right)^2z}\Pi_{\beta}(z)\,dz,\end{array}\right. \tag{23}$$

where $\Pi_{\chi^2}(y)$ and $\Pi_{\beta}(z)$ are respectively the probability density functions of a chi-square random variable $\chi^2\left(2T,\frac{\Sigma_1}{2}\right)$ and a beta random variable $\beta(1,2T-1)$:

$$\left\{\begin{array}{l}\Pi_{\chi^2}(y)=\frac{1}{2^{T-1}\Gamma(T)\Sigma_1^T}y^{T-1}e^{-\frac{y}{\Sigma_1}},\\ \Pi_{\beta}(z)=(2T-1)(1-z)^{2(T-1)}.\end{array}\right. \tag{24}$$

When $T\geq 2$, I_1 converges: it is a Gamma function. I_2 is a particular case of the integral representation of the Gauss hypergeometric function([5] pp. 556-565)

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt, \tag{25}$$

where $a=1$, $b=1$, $c=2T$ and $z=\left(\frac{h_2}{h_1}\right)^2$. (25) is finite when $T\geq 2$. Finally:

$$\left\{\begin{array}{l}I_1=\frac{1}{(T-1)\Sigma_1},\\ I_2={}_2F_1\left(1,1;2T;\left(\frac{h_2}{h_1}\right)^2\right).\end{array}\right. \tag{26}$$

Therefore, the variances of the first and second source (by symmetry) are:

$$\left\{\begin{array}{l}var(\tilde{\theta}_1)=\frac{{}_2F_1\left(1,1;2T;\left(\frac{h_2}{h_1}\right)^2\right)}{2(T-1)h_1\Sigma_1},\\ var(\tilde{\theta}_2)=\frac{{}_2F_1\left(1,1;2T;\left(\frac{h_2}{h_1}\right)^2\right)}{2(T-1)h_1\Sigma_2}.\end{array}\right. \tag{27}$$

It can be easily shown that $\Psi=0$ (19): it is the integral from minus infinity to plus infinity of an odd function.

According to (9) the SCRLB for the two sources case is:

$$\lim_{\sigma\to 0}\frac{1}{\sigma^2}\mathbf{B}_{\mathrm{STO}}=\frac{1}{2T}Diag\left\{\frac{1}{h_1\Sigma_1},\frac{1}{h_1\Sigma_2}\right\}. \tag{28}$$

Therefore, using (19), (27) and (28) we obtain (15). □

Corollary 2: The SML estimator is not asymptotically efficient since $\frac{T}{T-1} > 1$, and ${}_2F_1(1,1;c;z) \geq 1$ when $c > 2$ and $z \leq 1$ in 15. Furthermore, the minimum number of snapshots is $T = 2$. For otherwise integrals I_1 and I_2 in equation (23) diverge.

5. SIMULATIONS

Let us consider a Uniform Linear Array (ULA) of four sensors ($M = 4$) with half-wavelength spacing. We first investigate the single source case. Secondly, the two sources case is addressed. Monte-Carlo simulations are conducted with a Gauss Newton algorithm with 10000 independent realizations.

5.1 Single source case

Let us consider the DOA estimation of a single source located at zero degree ($\theta_0 = 0$) with respect to the array broadside. The number of snapshots is $T = 2$. Figure (1) represents the empirical variance of the SML estimator, the SCRLB and the theoretical variance given by (10). Here, the variance is twice the SCRLB since $\frac{T}{T-1} = 2$. There is a very good agreement between theoretical results and simulation.

5.2 Two sources case

Let us now consider the case of two sources with same power located at -7.5 degrees and 7.5 degrees (the half-power bandwidth is 22 degrees). The SML DOA estimation is performed with *T*=2 snapshots. We have reported in Figure (2) the evolution of the SML theoretical variance (27), and the SCRLB. We note a good agreement between theoretical results and simulations. We can also note the non-efficiency of SML at high SNR.

6. CONCLUSIONS

In the array processing framework, we have investigated the behavior of the SML estimator *for a finite number of samples* at high SNR. After recalling recent results on the non-Gaussiannity of the SML estimator in this context, we have addressed the study of the bearing estimates covariance. We have obtained an original analytical expression of the SML estimator covariance for a finite number of samples at high SNR in the two sources case. This expression can be interpreted as the product of the SCRLB with a magnification factor involving only the number of snapshots and the array geometry through a Gauss hypergeometric function.

REFERENCES

[1] H. Cramer, *Mathematical Methods of Statistics*. New York: Princeton University, Press, 1946.

[2] A. Renaux, F. Forster, E. Boyer, and P. Larzabal, "Non-efficiency and non-Gaussianity of a maximum likelihood estimator at high signal to noise ratio and finite number of samples," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, (Montreal, QC), May 2004.

[3] P. Stoica and A. Nehorai, "Performances study of conditional and unconditional direction of arrival estimation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 38, pp. 1783–1795, Oct. 1990.

[4] E. Boyer, P. Forster, and P. Larzabal, "Non asymptotic performances analysis of beamforming with stochastic signals," *IEEE Signal Processing Lett.*, vol. 11, pp. 23–25, Jan. 2004.

[5] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. New York: Dover Publications, inc., 1972.

[6] B. Ottersten, M. Viberg, P. Stoica, and A. Nehorai, "Exact and large sample maximum likelihood techniques for parameter estimation and detection in array processing," in *Radar Array Processing* (S. Haykin, J. Litva, and T. J. Shepherd, eds.), ch. 4, pp. 99–151, Berlin: Springer-Verlag, 1993.

[7] N. R. Goodman, "Statistical analysis based on certain multivariate complex Gaussian distribution (an introduction)," *Ann. Math. Statist.*, vol. 34, pp. 152–176, Mar. 1963.

[8] T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*. New York: Wiley, 1984.

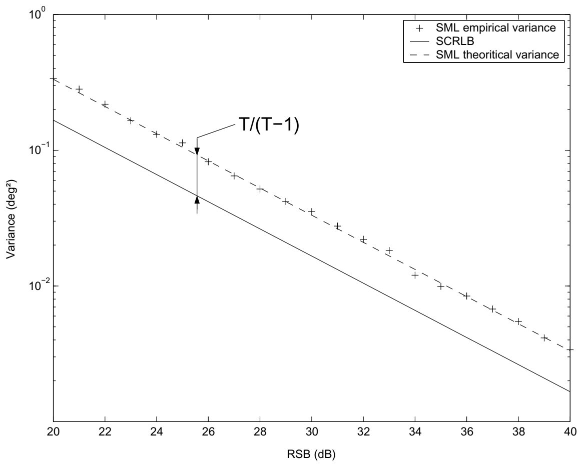

Figure 1: Asymptotic performance of SML estimator (one source case)

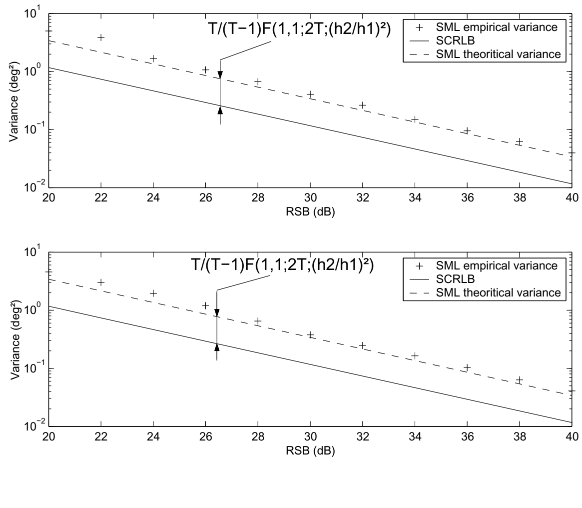

Figure 2: Asymptotic performance of SML estimator (two sources case)