UNIQUENESS OF REAL AND COMPLEX LINEAR INDEPENDENT COMPONENT ANALYSIS REVISITED

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ABSTRACT

Comon showed using the Darmois-Skitovitch theorem that under mild assumptions a real-valued random vector and its linear image are both independent if and only if the linear mapping is the product of a permutation and a scaling matrix. In this work, a much simpler, direct proof is given for this theorem and generalized to the case of random vectors with complex values. The idea is based on the fact that a random vector is independent if and only if locally the Hessian of its logarithmic density is diagonal.

1. INTRODUCTION

Independent component analysis (ICA) describes the task of transforming a random vector in order to make it independent. Blind source separation (BSS) tries to recover the original independent sources of a mixed random vector without knowing the mixing structure. In order to successfully perform BSS using ICA it is essential to know the indeterminacies of the problem that is to know how the separating model relates to the original mixing model (*separability*). For linear real-valued BSS Comon showed using the Darmois-Skitovitch theorem that the linear mixing matrix can be found except for permutation and scaling [4]. A generalization of his proof to complex random vectors based on the multivariate extension of the Darmois-Skitovitch theorem by Ghurye and Olkin [7] is used in [16] to show separability also in the complex case.

In this work, a much simpler, direct proof for separability of linearly mixed complex-valued (and hence also real-valued) random vectors is given. Without loss of generality we assume that the random vectors in question have twice continuously differentiable densities (instead of densities, we can use the characters $E(\exp iX_j)$ of the components, which are twice continuously differentiable if the variance of X_j exists). The proof uses the fact that a random vector is independent if and only if the Hessian of its logarithmic density is diagonal everywhere.

The paper is organized as follows: In the next section, basic terms and notations are introduced. Section 3 introduces the complex linear blind source separation model, which is proved in section 4.

2. NOTATION

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be either the real or the complex numbers. For $m, n \in \mathbb{N}$ let $\mathrm{Mat}(m \times n; \mathbb{K})$ be the $\mathbb{K}-$vectorspace of real respectively complex $m \times n$ matrices, $\mathrm{Gl}(n;\mathbb{K}) := \{\mathbf{W} \in \mathrm{Mat}(n \times n;\mathbb{K}) \mid \det(\mathbf{W}) \neq 0\}$ be the general linear group of $\mathbb{K}^n$. $\mathbf{I} \in \mathrm{Gl}(n;\mathbb{K})$ denotes the unit matrix. For $\alpha \in \mathbb{C}$ we write $\mathrm{Re}(\alpha)$ for its real and $\mathrm{Im}(\alpha)$ for its imaginary part.

An invertible matrix $\mathbf{L} \in \mathrm{Gl}(n;\mathbb{K})$ is said to be a *scaling matrix*, if it is diagonal. We say two matrices $\mathbf{B}, \mathbf{C} \in \mathrm{Mat}(m \times n;\mathbb{K})$ are $(\mathbb{K}-)$*equivalent*, $\mathbf{B} \sim \mathbf{C}$, if $\mathbf{C}$ can be written as $\mathbf{C} = \mathbf{BPL}$ with an scaling matrix $\mathbf{L} \in \mathrm{Gl}(n;\mathbb{K})$ and an invertible matrix with unit vectors in each row (*permutation matrix*) $\mathbf{P} \in \mathrm{Gl}(n;\mathbb{K})$. Note that $\mathbf{PL} = \mathbf{L}'\mathbf{P}$ for some scaling matrix $\mathbf{L}' \in \mathrm{Gl}(n;\mathbb{K})$, so the order of the permutation and the scaling matrix does not play a role for equivalence. Furthermore if $\mathbf{B} \in \mathrm{Gl}(n;\mathbb{K})$ with $\mathbf{B} \sim \mathbf{I}$, then also $\mathbf{B}^{-1} \sim \mathbf{I}$, and more general if $\mathbf{BC} \sim \mathbf{A}$, then $\mathbf{C} \sim \mathbf{B}^{-1}\mathbf{A}$. So two matrices are equivalent if and only if they differ by right-multiplication by a matrix with exactly one non-zero entry in each row and each column. If $\mathbb{K} = \mathbb{R}$, the two matrices are the same except for permutation, sign and scaling, if $\mathbb{K} = \mathbb{C}$, they are the same except for permutation, sign, scaling and phase-shift.

3. INDETERMINACIES OF COMPLEX ICA

Given a complex n-dimensional random vector $\mathbf{X}$, a matrix $\mathbf{W} \in \mathrm{Gl}(n;\mathbb{C})$ is called *(complex) ICA of* $\mathbf{X}$ if $\mathbf{WX}$ is independent (as a complex random vector). We will show that $\mathbf{W}$ and $\mathbf{V}$ are complex ICAs of $\mathbf{X}$ if and only if $\mathbf{W}^{-1} \sim \mathbf{V}^{-1}$ that is if they differ by right multiplication by a complex scaling and permutation matrix. This is equivalent to calculating the indeterminacies of the complex BSS model:

Consider the noiseless complex linear instantaneous blind source separation (BSS) model with as many sources as sensors

$$\mathbf{X} = \mathbf{AS}. \tag{1}$$

Here $\mathbf{S}$ is an independent complex-valued n-dimensional random vector and $\mathbf{A} \in \mathrm{Gl}(n;\mathbb{C})$ an invertible complex matrix.

The task of linear BSS is to find $\mathbf{A}$ and $\mathbf{S}$ given only $\mathbf{X}$. An obvious indeterminacy of this problem is that $\mathbf{A}$ can be found only up to equivalence because for scaling $\mathbf{L}$ and permutation matrix $\mathbf{P}$

$$\mathbf{X} = \mathbf{ALPP}^{-1}\mathbf{L}^{-1}\mathbf{S}$$

and $\mathbf{P}^{-1}\mathbf{L}^{-1}\mathbf{S}$ is also independent. We will show that under mild assumptions to $\mathbf{S}$ there are no further indeterminacies of complex BSS.

Various algorithms for solving the complex BSS problem have been proposed [2, 3, 6, 11, 14].

Theorem 3.1 (Separability of complex linear BSS). *Let* $\mathbf{A} \in \mathrm{Gl}(n;\mathbb{C})$ *and* $\mathbf{S}$ *a complex independent random vector. Assume that* $\mathbf{S}$ *has at most one gaussian component and the*

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(complex) covariance of $\mathbf{S}$ *exists. If* $\mathbf{AS}$ *is again independent*[1] *then* $\mathbf{A}$ *is equivalent to the identity.*

Here, the complex covariance of $\mathbf{S}$ is defined by

$$\mathrm{Cov}(\mathbf{S}) = \mathbf{E}\left((\mathbf{S}-\mathbf{E}(\mathbf{S}))(\mathbf{S}-\mathbf{E}(\mathbf{S}))^*\right),$$

where the asterix denotes the transposed and complexly conjugated vector. A complex random variable is said to be gaussian if both real and imaginary part are gaussians (possibly degenerate i.e. deterministic). The above theorem obviously includes separability of real-valued random vectors.

Comon has shown this for the real case [4]; for the complex case his proof can be generalized using a complex version of the Darmois-Skitovitch theorem [16]. In the following, we will give a proof without having to use the Darmois-Skitovitch theorem.

Theorem 3.1 indeed proves separability of the complex linear BSS model, because if $\mathbf{X} = \mathbf{AS}$ and $\mathbf{W}$ is a demixing matrix such that $\mathbf{WX}$ is independent, then $\mathbf{WA} \sim \mathbf{I}$, so $\mathbf{W}^{-1} \sim \mathbf{A}$ as desired. And it also calculates the indeterminacies of complex ICA, because if $\mathbf{W}$ and $\mathbf{V}$ are ICAs of $\mathbf{X}$, then both $\mathbf{VX}$ and $\mathbf{WV}^{-1}\mathbf{VX}$ are independent, so $\mathbf{WV}^{-1} \sim \mathbf{I}$ and hence $\mathbf{W} \sim \mathbf{V}$.

4. PROOF

Definition 4.1. *A function* $f : \mathbb{R}^{2n} \to \mathbb{R}$ *is said to be* 2-separated *if there exist two-dimensional functions* $g_1, \dots, g_n : \mathbb{R}^2 \to \mathbb{R}$ *such that* $f(x_1, \dots, x_{2n}) = g_1(x_1, x_2) \dots g_n(x_{2n-1}, x_{2n})$ *for all* $(x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$.

In this case we also write $f \equiv g_1 \otimes \dots \otimes g_n$.

Note that the functions g_i are uniquely determined by f up to a scalar factor. Obviously the character and the density (if it exists) of an independent random vector is separated.

Remark 4.2. *If* $f : \mathbb{R}^{2n} \to \mathbb{R}$ *is 2-separated and twice continuously differentiable, then*

$$f \frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \equiv 0$$

for $i \neq j, j+1$ *(odd j) respectively* $i \neq j-1, j$ *(even j).*

If f is positive, then the condition from remark 4.2 is equivalent to $\ln f$ having a 2-diagonal Hessian $\mathbf{H}_f$ everywhere i.e. a Hessian which is zero everywhere outside 2×2-matrices along the diagonal. Indeed, the converse — separability follows from an everywhere 2-diagonal logarithmic Hessian — is also true in this case.

4.1 Characterization of Gaussians

We show that among all densities the Gaussians satisfy a special differential equation.

Lemma 4.3. *Let* $\mathbf{X}$ *be a 2-dimensional random vector with twice continuously differentiable density* $p := p_{\mathbf{X}}$ *satisfying*

$$\mathbf{C}p^2 - p\mathbf{H}_p + \nabla p (\nabla p)^\top \equiv 0. \tag{2}$$

for a constant matrix $\mathbf{C} \in \mathbb{R}^{2\times 2}$. *Then* $\mathbf{X}$ *is a Gaussian.*

[1] Indeed, we only need that $\mathbf{AS}$ are pairwise mutually independent.

Here $\nabla f := (\partial f/\partial x_1, \dots, \partial f/\partial x_{2n})^\top$ denotes the gradient of f and $\mathbf{H}_f$ its Hessian.

Proof. First note that the differential equation 2 locally at non-zeros of p has the solution $\exp g(x_1, x_2)$, where g is a 2-dimensional polynomial of degree ≤ 2. For this let $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}$ with $p(\bar{x}_1, \bar{x}_2) \neq 0$. Then there exists a non-empty interval U containing $(\bar{x}_1, \bar{x}_2)$ such that $p|U > 0$. Set $g := \ln p|U$. Substituting $\exp g$ for p in equation 2 yields equations

$$c_{ij}\exp(2g) - \exp(g)\left(\frac{\partial^2 g}{\partial x_i \partial x_j} + \frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}\right)\exp(g) + \frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}\exp(2g) \equiv 0$$

for $i, j \in \{1, 2\}$, so

$$\frac{\partial^2 g}{\partial x_i \partial x_j} \equiv c_{ij}.$$

Hence g is a polynomial of degree ≤ 2, and $p = \exp g \neq 0$ on all of U. Therefore $p \neq 0$ everywhere, because of continuity and the fact that p is somewhere non-zero.

The local argument from above then shows that $p(x_1, x_2) = \exp g(x_1, x_2)$ everywhere. But $\int_{\mathbb{R}} p(x_1, x_2) dx_i = 1$, so g has second-order terms, hence p is a (possibly correlated) Gaussian, which finishes the proof. □

In the case of a one-dimensional random variable, this obviously reduces to the differential equation

$$cp^2 - pp'' + p'^2 \equiv 0.$$

4.2 A replacement of the multivariate Darmois-Skitovitch theorem

The original Skitovitch-Darmois theorem shows a non-trivial connection between gaussian distributions and stochastic independence. More precisely, it states that if two linear combinations of non-gaussian independent random variables are again independent, then each original random variable can appear only in one of the two linear combinations. It has been proved independently by Darmois [5] and Skitovitch [12]; in a more accessible form, the proof can be found in [10]. Skitovitch soon mentioned a multivariate extension of the Skitovitch-Darmois theorem [13], which was proved by Ghurye and Olkin [7]. Zinger gave a different proof for it in his PhD thesis [17].

Separability of linear BSS as shown by Comon [4] follows from this multivariate theorem [16]. In the following, we provide a lemma enabling us to show separability of linear complex BSS without having to resort to the Skitovitch-Darmois theorem. The proof is a multi-dimensional generalization of ideas presented in [15].

Lemma 4.4. *Let* $f : \mathbb{R}^{2n} \to \mathbb{R}$ *be a 2-separated square-integrable density function such that also* $f(\mathbf{Ax})$ *is 2-separated. If* $\mathbf{A}^{-1}$ *has two invertible* (2×2)*-submatrices at indices* $(2k-1, i)$ *and* $(2k-1, j)$ *with odd* $i \neq j$*, then the k-th 2-component of f is a Gaussian.*

Proof. We assume without loss of generality that f is twice continuously differentiable — otherwise, instead of densities, we can use the characters $E(\exp iX_j)$ of the random variables, which are twice continuously differentiable if the variance of X_j exists.

$f \circ \mathbf{A} : \mathbb{R}^{2n} \to \mathbb{R}$ is 2-separated, so $f(\mathbf{Ax}) = g_1 \otimes \ldots \otimes g_n(\mathbf{x})$. $f(\mathbf{x}) = g_1 \otimes \ldots \otimes g_n(\mathbf{Bx})$ is also assumed to be 2-separated (here $\mathbf{B} := \mathbf{A}^{-1}$), so by remark 4.2 we get for $i \neq j, j+1$ (odd j) respectively $i \neq j-1, j$ (even j):

$$0 = \left(f \frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \right)(\mathbf{x})$$

The ingredients of this equation can be calculated for $i < j$ as follows:

$$\begin{aligned}
\frac{\partial f}{\partial x_i}(\mathbf{x}) &= \sum_{k=1}^{n} g_1 \otimes \ldots \otimes \frac{\partial g_k}{\partial x_i} \otimes \ldots \otimes g_n(\mathbf{Bx}) \\
\frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}(\mathbf{x}) &= \sum_{k,l} (g_1 \otimes \ldots \otimes \frac{\partial g_k}{\partial x_i} \otimes \ldots \otimes g_n) \\
&\quad (g_1 \otimes \ldots \otimes \frac{\partial g_l}{\partial x_j} \otimes \ldots \otimes g_n)(\mathbf{Bx}) \\
\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) &= \sum_k (g_1 \otimes \ldots \otimes \frac{\partial^2 g_k}{\partial x_i \partial x_j} \otimes \ldots \otimes g_n + \\
&\quad \sum_{l \neq k} g_1 \otimes \ldots \otimes \frac{\partial g_k}{\partial x_i} \otimes \ldots \otimes \frac{\partial g_l}{\partial x_j} \otimes \ldots \otimes g_n)(\mathbf{Bx})
\end{aligned}$$

Plugging this into the above equation yields

$$\begin{aligned}
0 &= \sum_k (g_1^2 \otimes \ldots \otimes g_k \frac{\partial^2 g_k}{\partial x_i \partial x_j} \otimes \ldots \otimes g_n^2 - \\
&\quad g_1^2 \otimes \ldots \otimes \frac{\partial g_k}{\partial x_i} \frac{\partial g_k}{\partial x_j} \otimes \ldots \otimes g_n^2)(\mathbf{Bx}) \\
&= \sum_k g_1^2 \otimes \ldots \otimes g_{k-1}^2 \otimes \left(g_k \frac{\partial^2 g_k}{\partial x_i \partial x_j} - \frac{\partial g_k}{\partial x_i} \frac{\partial g_k}{\partial x_j} \right) \otimes \\
&\quad g_{k+1}^2 \otimes \ldots \otimes g_n^2(\mathbf{Bx})
\end{aligned}$$

for $\mathbf{x} \in \mathbb{R}^{2n}$. We want to calculate the term in the brackets. For this note that

$$\begin{aligned}
\frac{\partial}{\partial x_i} g_k(\mathbf{Bx}) &= (b_{2k-1,i} \frac{\partial g_k}{\partial x_{2k-1}} + b_{2k,i} \frac{\partial g_k}{\partial x_{2k}}) \mid_{\mathbf{Bx}} \\
&= (b_{2k-1,i}, b_{2k,i}) \nabla g_k \mid_{\mathbf{Bx}}
\end{aligned}$$

So, the term in the brackets can be calculated as

$$(g_k \tfrac{\partial^2 g_k}{\partial x_i \partial x_j} - \tfrac{\partial g_k}{\partial x_i} \tfrac{\partial g_k}{\partial x_j})(\mathbf{Bx}) =$$
$$(b_{2k-1,i}, b_{2k,i})(g_k H_{g_k} - \nabla g_k (\nabla g_k)^\top) \mid_{\mathbf{Bx}} \begin{pmatrix} b_{2k-1,j} \\ b_{2k,j} \end{pmatrix} =: h_{ijk}(\mathbf{Bx})$$

and we get

$$0 = \sum_k g_1^2 \otimes \ldots \otimes g_{k-1}^2 \otimes h_{ijk} \otimes g_{k+1}^2 \otimes \ldots \otimes g_n^2(\mathbf{Bx})$$

$\mathbf{B}$ is invertible, so the whole function is zero:

$$0 \equiv \sum_k g_1^2 \otimes \ldots \otimes g_{k-1}^2 \otimes h_{ijk} \otimes g_{k+1}^2 \otimes \ldots \otimes g_n^2 \tag{3}$$

Choose $\mathbf{x} \in \mathbb{R}^{2n}$ with $g_k(x_{2k-1}, x_{2k}) \neq 0$ for $k = 1, \ldots, n$. Evaluating equation 3 at $(x_1, \ldots, x_{2l-2}, \mathbf{y}, x_{2l+1}, \ldots, x_n)$ for variable $\mathbf{y} \in \mathbb{R}^2$ and dividing the resulting equation by the constant $g_1^2(x_1, x_2) \ldots g_{k-1}^2(x_{2l-3}, x_{2l-2})$ $g_{l+1}^2(x_{2l+1}, x_{2l+2}) \ldots g_n^2(x_{2n-1}, x_{2n})$ shows

$$h_{ijl}(\mathbf{y}) = -\left(\sum_{k \neq l} h_{ijk}(x_{2k-1}, x_{2k}) \right) g_l^2(\mathbf{y}) =: c_{ijl} g_l^2(\mathbf{y}) \tag{4}$$

for $\mathbf{y} \in \mathbb{R}^2$.

Now let $i \neq j$, both odd. Equation 4 for $(i,j,k), (i+1,j,k), (i,j+1,k), (i+1,j+1,k)$ can then be gathered into a matrix to read

$$\begin{pmatrix} b_{2k-1,i} & b_{2k-1,i+1} \\ b_{2k,i} & b_{2k,i+1} \end{pmatrix}^\top (g_k H_{g_k} - \nabla g_k (\nabla g_k)^\top)$$
$$\begin{pmatrix} b_{2k-1,j} & b_{2k-1,j+1} \\ b_{2k,j} & b_{2k,j+1} \end{pmatrix} \equiv \mathbf{C} g_k$$

If now the two submatrices of $\mathbf{B}$ in this equation are invertible, then

$$g_k H_{g_k} - \nabla g_k (\nabla g_k) \equiv \mathbf{C}' g_k,$$

so g_k fulfills precisely the differential equation from lemma 4.3. According to this lemma, g_k is a Gaussian. □

4.3 Proof of theorem 3.1

Denote $\mathbf{X} := \mathbf{AS}$ with $\mathbf{A} \in \mathrm{Gl}(n, \mathbb{C})$.

First we will show using complex decorrelation that we can assume $\mathbf{A}$ to be unitary. Without loss of generality assume that all random vectors are centered. By assumption $\mathrm{Cov}(\mathbf{X})$ is diagonal, so let $\mathbf{D}_1$ be diagonal invertible with $\mathrm{Cov}(\mathbf{X}) = \mathbf{D}_1^2$. Note that $\mathbf{D}_1$ is real. Similarly let $\mathbf{D}_2$ be diagonal invertible with $\mathrm{Cov}(\mathbf{S}) = \mathbf{D}_2^2$. Set $\mathbf{Y} := \mathbf{D}_1^{-1}\mathbf{X}$ and $\mathbf{T} := \mathbf{D}_2^{-1}\mathbf{S}$ that is normalize $\mathbf{X}$ and $\mathbf{S}$ to covariance $\mathbf{I}$. Then

$$\mathbf{Y} = \mathbf{D}_1^{-1}\mathbf{X} = \mathbf{D}_1^{-1}\mathbf{AS} = \mathbf{D}_1^{-1}\mathbf{AD}_2\mathbf{T}$$

and $\mathbf{T}$, $\mathbf{D}_1^{-1}\mathbf{AD}_2$ and $\mathbf{Y}$ satisfy the assumption and $\mathbf{D}_1^{-1}\mathbf{AD}_2$ is unitary because

$$\begin{aligned}
\mathbf{I} &= \mathrm{Cov}(\mathbf{Y}) \\
&= \mathbf{E}(\mathbf{YY}^*) \\
&= \mathbf{E}(\mathbf{D}_1^{-1}\mathbf{AD}_2\mathbf{TT}^*\mathbf{D}_2\mathbf{A}^*\mathbf{D}_1^{-1}) \\
&= (\mathbf{D}_1^{-1}\mathbf{AD}_2)(\mathbf{D}_1^{-1}\mathbf{AD}_2)^*.
\end{aligned}$$

If we assume $\mathbf{A} \not\sim \mathbf{I}$, then also $\mathbf{B} := \mathbf{A}^{-1} \not\sim \mathbf{I}$. Using the fact that $\mathbf{A}$ hence also $\mathbf{B}$ is unitary there exist indices $i_1 \neq i_2$ and $j_1 \neq j_2$ with $b_{i_* j_*} \neq 0$.

We can now interpret the independent complex random vector $\mathbf{X}$ as $2n$-dimensional real random vector with 2-separated density. Multiplication by a complex number α either ($\alpha \neq 0$) is a multiplication by the real invertible matrix

$$\begin{pmatrix} \mathrm{Re}(\alpha) & -\mathrm{Im}(\alpha) \\ \mathrm{Im}(\alpha) & \mathrm{Re}(\alpha) \end{pmatrix}$$

or ($\alpha = 0$) multiplication by the 0-matrix.

If we interpret $\mathbf{B}$ as real $(2n \times 2n)$-matrix, this means that the four (2×2)-matrices at the positions $(2i_* - 1, 2j_* - 1)$ are invertible. Applying lemma 4.4 twice therefore shows that the two complex source variables S_{j_1} and S_{j_2} are gaussian, which is a contradiction to the fact that at most one source is gaussian. This finished the proof of theorem 3.1.

5. CONCLUSION

Separability in nonlinear situations has turned out to be a hard problem — ill-posed in the most general case [8] — and not many non-trivial results exist for restricted models [1, 8], all only two-dimensional. This is partially due to the fact that the rather non-trivial proof of the Darmois-Skitovitch theorem and as corollary theorem 3.1 are not at all easily generalized to more general settings [9]. By introducing separated functions, we are able to give a much easier proof for linear real and complex separability; we expect that these ideas will be used to show separability in other more complicated situations.

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