GENERALIZED METHOD FOR IMPROVING THE STFT VIA LINEAR CANONICAL TRANSFORMS

Huseyin Emre Guven and Orhan Arikan

Bilkent University,
Department of Electrical and Electronics Engineering,
TR-06800, Bilkent, Ankara, Turkey.
e-mail: hguven@ee.bilkent.edu.tr and oarikan@ee.bilkent.edu.tr

ABSTRACT

It has been previously shown that chirp signals can be manipulated into a minimal time-bandwidth product (TBP) form in its so called natural domain, via fractionally Fourier transforming. Here we show that the time-bandwidth product (TBP) of the signal can be achieved in various other ways, under the class of Linear Canonical Transforms (LCT), which can be decomposed into shearing operations in time and frequency of the energy support of the signal. A general background on LCT is built, and its possible uses for obtaining the GTBP are presented.

1. INTRODUCTION

Time-frequency distributions providing high resolution has long been desirable. Wigner distribution maintains high localization, however it desperately suffers from spurious values in the presence of multi-components or noise. Short-time Fourier transform offers an alternative, albeit with an inherent localization problem. Adaptive and heuristic methods have been used to search for an optimal window [1, 2, 3, 4, 5, 6]. Recently, a new method has emerged [7], which attains high localization properties using fractional Fourier transform.

In this work, starting from basic shearing operations, we arrive at a generalization to the group of linear canonical transforms (LCT). Signals with smaller time-frequency support are represented with higher resolution, and the minimum time-bandwidth product form of the signal be achieved in several ways, either rotating the support of chirp-like signals at a suitable angle [7], or simply shearing it. In fact, both operations are particular forms of LCT. Here we present a generalized method for improving the STFT using LCT.

General background on LCT is constructed in the next section, followed by third section introducing the use of LCT. The results are discussed in the fourth section, followed by the future work in the fifth section.

2. BACKGROUND ON LINEAR CANONICAL TRANSFORMS

The linear canonical transform of $f(u)$, denoted $f_{\mathbf{M}}(u) = \mathscr{C}_{\mathbf{M}} f(u)$ is defined as [8]:

$$\mathscr{C}_{\mathbf{M}} f(u) = \int C_{\mathbf{M}}(u,u') f(u') du', \tag{1}$$

$$C_{\mathbf{M}}(u,u') = A_{\mathbf{M}} \exp[i\pi(\alpha u^2 - 2\beta u u' + \gamma u'^2)],$$

where $A_{\mathbf{M}} = \sqrt{\beta}\, e^{-j\pi/4}$. The linear canonical transform operator $\mathscr{C}_{\mathbf{M}}$ is defined by three real parameters α, β, γ, which can be conveniently represented in matrix form $\mathbf{M}$ as:

$$\mathbf{M} \equiv \begin{bmatrix} \gamma/\beta & 1/\beta \\ -\beta + \alpha\gamma/\beta & \alpha/\beta \end{bmatrix} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{2}$$

This matrix representation is particularly useful for providing insight to decompositions of LCT. Matrices of the following forms are of particular interest to us:

$$\mathbf{R}_r = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \tag{3}$$

$$\mathbf{Q}_q = \begin{bmatrix} 1 & 0 \\ -q & 1 \end{bmatrix} \tag{4}$$

The transforms associated with these two are as follows [8]:

$$\mathscr{C}_{\mathbf{R}_r} f(u) = f(u) * e^{-j\pi/4} \sqrt{1/r}\, e^{j\pi u^2/r} \tag{5}$$

$$\mathscr{C}_{\mathbf{Q}_q} f(u) = f(u)\, e^{-j\pi q u^2} \tag{6}$$

The first one is known to be chirp convolution and it shears the support of the signal in time, i.e. parallel to time axis; whereas the second one -chirp multiplication- shears the signal support in frequency, i.e. parallel to frequency axis (Figures 4-5). If the signal is transformed through $\mathscr{C}_{\mathbf{M}_q}$, then the Wigner distribution is affected as [8] :

$$W_{f_{\mathbf{M}}}\left(\begin{bmatrix} u \\ \mu \end{bmatrix} \right) = W_f\left(\mathbf{M}^{-1} \begin{bmatrix} u \\ \mu \end{bmatrix} \right). \tag{7}$$

It is well known that any linear canonical transform can be broken down to combinations of chirp multiplications and chirp convolutions. In fact, more than 3 of these operations are redundant, since the space of the unit-determinant 2×2 matrices $\mathbf{M}$ can be spanned by suitably chosen parameters for $\mathbf{Q}$ and $\mathbf{R}$ as

$$\mathbf{M} = \mathbf{Q}_{q_1} \mathbf{R}_r \mathbf{Q}_{q_2} \tag{8}$$

We now have enough tools of manipulating the signal support using time and frequency shearing. For more on the effect of LCT on energy distributions and other matrix decompositions, the reader is referred to [8]. In the next section, we will present how to use shearing in obtaining STFT of higher precision.

3. STFT PROCEDURE WITH LCT

Up to now we have seen the effect of shearing on the signal's energy distribution. Rotation property [9] of the fractional Fourier transform allowed us to implement the GTBP-optimal STFT at a single step with suitably chosen window [7]. We propose an equivalence relationship for the inverse-mapped TBP-optimal STFT of the sheared signal with the GTBP-optimal STFT. In fact, this is true for any linear canonical transform used to achieve the minimal TBP, as we will show later. The TBP of the intermediate signal $x_q(t)$, is the same as GTBP for any chirp signal. We introduce the foretold STFT procedure hereby.

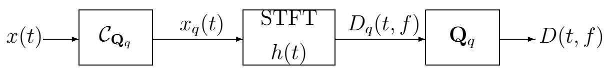

Figure 1: STFT procedure with frequency shearing.

Suppose that $x(t)$ is a chirp signal with rate q. We first multiply it with the chirp $e^{-i\pi qt^2}$, shearing in frequency. Now the signal, lying parallel to time axis in time-frequency plane, has the minimum possible TBP. We know that the optimal window for the STFT of this signal is given by [7]:

$$h(t) = e^{-\gamma\pi t^2} \tag{9}$$

$$\gamma = \frac{B_{x_q}}{T_{x_q}} \tag{10}$$

where B_{x_q} and T_{x_q} are the bandwidth and duration of the sheared signal. Henceforth, the effect of the chirp multiplication is to be undone. But this is realized simply by coordinate transformation using the matrix $\mathbf{Q}_q$ with a change of the window function, relying on Theorem 1 in the Appendix.

$$D(t,f) = e^{j\pi qt^2}\, STFT_x^{(\mathscr{Q}_{-q}h)}(t,f) \tag{11}$$

In equation (11), we can ignore the unit magnitude $e^{j\pi qt^2}$ term; since we are interested only in the magnitude. Thus, output to any system using the above procedure will be computable at the same cost as a single STFT. There is an equivalent result for the chirp convolution by Theorem 2.

$$D_r(t,f) = e^{j\pi rf^2}\, STFT_x^{(\mathscr{R}_{-r}h)}(t,f) \tag{12}$$

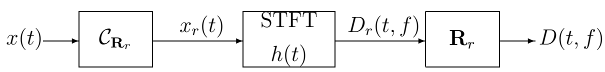

Figure 2: STFT procedure with time shearing.

Finally, from Theorems 1,2 and equation (8) we have the following lemma:

Lemma 1. *The following system, $\mathscr{C}_{\mathbf{M}}$ being any linear canonical transform, can be implemented as a single STFT.*

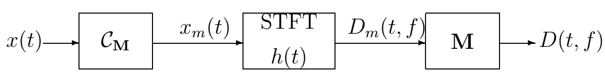

Figure 3: STFT using linear canonical transform.

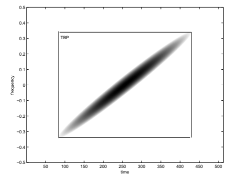

Figure 4: TBP of a chirp signal

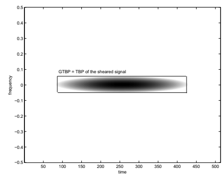

Figure 5: TBP of the sheared chirp

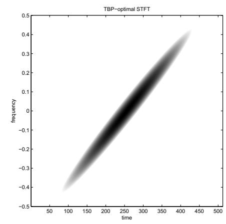

Figure 6: TBP-optimal STFT

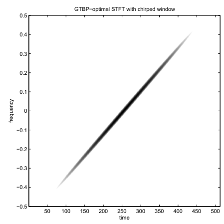

Figure 7: GTBP-optimal STFT with frequency shearing

Proof. Since any LCT can be decomposed into following operators as defined by equation (8),

$$\mathscr{C}_{\mathbf{M}} f(u) = \mathscr{C}_{\mathbf{Q}_{q_1}}\{\mathscr{C}_{\mathbf{R}_r}\{\mathscr{C}_{\mathbf{Q}_{q_2}} f\}\}(u) \tag{13}$$

we can consider the system as a cascade of chirp multiplication and convolutions as we previously defined, which can be simply turned into a single STFT by inversely operating on the window. Therefore, the above system can be simply realized as $STFT_x^{(\mathscr{M}^{-1}h)}(t,f)$. □

Figures 6-7 shows an example of the above procedure with a chirp signal. We consider the incident as shearing the signal in frequency, so that the chirp multiplied signal's energy support is parallel to time axis. We also estimate the optimal γ parameter which should yield the maximum resolution. Calculations reveal that the TBP achieved this way is equal to GTBP.

4. DISCUSSION

We have demonstrated why chirped-windows perform better than ordinary gaussian windows in the case of chirp-like signals, and the equivalence of the methods of using chirped windows [6] and TF-rotated windows [7] by presenting a general relationship between the class of linear canonical transforms and the short-time Fourier transform. Linear canonical transforms provide an elegant class of tools for, since they are symplectic and linear, as their name suggests. They can be used as a basis for manipulating the signal support to achieve reduced generalized time–bandwidth product while guaranteeing applicability for composite signals as well as monocomponent signals.

In application, however, one should note that we cannot arbitrarily use chirp multiplication (or convolution) with a very high chirp rate. One reason is that aliasing in frequency (or time) occurs after this operation. It is still possible to gain advantage at a reasonable cost of oversampling. Secondly, although it seems very high chirp rates will bring more advantage since the signal support approximates to a straighter shape. In reality, the inversion matrix of the linear canonical transform has reciprocal eigenvalues so that the condition number is proportional to the square of the chirp rate, hence corresponding to an ill-conditioned system. Again, sampling rate becomes the determining factor of the availability for highly straightened (reduced TBP) signal supports.

5. CONCLUSION

In this work we have provided a broad class of transforms for possible use with short-time Fourier transform systems, so that one can reduce the effective time-bandwidth product that determines the resolution (instantaneous bandwidth) of the time-frequency representation. The transformations can be easily realized as cascaded chirp multiplications and convolutions, each having a computational complexity of $O(N)$. The overall system can always be performed as a single short-time Fourier transform, as the window goes through the inverse of the linear canonical transform that reduces the time-bandwidth product of the analyzed signal. Our system makes no assumptions on the signal type, so that it applies to both monocomponent and composite signals.

6. FUTURE WORK

Although a broad class of operators are included with this procedure, it is computationally expensive to search for the optimum linear canonical transform that brings the time-bandwidth product to the global minimum. Further research is required on generating a suboptimum algorithm that is easy to apply. Once the required parameters are known, a time-frequency projection of linear time-invariant systems can take place as a subsystem within the given short-time Fourier transform procedure.

A. APPENDIX

Theorem 1. *Let $y(t) = \mathscr{Q}_{-q}x(t)$ so that $y(t) = e^{j\pi qt^2}x(t)$, then magnitude-wise*

$$STFT_y^h(t,f) = STFT_x^{(\mathscr{Q}_q h)}(t, -qt+f) \tag{14}$$

Proof. Starting from the STFT of $y(t)$:

$$\begin{aligned} STFT_y^h(t,f) &= \int y(\tau)h^*(\tau-t)\, e^{-j2\pi f\tau}\, d\tau && (15)\\ &= \int x(\tau)e^{j\pi q\tau^2}h^*(\tau-t) \\ &\quad e^{j\pi q(t^2-t^2+2t\tau-2t\tau)}e^{-j2\pi f\tau}\, d\tau && (16)\\ &= e^{-j\pi qt^2}\int x(\tau)h^*(\tau-t) \\ &\quad e^{j\pi q(\tau-t)^2}\, e^{-j2\pi(-qt+f)\tau}\, d\tau && (17)\\ &= e^{-j\pi qt^2}\, STFT_x^{(\mathscr{Q}_q h)}(t,-qt+f) \quad \square \end{aligned}$$

Theorem 2. *Let $y(t) = \mathscr{R}_{-r}x(t)$ so that $Y(f) = e^{j\pi rf^2}X(f)$, where*

$$Y(f) = \mathscr{F}\{y(t)\}, X(f) = \mathscr{F}\{x(t)\}$$

Then magnitude-wise

$$STFT_y^h(t,f) = STFT_x^{(\mathscr{R}_r h)}(rf+t, f) \tag{18}$$

Proof. Writing down the first expression:

$$\begin{aligned} STFT_y^h(t,f) &= \int Y(f')H^*(f'-f)\, e^{j2\pi f't}\, df' && (19)\\ &= \int X(f')e^{j\pi rf'^2}H^*(f'-f) \\ &\quad e^{j\pi r(f^2-f^2+2ff'-2ff')}e^{j2\pi f't}\, df' && (20)\\ &= e^{-j\pi rf^2}\int X(f')H^*(f'-f) \\ &\quad e^{j\pi r(f'-f)^2}\, e^{j2\pi(rf+t)f'}\, df' && (21)\\ &= e^{-j\pi rf^2}\, STFT_x^{(\mathscr{R}_r h)}(rf+t,f) \quad \square \end{aligned}$$

REFERENCES

[1] F. Hlawatsch and G. F. Boudreaux-Bartels. Linear and Quadratic Time-Frequency Signal Representations. *IEEE Signal Processing Magazine*. 2:21-67, 1992

[2] Leon Cohen. Time–Frequency Representations – A review. *Proceedings of the IEEE Signal Processing Letters*. 7:941-981, 1989.

[3] Douglas L. Jones and Richard G. Baraniuk. A Simple Scheme for Adapting Time-Frequency Representations. *IEEE Trans. on Signal Processing*. 12:3530-3535, 1994.

[4] Douglas L. Jones and Richard G. Baraniuk. An Adaptive Optimal-Kernel Time-Frequency Representation. *IEEE Trans. on Signal Processing*. 10:2361-2371, 1995.

[5] Douglas L. Jones and Thomas W. Parks. A Resolution Comparison of Several Time-Frequency Representations. *IEEE Trans. on Signal Processing*. 2:413-420, 1992.

[6] Henry K. Kwok and Douglas L. Jones. Improved Instantaneous Frequency Estimation Using an Adaptive Short-Time Fourier Transform. *IEEE Trans. on Signal Processing*. 10:2964-2972, 2000

[7] Lutfiye Durak and Orhan Arikan. Short-Time Fourier Transform: Two Fundamental Properties and An Optimal Implementation. *IEEE Trans. on Signal Processing*, 5:1231–1242, 2003.

[8] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay. *The Fractional Fourier Transform with Applications in Optics and Signal Processing*. Wiley, 2001.

[9] Luis B. Almeida. The Fractional Fourier Transform and Time-Frequency Representations. *IEEE Trans. on Signal Processing*, 11:3084–3091, 1994.

[10] Haldun M. Ozaktas, Orhan Arikan, M. Alper Kutay and Gozde Bozdagi. Digital Computation of the Fractional Fourier Transform. *IEEE Trans. on Signal Processing*. 9:2141-2150, 1996.