GEODESIC HOMOTOPIES

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ABSTRACT

In this paper we wish to endow the manifold M of smooth curves in $\mathbb{R}^n$ (either closed curves or open curves with fixed endpoints) with a Riemannian structure that allows us to treat homotopies between two curves C_0 and C_1 as trajectories with computable lengths. If we regard a curve as all possible reparameterizations of a C^1 mapping of the unit interval into $\mathbb{R}^n$, then the tangent space at a point on M corresponds to all possible non-tangential flows of the underlying curve. We note that a Riemannian metric corresponds to a choice of inner-product on this tangent space. Once this is defined, we can compute distances between curves and consider the natural problem of finding geodesics which will yield minimal length homotopies between C_0 and C_1. We begin by noting that a seemingly natural inner-product corresponding to a geometric version of ℓ_2 does not yield useful geodesics and will instead introduce a sequence of conformally modfied inner-products which has interesting limit properties.

1. PROBLEM FORMULATION

1.1 Preliminary notation

Let $C : [0,1] \times [0,1] \to \mathbb{R}^n$, denote a homotopy $C(u,v)$ of either closed curves[1] or open curves with common end-points[2] between two boundary curves C_0 and C_1 where u parameterizes an individual curve in the homotopy and v parameterizes the homotopy itself.[3] In what follows, we will let $T(u,v) = C_u/\|C_u\|$ denote the unit tangent vector at a point $C(u,v)$ along a particular curve in the homotopy, and we will let $L(v)$ denote the total arclength of a particular curve in the homotopy.

1.2 The "Homotopy Surface"

To the parameterized homotopy $C(u,v)$ in $\mathbb{R}^n$ we may associate a unique two dimensional surface in $\mathbb{R}^{n+1}$ defined by $S(u,v) = \big(C(u,v), v\big)$. Note that any tangent vector C_u of any curve in the homotopy is also a tangent vector $S_u = (C_u, 0)$ of the surface under the inclusion map. The tangent space on S is two dimensional, however, and we may form another linearly indepedent tanget vector $S_v = (C_v, 1)$. We note, on the other hand, that both of these tangent vectors and the corresponding differential operators $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ are dependent upon the parameterization of the curves in the homotopy.[4]

[1] in which case $C(0,v) = C(1,v)$

[2] in which case $C(0,v)$ and $C(1,v)$ are constants

[3] i.e. $C(u,0) = C_0(u)$ and $C(u,1) = C_1(u)$

[4] While it is obvious that S_u depends upon the parameterization of the curves $C(u,v)$ in the homotopy, it may be less obvious to see the dependence of S_v. Note, however that $C(u^{(1+v)}, v)$ constitutes the same homotopy geometrically in the sense that both give rise to the same homotopy surface and yet this reparameterization will clearly alter the tangent vectors S_v.

1.3 Geometric coordinates (s, v_*)

We can construct a basis for the tangent space of S which is independent of the parameterization of the curves $C(u,v)$ making up the homotopy surface by normalizing S_u and taking only the orthogonal component of S_v. We will denote these new tangent vectors by $S_s = S_u/\|S_u\| = (T, 0)$ and $S_{v_*} = S_v - (S_v \cdot S_s)S_s$. Note that S_s is a unit vector (with s as the arclength parameter of the corresponding curve) but that S_{v_*} is in general *not* a unit vector. It is orthogonal to S_s and yields a transport from curve to curve along the homotopy with the same direction and speed as S_v but possibly (if $S_v \cdot S_s \neq 0$) through a different sequence of points on each of the individual curves. Stated intuitively, S_{v_*} indicates the most efficient direction to move from one curve to another along the homotopy surface. These new tangent vectors, and the associated differential operators $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial v_*}$ depend exclusively upon the geometry of the homotopy surface and may be expressed in terms of the parameterization (u,v) as follows:

$$\frac{\partial}{\partial s} = \frac{1}{\|C_u\|}\frac{\partial}{\partial u} \tag{1}$$

$$\frac{\partial}{\partial v_*} = \frac{\partial}{\partial v} - (C_v \cdot T)\frac{\partial}{\partial s} = \frac{\partial}{\partial v} - \left(\frac{C_v \cdot C_u}{C_u \cdot C_u}\right)\frac{\partial}{\partial u} \tag{2}$$

1.4 Riemannian metrics

We may regard the homotopy $C(u,v)$ itself as a parameterized (by v) abstract curve on the infinite dimensional manifold of space curves (where a point on this manifold corresponds to all possible reparameterizations of a C^1 mapping from the unit interval into $\mathbb{R}^n$). A seemingly natural metric on this manifold comes from a geometric version[5] of the ℓ_2 inner product. We will denote this inner product by $<>_0$. Noting that a tangent vector on this manifold consists of a non-tangential flow of the underlying space curve, and letting $\mathscr{V}_1$ and $\mathscr{V}_2$ denote two such flows, we may write the inner product as follows:

$$\left\langle \mathscr{V}_1, \mathscr{V}_2 \right\rangle_0 = \int_0^L \mathscr{V}_1 \cdot \mathscr{V}_2 \, ds \tag{3}$$

where s denotes the arclength parameter of the curve and L denotes its total arclength.

Given this metric, it is now meaningful to talk about the length of the homotopy itself (as a curve on this Riemmannian manifold). We will denote this length by $\mathscr{L}_0$ to distinguish it from the arclengths $L(v)$ of the individual space

[5] "geometric version" since the arclength measure is used in the integral making it invariant to the curve's parameterization

curves which comprise the actual homotopy. Before writing the arclength formula, we note that differentiation and integration along the homotopy should be performed with respect to the geometric parameter v_* to ensure invariance with respect to the parameterizations of the space curves along the homotopy. In general C_v does not constitute a tangent vector on this manifold since it is not orthogonal to the tangent field of the underlying space curve. $C_{v_*} = C_v - (C_v \cdot T)T$ on the other hand, is orthogonal to the curve's tangent field and therefore constitutes a true tangent vector.

$$\mathscr{L}_0 = \int_0^1 \sqrt{\left(\int_0^L \left\| \frac{\partial C}{\partial v_*} \right\|^2 ds\right)} \, dv_* \tag{4}$$

Unfortunately, while this metric seems natural, it does not yield geodesics. It can be shown that the infimum of the distance between any two curves is zero according to this metric. Said another way, a sequence $C(u,v,t)$ of homotopies (where t is the sequence index) between two fixed boundary curves may be found such that $\mathscr{L}(t) \to 0$. This may occur by choosing homotopies consisting of extremly high arclength curves strategically chosen to minimize the total motion orthogonal to the curves as one proceeds along the homotopy between its two boundaries. To regularize this, we could introduce a penalty on arclength in the inner product or, as Michor and Mumford[3], weight the inner-product more along arcs of high curvature. We, instead, will take a different approach to discourage excessive arclength of curves along minimal length homotopies by multiplying the above inner product by a positive power of the total arclength L of the curve. If we consider positive integer powers $k = \{1,2,3,\ldots\}$ then we obtain a sequence of inner-products $<>_k$ given by

$$\left\langle \mathscr{V}_1, \mathscr{V}_2 \right\rangle_k = L^k \int_0^L \mathscr{V}_1 \cdot \mathscr{V}_2 \, ds \tag{5}$$

which in turn gives rise to a sequence of possible lengths $\mathscr{L}_k$ for the homotopy:

$$\mathscr{L}_k = \int_0^1 \sqrt{L^k \left(\int_0^L \left\| \frac{\partial C}{\partial v_*} \right\|^2 ds\right)} \, dv_* \tag{6}$$

2. VARIATIONAL APPROACH

We now outline a procedure to compute homotopies which comprise geodesics between their boundary curves by starting with an arbitrary homotopy and applying a "homotopy shortening flow" until it converges to a minimal length homotopy between the same two boundary curves C_0 and C_1.

2.1 Energy functional

Consider the energy

$$E_k = \int_0^1 \left\langle \frac{\partial C}{\partial v_*}, \frac{\partial C}{\partial v_*} \right\rangle_k dv_* = \int_0^1 L^k \int_0^L \|C_{v_*}\|^2 \, ds \, dv_* \tag{7}$$
$$= \int_0^1 L^k \int_0^1 \left(C_v \cdot C_v - (C_v \cdot T)^2\right) \|C_u\| \, du \, dv$$

that is associated to the homotopy length $\mathscr{L}_k$ given by (6). To find a geodesic connecting two end curves C_0 and C_1, we minimize the Energy E_k in the family of all homotopies C between two end curves C_0 and C_1; by standard knowledge,[6] the minimizing geodesic also provides a minimum for the length $\mathscr{L}_k$.

2.2 The limit energy

There is no difference in minimizing E_k or $\sqrt[k]{E_k}$; we can provide a limit for the latter

Proposition 1 *For any fixed homotopy of class C^1,*

$$\lim_{k\to\infty} \sqrt[k]{E_k} = \sup_{v\in[0,1]} L(v) \tag{8}$$

Proof. We recall this proposition: if μ is any measure on $[0,1]$, and $f : [0,1] \to \mathbb{R}^+$ is Borel measurable, then

$$\lim_{k\to\infty} \sqrt[k]{\int f(v)^k \, dv} = \mu - \operatorname*{supess}_{v\in[0,1]} f(v)$$

where

$$\mu - \operatorname*{supess}_{v\in[0,1]} f(v) \doteq \inf\{t : \mu\{f > t\} = 0\} \quad .$$

Let

$$\rho(v) \doteq \int_0^L \|C_{v_*}\|^2 \, ds$$

Since the homotopy is C^1, then ρ and L are continuous. We define $\mu \doteq \rho \cdot dv$ (that is, μ is defined by $\mu(A) = \int_A \rho dv$ for all Borel sets $A \subset [0,1]$): then

$$\lim_{k\to\infty} \sqrt[k]{E_k} = \mu - \operatorname*{supess}_{v\in[0,1]} L(v) \tag{9}$$

Since L is continuous, then

$$\mu - \operatorname*{supess}_{v\in[0,1]} L(v) = \sup_{v\in[0,1],\rho(v)>0} L(v) \quad .$$

We want to prove that

$$\sup_{v\in[0,1],\rho(v)>0} L(v) = \sup_{v\in[0,1]} L(v) \quad .$$

Consider $Z = \{\rho = 0\}$, and any connected component of Z. If it is a single point, then this point is limit of points where $\rho(v) > 0$. If the connected component of Z is a closed interval $[a,b]$ then, since $\rho = 0$, $L(a) = L(b) = L(v)$ for all $v \in [a,b]$. ☐

2.3 Gradient flow

Our approach to computing a geodesic homotopy between two curves C_0 and C_1 will be to start with an easily constructed homotopy (by linear interpolation for example) and evolve it in time via a length shortening flow until it converges to a geodesic. We will therefore now consider a family of homotopies $C(u,v,t)$ where u and v continue to parameterize each individual homotopy as before, but where $t \in \mathbb{R}^+$ now parameterizes the family and may be interpreted as an

[6] see for example proposition 2 in [2]

artificial time parameter. Differentiating (7) with respect to this new parameter yields (after a bit of work)

$$\frac{dE_k}{dt} = -2\int_0^1 \left\langle C_t, C_{v_*v_*} - (C_{v_*v_*}\cdot T)T - C_{v_*}\left(k\frac{\int_0^L C_{v_*}\cdot C_{ss}ds}{L} + C_{v_*}\cdot C_{ss}\right)\right.$$
$$\left. + \frac{1}{2}C_{ss}\left(k\frac{\int_0^L \|C_{v_*}\|^2 ds}{L} + \|C_{v_*}\|^2\right)\right\rangle_k dv_*$$

From this we see that the gradient flow for the family of curves C is given by[7]

$$\begin{aligned}\frac{\partial C}{\partial t} = {} & C_{v_*v_*} - C_{v_*}\left(k\frac{\int_0^L C_{v_*}\cdot C_{ss}ds}{L} + C_{v_*}\cdot C_{ss}\right) \\ & + \frac{1}{2}C_{ss}\left(k\frac{\int_0^L \|C_{v_*}\|^2 ds}{L} + \|C_{v_*}\|^2\right)\end{aligned} \qquad (10)$$

which converges to the following limit as k increases

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{k}\frac{\partial C}{\partial t} &= -C_{v_*}\left(\frac{\int_0^L C_{v_*}\cdot C_{ss}ds}{L}\right) + \frac{1}{2}C_{ss}\left(\frac{\int_0^L \|C_{v_*}\|^2 ds}{L}\right) \\ &= C_{v_*}\frac{L_{v_*}}{L} + \frac{1}{2}C_{ss}\left(\frac{\int_0^L \|C_{v_*}\|^2 ds}{L}\right)\end{aligned} \qquad (11)$$

3. PLANAR CURVES

We now consider the special case of planar embedded curves, which admit straight-forward implicit representations. For such curves we may used Level Set Methods [1] in the numerical implementation of the flow (10).

3.1 Gradient flow

Note that for planar curves C_{v_*} and C_{ss} are linearly dependent vectors since both are perpendicular to the unit tangent vector T and since the codimension is one. As such, $C_{v_*}(C_{v_*}\cdot C_{ss}) = C_{ss}(C_{v_*}\cdot C_{v_*})$, which after substitution into (10) yields

$$\begin{aligned}\frac{\partial C}{\partial t} = {} & C_{v_*v_*} + \frac{1}{2}C_{ss}\left(k\frac{\int_0^L \|C_{v_*}\|^2 ds}{L} - \|C_{v_*}\|^2\right) \\ & - C_{v_*}\left(k\frac{\int_0^L C_{v_*}\cdot C_{ss}ds}{L}\right) \\ = {} & \left(\alpha_v - \alpha_s(C_v\cdot T) + \frac{1}{2}\kappa\left(\alpha^2 - k\overline{\alpha^2}\right) + k\alpha\overline{\alpha\kappa}\right)N \\ & \left[\overline{\bullet} \text{ denotes the mean of } \bullet \text{ over } C = \frac{1}{L}\int_0^L \bullet\, ds\right]\end{aligned} \qquad (12)$$

where κ denotes the signed curvature and N denotes the outward[8] unit normal of the curve and $\alpha = C_v\cdot N$ denotes the normal velocity of the curve as we move along the homotopy. Notice that as long as the normal velocity α is bounded the diffusion coefficient multiplying the curvature term will become negative everywhere (thereby yielding a well-posed curvature-based diffusion along the outward normal) if k is sufficiently large. To ensure this, we consider the limiting gradient flow as $k\to\infty$.

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{k}\frac{\partial C}{\partial t} &= \frac{1}{L}\left(\alpha L_v - \frac{1}{2}\kappa\int_0^L \alpha^2\, ds\right)N \\ &= \left(\alpha\overline{\alpha\kappa} - \frac{1}{2}\kappa\overline{\alpha^2}\right)N\end{aligned} \qquad (13)$$

3.2 Level Set Implementation

To implement the gradient flow (12) we will represent the evolving homotopy $C(u,v,t)$ by embedding the corresponding evolving homotopy surface $S(u,v,t)$ as the zero level set of a 4D scalar function $\psi(x,y,v,t)$. In other words, ψ will be chosen such that $\psi\big(x(u,v,t), y(u,v,t), v, t\big) = 0$ where $\big(x(u,v,t), y(u,v,t)\big) = C(u,v,t)$. Our goal now is to determine the evolution equation for ψ which yields the evolution (12) for each of its 2D cross-sectional zero level sets.

Differentiating

$$\frac{d}{dt}\Big(\psi\big(x(u,v,t), y(u,v,t), v, t\big) = 0\Big) \quad\longrightarrow\quad \psi_t + \nabla\psi\cdot C_t = 0$$

where $\nabla\psi = (\psi_x, \psi_y)$ denotes the 2D spatial gradient of each cross-section of ψ, and substituting (12), noting that $N = \nabla\psi/\|\nabla\psi\|$, yields

$$\psi_t + \left(\alpha_v - \alpha_s\langle C_v, T\rangle + \frac{1}{2}\kappa(\alpha^2 - k\overline{\alpha^2}) + k\alpha\overline{\alpha\kappa}\right)\|\nabla\psi\| = 0. \qquad (14)$$

Note that since all of the terms above are geometric (i.e. independent of the parameterization of each curve) they may be expressed directly in terms of the function ψ. Computing such expressions and subsitituting these expressions into (14) yields the following time evolution PDE for ψ:

$$\begin{aligned}\psi_t = {} & \psi_{vv} - \frac{2\psi_v}{\|\nabla\psi\|^2}(\nabla\psi_v\cdot\nabla\psi) + \frac{\psi_v^2}{\|\nabla\psi\|^4}\left(\nabla^2\psi\nabla\psi\right)\cdot\nabla\psi \\ & + \frac{1}{2}\left(k\overline{\alpha^2} - \frac{\psi_v^2}{\|\nabla\psi\|^2}\right)\nabla\cdot\left(\frac{\nabla\psi}{\|\nabla\psi\|}\right)\|\nabla\psi\| + k\overline{\alpha\kappa}\psi_v\end{aligned} \qquad (15)$$

4. EXPERIMENTAL RESULTS

In this section we show experimental results of using the time evolution PDE (15) to compute the geodesic homotopy between two rather different closed curves C_0 and C_1. The two boundary curves C_0 and C_1 are displayed in figure 1 and the geodesic homotopy computed between these two curves is displayed in figure 2 in a left-to-right then top-to-bottom visualization. Finally, in figure 3 we show the homotopy surface S represented by the zero level-set of the function $\psi(x,y,v,t)$ after running the evolution (15) to steady state. Note that the curves visualized in figure 2 we obtained by taking cross-sections of this homotopy surface at evenly spaced values of v. The initial condition used by the evolution equation (15) was a simple linear interpolation of the signed distance transforms of the two boundary curves C_0 and C_1.

[7] We have added a tangential term $(C_{v_*v_*}\cdot T)T$, which does not affect the geometry of the flow.

[8] In the case of a non-closed curve, the sign of the curvature κ and the direction of the normal N follow the convention $C_{ss} = -\kappa N$.

Figure 1: Boundary curves C_0 and C_1

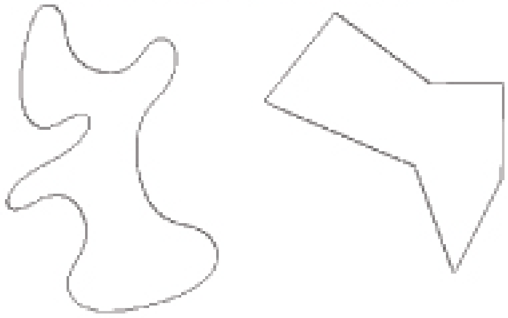


Figure 2: Visualization of the geodesic homotopy.

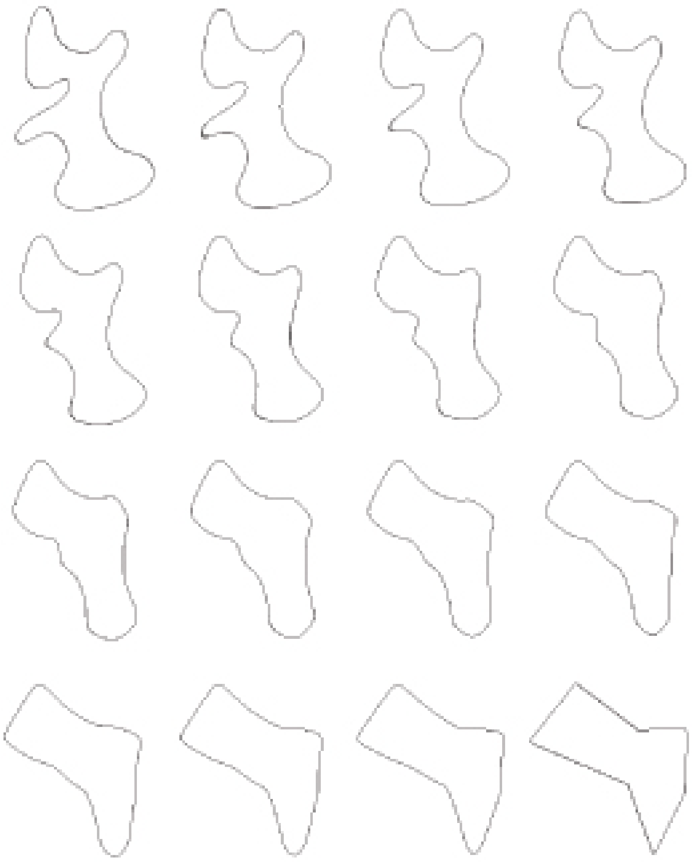
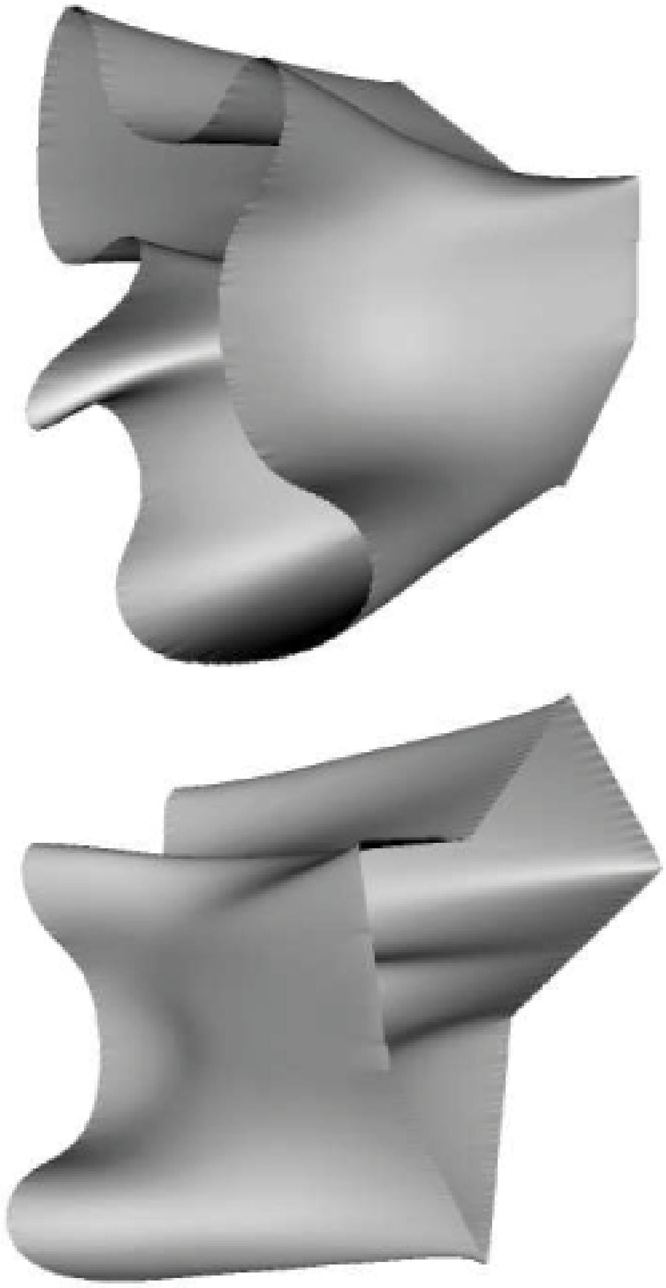


Figure 3: Computed homotopy surface (two views).



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