TENSION MODULATED NONLINEAR 2D MODELS FOR DIGITAL SOUND SYNTHESIS WITH THE FUNCTIONAL TRANSFORMATION METHOD

Stefan Petrausch and Rudolf Rabenstein

University of Erlangen-Nuremberg
Multimedia Communications and Signal Processing
Cauerstraße 7, D-91058 Erlangen, Germany
phone: + (49) 9131 85 28904, fax: + (49) 9131 85 28849, email: {stepe, rabe}@LNT.de
web: http://www.LNT.de/∼stepe

ABSTRACT

Tension modulated nonlinearities for the modeling of string instruments are well known to increase the quality of synthesized sounds significantly. These models consider the nonlinear feedback from the string's deflection to one of its physical parameters, the string tension. Obviously this effect occurs for two dimensional models, drums or plates for instance, too, however so far only non real-time implementations are available. Therefore in this paper a new approach is presented, where the functional transformation method (FTM) is applied. The mathematical model of a dispersive and damped membrane is set up including an additional term for the tension modulated nonlinearity. Using some slight simplification this model is solved with the FTM and, thanks to the scalability of the FTM, implemented in real-time.

1. INTRODUCTION

Physical modeling is one of the major current developments for digital sound synthesis, which becomes more and more part of commercial products (so far mainly in hybrid forms). Especially for the reproduction of musical instruments with strong nonlinearities physical modeling is advantageous, as such instruments sound different for every level of sound intensity they are played with.

In this scope one typical and common effect for string instruments are tension modulated nonlinearities (see [1]). The tension of the string is assumed to be a superposition of the string's tension in equilibrium plus an additional nonlinear term, that is a function of the string's deflection. For strings there are already several implementations for a number of different underlying modeling techniques, as for the digital waveguide (DWG) method [2], the functional transformation method (FTM) [3], and finite difference time domain (FDTD) schemes (see [4] for instance). All of them enhance the quality of the synthesized sounds significantly. Certainly the same principle can be used for sound synthesis of two-dimensional objects, what increases the sound quality all the more, as these nonlinear effects are often used on purpose by the musician (the drummer for instance).

Therefore a novel algorithm based on the FTM is presented in this paper. Thanks to the generality of the FTM it is a straightforward extension respectively combination of prior work on tension modulated nonlinear strings in [5] and linear models of drums in [6]. In detail the model of a rectangular, dispersive, and damped membrane with additional solution-dependent surface tension is formulated in form of a partial differential equation (PDE). This PDE is solved by means of transfer function models with the FTM, consequently accounting for the nonlinear term. After discretization and inverse transformation of the transfer function model, a discrete realization is achieved, that thanks to the scalability of the FTM can run in real-time on current hardware.

The paper is organized as follows: in section 2 the basic physical properties are discussed and the nonlinear model and some simplifying assumptions are introduced. After that the FTM is applied on the resulting PDE in section 3 yielding a discrete implementation. The results of this implementation are presented in section 4 and section 5 concludes this paper.

2. NONLINEAR SURFACE TENSION

A schematic of the model under consideration can be seen in figure 1. A thin rectangular membrane is defined on the region V, bounded by ∂V. The length along the x_1 axis is L_1 and along the x_2 axis L_2. The deflection is abbreviated by $y = y(x_1, x_2, t)$. The membrane is supported at the boundary ∂V with a certain surface tension T_0 in equilibrium.

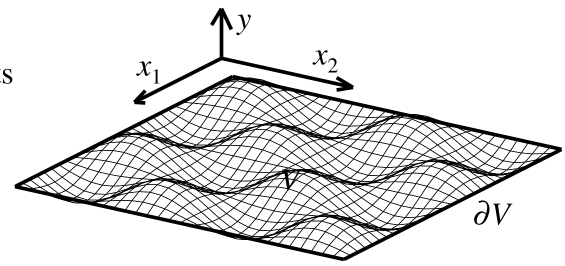

Figure 1: Schematic plot of a thin rectangular object defined on the region V with the boundary region ∂V.

The derivation of the basic physical principles is a combination of the physical principles of linear membranes (see [7]) and nonlinear strings (see [5]). First a small section (denoted by the upper index S) of the membrane has to fulfill Hook's law, what introduces Poisson's ratio p for two-dimensional systems:

$$\varepsilon_1 = \frac{\Delta l_1}{l_1^{\mathrm{S}}} = \frac{T_1}{E} - p\frac{T_2}{E} \tag{1}$$

$$\varepsilon_2 = \frac{\Delta l_2}{l_2^{\mathrm{S}}} = \frac{T_2}{E} - p\frac{T_1}{E}\,. \tag{2}$$

Note, that the subscripts (1 or 2) denote the spatial dimension (x_1 or x_2) and Δl_i denotes the elongation in x_i caused by the surface tension. Thus T_1 is the membranes surface tension in x_1 direction, that causes the elongation Δl_1. E is Young's modulus.

The next step is specific for multidimensional systems: we have to relate the elongation in x_1 and x_2 with the change of the surface area ΔA. For small segments this is done by

$$\Delta A^{\mathrm{S}} = A^{\mathrm{S}} - A_0^{\mathrm{S}} = l_1^{\mathrm{S}} \Delta l_2 + \Delta l_1 l_2^{\mathrm{S}} + \Delta l_1 \Delta l_2 \,, \quad (3)$$

there $A_0^{\mathrm{S}} = l_1^{\mathrm{S}} l_2^{\mathrm{S}}$ is the original surface area of the segment at equilibrium. To achieve a relation for the complete surface area A, we have to integrate over all infinitesimal small surface segments

$$\begin{aligned} A &= \int_0^{L_2}\int_0^{L_1} \mathrm{d}l_1 \mathrm{d}l_2 \\ &= \int_0^{L_2}\int_0^{L_1} \sqrt{1+\left(\frac{\partial y}{\partial x_1}\right)^2}\sqrt{1+\left(\frac{\partial y}{\partial x_2}\right)^2}\,\mathrm{d}x_1\mathrm{d}x_2 \,. \quad (4) \end{aligned}$$

The following steps, that finally lead to an expression for the nonlinear surface tension $T_{\mathrm{NL}}(y)$, are based on three assumption (according to [1, 2, 5]) that simplify the derivation considerably:

1. The elongation in both dimensions is assumed to be small compared to the original length, $\Delta l_i \ll l_i^{\mathrm{S}}$, so that an approximation by a Taylor series is possible, both in equation (4) and in equation (3).
2. The surface tensions T_1 and T_2 in equation (1) and (2) are assumed to be constant throughout the complete area.
3. Furthermore these tensions are assumed to be equal $T_1 = T_2$. This assumption presumes an uniform clamping, what holds for most drums.

Using these assumptions we can derive from (1) - (4) the overall nonlinear surface tension to be

$$\begin{aligned} T_{\mathrm{NL}}(y) &= T_0 + T_1(y) = \\ &= T_0 + \frac{E}{2(1-p)}\frac{1}{2L_1L_2}\cdot\int_0^{L_2}\int_0^{L_1}\left(\left(\frac{\partial y}{\partial x_1}\right)^2 + \right. \\ &\quad \left. +\left(\frac{\partial y}{\partial x_2}\right)^2 + \frac{1}{2}\left(\frac{\partial y}{\partial x_1}\right)^2\left(\frac{\partial y}{\partial x_2}\right)^2\right)\mathrm{d}x_1\mathrm{d}x_2 \,. \quad (5) \end{aligned}$$

The remaining steps towards the desired PDE include the equation of motion, an equation of bending, and the introduction of damping terms. Details on this derivation and the associated physical parameters can be found in [7] or [8] and are not described here. Important in this scope is, that the nonlinear part of the surface tension $T_1(y)$ simply produces one additional term in the equation:

$$-\frac{T_1(y)}{\sigma}\nabla^2 y - \frac{T_0}{\sigma}\nabla^2 y + S^4\nabla^4 y + d_1\dot{y} - d_3\nabla^2\dot{y} + \ddot{y} = f_{\mathrm{e}}(x_1,x_2,t) \,, \quad (6)$$

where $\dot{y}$ denotes first order temporal derivation and the Nabla-operator ∇ denotes first order spatial derivation; $f_{\mathrm{e}}(x_1,x_2,t)$ is an arbitrary excitation force.

3. APPLICATION OF THE FTM

The application of the FTM follows completely the procedure described in [6] or [8], thereby the nonlinear term in equation (6) is regarded as an additional excitation. In particular the eigenvalue problem of the FTM is created and solved completely disregarding the nonlinear term. In result, eigenvalues and transformation kernel are identical to the linear model as described in [6] yielding the transfer function model

$$\bar{Y}(\mu_1,\mu_2,s) = \frac{1}{s^2 + s\sigma_\mu + (\omega_\mu^2 + \sigma_\mu^2)} \cdot \left(\bar{F}_{\mathrm{e}}(\mu_1,\mu_2,s) + \frac{1}{\sigma}\bar{B}(\eta_\mu,Y,\bar{Y})\right) \,, \quad (7)$$

with its corresponding eigenfrequencies

$$\begin{aligned} \beta_\mu &= \sigma_\mu + j\omega_\mu \quad (8) \\ \sigma_\mu &= d_1 + d_3\pi^2\left(\frac{\mu_1^2}{L_1^2} + \frac{\mu_2^2}{L_2^2}\right) \\ \omega_\mu &= \sqrt{\frac{T_0}{\sigma}\pi^2\left(\frac{\mu_1^2}{L_1^2}+\frac{\mu_2^2}{L_2^2}\right) + S^4\pi^4\left(\frac{\mu_1^4}{L_1^4}+\frac{\mu_2^4}{L_2^4}\right) - \sigma_\mu^2} \,. \end{aligned}$$

Note, that due to the finite dimensions of the region V, only discrete values are adopted for the eigenfrequencies β_μ. This fact is indicated by the integer values $\mu_1, \mu_2 \in \mathbb{N}_0$ and $\mu = \mu(\mu_1,\mu_2)$ as an arbitrary but invertible mapping of $\mathbb{N}_0^2 \leftrightarrow \mathbb{N}_0$.

Nevertheless, both integral transformations that yielded the transfer function model (7) have to be applied on the nonlinear term $T_1(y)\nabla^2 y$ too. This includes the Laplace transformation (denoted by $\bar{Y} = \mathscr{L}\{\bar{y}\}$) and the Sturm-Liouville transformation (denoted by $\bar{y} = \mathscr{T}\{y\}$)

$$\bar{B}(y) := \mathscr{L}\left\{\bar{b}(y)\right\} := \mathscr{L}\left\{\mathscr{T}\left\{T_1(y)\nabla^2 y\right\}\right\} \,. \quad (9)$$

Fortunately it is possible to simplify the evaluation of (9) significantly by

$$\bar{b}(\eta_\mu,y,\bar{y}) := \mathscr{T}\left\{T_1(y)\nabla^2 y\right\} = \eta_\mu^2 T_1(y)\bar{y} \,, \quad (10)$$

with $\eta_\mu^2 = -\left(\frac{\pi\mu_1}{L_1}\right)^2 - \left(\frac{\pi\mu_2}{L_2}\right)^2$. The explicit procedure to achieve equation (10) is quite complex and similar to the procedure described in [5]and therefore not performed here.

To achieve a discrete implementation the transfer function model (7) is discretized with the impulse invariant transformation, and transformed back, both in space (with the inverse Sturm-Liouville transformation) and in time (with the inverse Laplace transformation). The nonlinear term $\bar{B}(\eta_\mu,Y,\bar{Y})$ thereby is treated as an external excitation function which is known to the algorithm.

Again the procedure is quite similar to [5] and not described in detail here. The crucial point is, that due to the shifting theorem of the impulse invariant transformation only past values of $\bar{b}^{\mathrm{d}}(\eta_\mu,y^{\mathrm{d}},\bar{y}^{\mathrm{d}})$ are needed for the evaluation of the actual output $y^{\mathrm{d}}(x_1,x_2,k)$ (k denotes the discrete time and upper d denotes discrete time values). Values that are delayed by exactly one sample. Furthermore, as $T_1(y)$ is assumed to be constant over the complete area $\bar{b}^{\mathrm{d}}(\eta_\mu,y^{\mathrm{d}},\bar{y}^{\mathrm{d}})$

is also constant over the complete area (see equation (10)). As the Sturm-Liouville transformation from $y^{\mathrm{d}}(x_1,x_2,k)$ to $\bar{y}^{\mathrm{d}}(\mu_1,\mu_2,k)$ is a one to one mapping, $\bar{b}^{\mathrm{d}}(\eta_\mu,y,\bar{y})$ can be attained by $\bar{y}^{\mathrm{d}}(\mu_1,\mu_2,k)$ alone:

$$\bar{b}^{\mathrm{d}}(\eta_\mu,y,\bar{y}) = \bar{b}^{\mathrm{d}}\left(\eta_\mu, \mathscr{T}^{-1}\left\{\bar{y}^{\mathrm{d}}(\mu_1,\mu_2,k)\right\}, \bar{y}^{\mathrm{d}}(\mu_1,\mu_2,k)\right) .$$

These considerations can also be proven by insertion of all known values into $\bar{b}^{\mathrm{d}}(\eta_\mu,y,\bar{y})$ and some tedious reformulation process which makes use of the orthogonality of the transformation kernels of the Sturm-Liouville transformation. Finally it results in

PSfrag replacements

$$\bar{b}^{\mathrm{d}}(\eta_\mu,y,\bar{y}) = \eta_\mu^2 \bar{y}^{\mathrm{d}}(\mu_1,\mu_2,k)\frac{E}{(1-p)L_1^2L_2^2} \cdot \\ \cdot \sum_{\nu_1=0}^{N_1}\sum_{\nu_2=0}^{N_2}\left(\pi^2\left(\frac{\nu_1^2}{L_1^2}+\frac{\nu_2^2}{L_2^2}\right)\left(\bar{y}^{\mathrm{d}}(\nu_1,\nu_2,k)\right)^2 + \right. \\ \left. + \frac{\pi^4\nu_1^4\nu_2^4}{2L_1^4L_2^4}\left(\bar{y}^{\mathrm{d}}(\nu_1,\nu_2,k)\right)^4\right) , \quad (11)$$

where N_1 and N_2 are finite integers, that determine the order in each spatial dimension and whose product is the absolute number of harmonics $N = N_1 \cdot N_2$.

The complete implementation of an oscillating membrane with tension modulated nonlinearity can be seen in figure 2. Its structure is almost identical to the linear implementation, proposed in [3], only the double bounded box is added. In detail one can see several second order recursive systems, which implement the second order transfer function in equation (7). The constant $c_1(\mu)$ and $c_2(\mu)$ result from the shifting theorem of the inverse Z-transformation and can be calculated by

$$\begin{aligned} c_1(\mu) &= 2\cdot e^{-\sigma_\mu T}\cdot\cos(\omega_\mu T) \\ c_2(\mu) &= -e^{-2\sigma_\mu T} , \end{aligned}$$

where T is the sampling interval of the impulse invariant discretization.

These recursive systems produce the Sturm-Liouville transformed outcome $\bar{y}^{\mathrm{d}}(\mu,k)$. The inverse Sturm-Liouville transformation is simply the summation over all eigenfrequencies β_μ (and consequently μ) weighted by the transformation kernel $K(x_1,x_2,\mu)$ (which can be given analytically in this scenario, see [6] for details). Important in this scope is the excitation of the recursive systems. On the one hand (as also denoted in equation (7)) the system is excited by the discretized Sturm-Liouville transformed excitation function $\bar{f}_{\mathrm{e}}^{\mathrm{d}}(\mu,k)$, which can be achieved by (again, see [6])

$$\bar{f}_{\mathrm{e}}^{\mathrm{d}}(\mu,k) = e^{-\sigma_\mu T}\cdot\frac{\sin(\omega_\mu T)}{\omega_\mu}\cdot \\ \cdot\iint_V K(x_1,x_2,\mu)\cdot f_{\mathrm{e}}(x_1,x_2,kT)\,\mathrm{d}x_1\,\mathrm{d}x_2 .$$

One the other hand it is also excited by the nonlinear term $\frac{1}{\sigma}\bar{b}(\eta_\mu,y,\bar{y})$ which is calculated by equation (11). In figure 2 this term is included in the double bounded box, where $(\cdot)$ is an abbreviation for the discrete Sturm-Liouville transformed outcome $\bar{y}^{\mathrm{d}}(\mu,k)$ and the constants $\chi(\mu)$ follow from equation (11) to

$$\chi(\mu) = \frac{\eta_\mu^2 E}{\sigma(1-p)L_1^2L_2^2}\cdot e^{-\sigma_\mu T}\cdot\frac{\sin(\omega_\mu T}{\omega_\mu} . \quad (12)$$

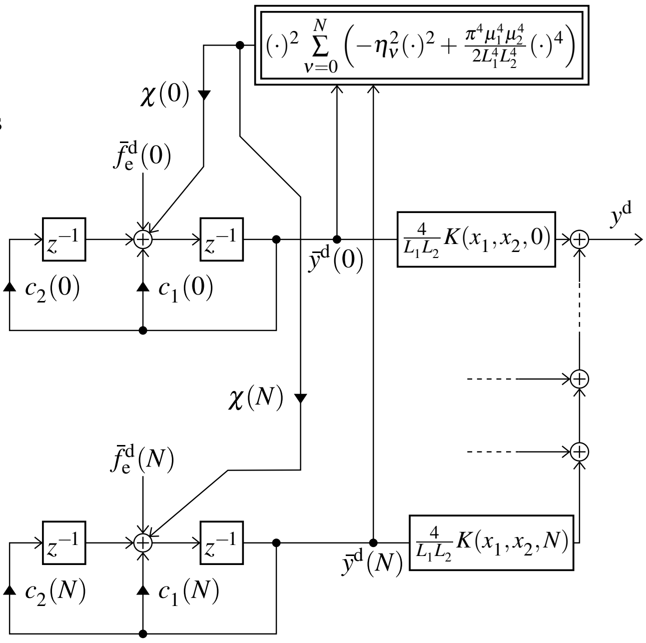

Figure 2: Block diagram of the implementation of an oscillating membrane with tension modulated nonlinearities. The constants $c_1(\mu)$ and $c_2(\mu)$ are identical to the linear implementation described in [6], the transformation kernel $K(x_1,x_2,\mu)$ too. The constants $\chi(\mu)$ are given in equation (12). All signals are function of discrete time k, what is omitted for concise notation. The nonlinear term in the double bounded box is a function of $\bar{y}^{\mathrm{d}}(\mu,k)$ (denoted by the dots).

4. RESULTS

Simulation results can be seen in figure 3. The upper plots are simulations without the nonlinear terms and the lower plots includes the nonlinearity. The simulations on the right were excited with a five times stronger excitation force compared to the simulations on the left.

It can be clearly seen, that the first few microseconds after the excitation are identical in all plots. However, the deflection of the nonlinear models is limited by an increasing, backward driving surface tension. This effect is even more impressive in the right hand side plots in figure 3, where the results of the linear model are obviously just scaled, while the nonlinear effect in the lower right plot limits the deflection below 0.3mm.

However, one major benefit of the approach is the scalability. The proposed structure in figure 2 can be easily scaled down by limiting the number of calculated modes N. An upper limit for the number of modes is obviously the Nyquist frequency, however less modes are also possible, mostly without noticeable differences. On current hardware lacements

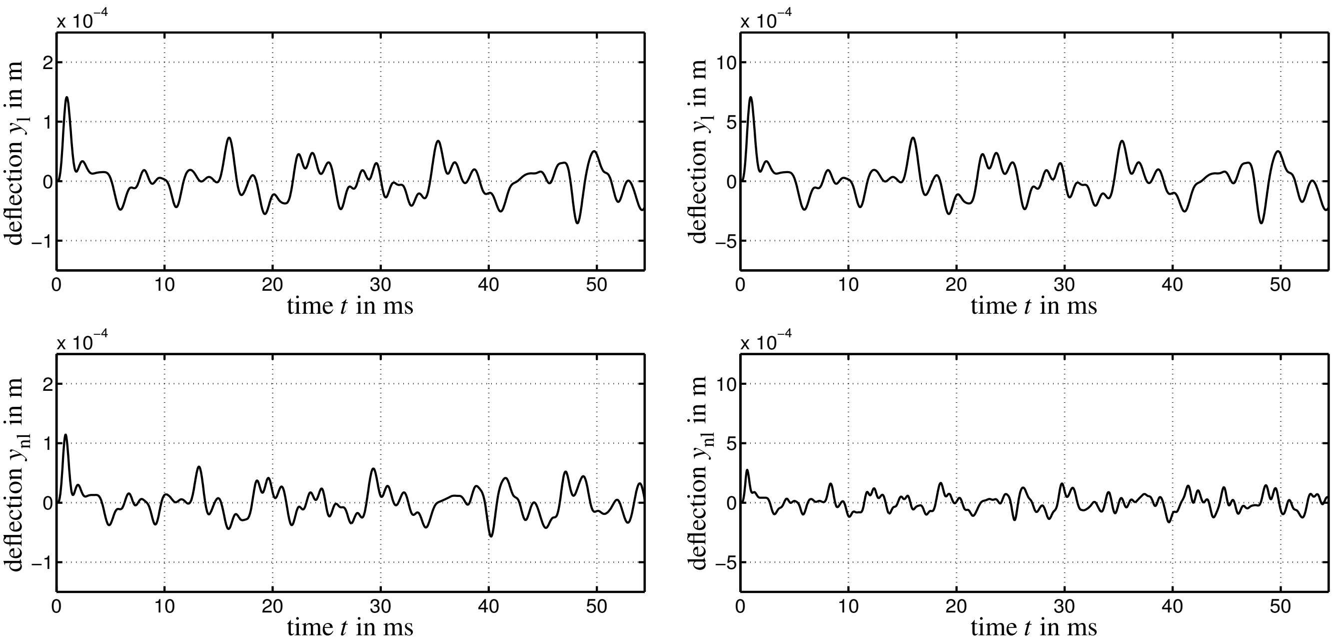

Figure 3: Simulation of a membrane's deflection after it was excited by a band-limited impulse. The upper plots are the simulations of a linear model without tension modulation. The lower plots are the simulations of the same model with tension modulated nonlinearities. The simulations on the right were excited five times stronger.

over 1000 harmonics can be calculated in real-time at full audio sample rate (44100 Hz) without any algorithmic delay. This number can cover all audible harmonics for wooden and metal plates. However for typical kettle drum parameters it only reaches up to 1 or 2 kHz. Nevertheless, one easily can take advantage of psychoacoustic effects by skipping low excited modes and thus obtain a satisfactory real-time sound output.

5. CONCLUSIONS

In this paper a new approach for sound synthesis with nonlinear two-dimensional physical models was presented. In doing so, the well known approach of tension modulated nonlinearities for one-dimensional string models was extended to two-dimensional membranes. A few simplifications on the model were introduced to keep the evaluation of the nonlinearity practicable. The model was solved in the frequency domain with the functional transformation method (FTM), whereas the nonlinear term was treated as an additional excitation. According to the procedure of the FTM, the transfer function was discretized and transformed back, both in time and space frequency domain yielding a discrete system for the linear model. The nonlinear term was reformulated to minimize the computational effort for its evaluation and it was introduced as an additional block in the discrete system of the linear part. The complete structure was implemented on a common personal computer and with some mild restrictions was proven to run in real-time.

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