ANALYSIS BANKS, SYNTHESIS BANKS, LPTV FILTERS: PROPOSITION OF AN EQUIVALENCE DEFINITION APPLICATION TO THE DESIGN OF INVERTIBLE LPTV FILTERS

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ABSTRACT

Linear Periodical Time Varying (**LPTV**) filters suffer from a lack of theoretical studies chiefly regarding construction methods of invertible LPTV filters. Conversely, filter banks have received considerable attention because of large applications in digital communications. In this paper, analysis and synthesis banks of Maximally Decimated Filter Banks (**MDFB**) are separately related to LPTV filters. Thanks to this definition, perfect reconstruction for a MDFB is mathematically related to invertibility of LPTV filters. We then make the most of these results to propose a new class of invertible LPTV filters relying on a lossless matrix. In a spread spectrum framework, lossless property results in no Bit Error Rate degradation when Spreading is achieved by lossless LPTV filters.

1. INTRODUCTION

Linear Periodically Time Varying (**LPTV**) filters are characterized by a periodical time varying impulse response and a large number of applications turn out to explicitly rely on these filters in digital signal processing: spread spectrum [5],[3], transmultiplexing [1], blind channel estimation or spectral scrambling.

Many applications like spread spectrum require invertible LPTV filters, and although the invertibility necessary and sufficient condition is well known [7], it does not provide any solution for the construction method of an a priori invertible LPTV filter. This design issue suffers from a lack of theoretical studies since only [10] adressed this problem. This bibliographic observation motivated a previous paper [4] in which we introduced a subset of invertible LPTV filters defined by a circulant modulator matrix. This subset turns out to be quite restrictive.

In a multirate framework, Maximally Decimated Filter Banks (**MDFB**) have received a large interest. MDFB are made of the cascade of an analysis bank (**AB**) and a synthesis bank (**SB**), but, because of the LPTV nature [2] of interpolation and decimation used in multirate processing, MDFB turn out to have LPTV properties. Actually, [8] has already pointed out these LPTV properties emphasizing that the whole MDFB (casacade of analysis and synthesis banks) could be described by an LPTV filter.

We propose here to separately point out LPTV nature of Analysis and Synthesis Banks. Although AB and SB are not exactly LPTV filters according to representations [6], we introduce two equivalence definitions between an LPTV filter and an AB and an SB respectively. In the light of these equivalences, it is then possible to relate the LPTV filter invertibility and Perfect Reconstruction (**PR**) for an MDFB.

This result is of great interest in that by generalizing results [9] on MDFB, we can propose a new method for designing non trivial invertible LPTV filters that we call LossLess (**LL**) LPTV filters relying on lossless matrices. In a spread spectrum framework, previous works, [5], [3], used trivial LPTV filters (interleavers). Here, our theoretical results provide a simple method to carry out an efficient Spread Spectrum system relying on a large subset of LPTV filters. In addition, the lossless property ensures that the introduction of the LPTV filter does not degrade Bit Error Rate (**BER**) performance of the system if the channel is an Additive White Gaussian Noise (**AWGN**) channel.

Part II and III are respectively devoted to a review of LPTV filters and MDFB. In part IV, these two domains are mathematically related together through the proposition of two equivalence definitions. Thanks to these definitions, part V proposes to extend results [9] on MDFB to the construction of invertible LPTV filters. Finally, part VI illustrates how this result makes possible an efficient spread spectrum system relying on LL LPTV filters.

2. REVIEW OF LPTV FILTERS

In this part, we present LPTV filters and the invertibility definition.

2.1 Notations

Given two integers (n, N), $n = N\overline{n}^N + \underline{n}_N$ stands for the Euclidian division of n by N (i.e $0 \leq \underline{n}_N \leq N-1$). The z-transform of a digital signal $x(n)$ is defined by $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$. N stands for LPTV filter period in the general case. Given a matrix $\mathbf{A}$, $\mathbf{A}^t$ stands for transposate operation. Given a z-transform matrix $\mathbf{A}$(z), $\mathbf{A}_*$(z) denotes conjugation of the coefficients without conjugating z.

Given a signal $x(n)$, we define two kinds of polyphase components, $x_k(n)$ and $x'_k(n)$. The first polyphase components $x_k(n)$, are the N decimated versions of $x(n+k)$ (1).

$$x_k(n) = x(nN + k) \quad \text{for} \quad 0 \leq k \leq N-1 \tag{1}$$

The second polyphase components $x'_k(n)$ are defined by (2) where $\mathbf{0}_{N-1}$ denotes a row vector of $N-1$ zeros. It is relevant to introduce the corresponding z-transforms (3).

$$x'_k(n) = \delta(\underline{n-k}_N)x(n) = [...x(k-N)\ \mathbf{0}_{N-1}\ x(k)\ \mathbf{0}_{N-1}...] \tag{2}$$

$$X'_k(z) = \sum_{n=-\infty}^{\infty} x'_k(n)z^{-n} \tag{3}$$

Finally, $\hat{\mathbf{X}}'(z) = \begin{bmatrix} X'_0(z) & X'_1(z) & ..... & X'_{N-1}(z) \end{bmatrix}^t$ represents the polyphase vector.

2.2 Definition of an LPTV filter

An LPTV filter with period N is a filter whose impulse response is an N periodic function. Many equivalent representations, instructively summed up in [6], are possible for an LPTV filter. One of interest in the following is the Multiple Input Multiple Output (MIMO) representation. In such a representation, output and input polyphase vectors are related together according to the matrix relation (4) where $\mathbf{H}(z)$ is the $N \times N$ $z-$transform MIMO matrix (Figure 1). F_e stands for the input signal sampling frequency.

$$\hat{\mathbf{Y}}\text{'}(z) = \mathbf{H}(z)\hat{\mathbf{X}}\text{'}(z) \tag{4}$$

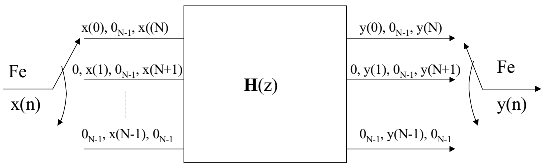

Figure 1: MIMO representation of an LPTV filter

2.3 Invertibility of an LPTV filter

Given an LPTV filter, with period N, defined by its MIMO matrix $\mathbf{H}_1(z)$, we say that this LPTV filter is invertible if there is an LPTV filter with a MIMO matrix $\mathbf{H}_2(z)$ and an integer r_0 such that relation (5) is fulfilled where $\mathbf{I}_N$ stands for the N identity matrix. In other words, the cascade of these two LPTV filters results in a delay r_0.

$$\mathbf{H}_2(z)\mathbf{H}_1(z) = z^{-Nr_0}\mathbf{I}_N \tag{5}$$

Actually, LPTV invertibility leads to a matrix invertibility problem for z belonging to the complex unit circle. A Necessary and Sufficient Condition (**NSC**) is given in [7]. In addition, with such a definition, because of the introduction of any delay r_0, providing an LPTV filter is invertible, there is not only one inverse LPTV filter.

3. REVIEW OF MAXIMALLY DECIMATED FILTER BANKS

In this part, we restate the description of MDFB. Then, the notion of perfect reconstruction is emphasized.

3.1 Definition of Maximally Decimated Filter Banks

Maximally Decimated Filter Banks are defined according to figure 2 by two sets of N Linear Time Invariant (**LTI**) filters $\{F_i(z)\}$ and $\{G_i(z)\}$ repectively called analysis and synthesis filters. Analysis filters aim at splitting the input signal $u(n)$ into N decimated subband signals $u^i(n)$ that are then processed according to the application. Conversely, synthesis filters aim at reconstructing the signal $v(n)$ from the N decimated subband signals $v^i(n)$.

It is relevant for the following to decompose analysis filters and synthesis filters respectively according to relations (6) and (7). We denote $\mathbf{F}(z)$ and $\mathbf{G}(z)$ as the polyphase matrices defined by $[\mathbf{F}(z)]_{i,k} = F_{i,k}(z)$ and $[\mathbf{G}(z)]_{i,k} = G_{i,k}(z)$.

$$F_i(z) = \sum_{k=0}^{N-1} z^{-k} F_{i,k}(z^N) \tag{6}$$

$$G_i(z) = \sum_{k=0}^{N-1} z^{-(N-1-k)} G_{k,i}(z^N) \tag{7}$$

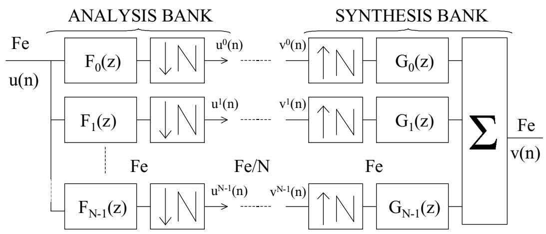

Figure 2: Structure of Maximally Decimated Filter Banks

3.2 Perfect Reconstruction (PR) for MDFB

An usual property of interest for a Maximally Decimated Filter Bank is its ability to recover a signal $v(n)$ closest to $u(n)$. In the absence of processing (ie. $u^i(n) = v^i(n)$) and using basic results on decimation and interpolation, it is possible [8] to relate z-transforms $U(z)$ and $V(z)$ according to expression (8) where $W_N = \exp(-\frac{2i\pi}{N})$.

$$V(z) = \frac{1}{N} \sum_{n=0}^{N-1} U(zW_N^{-n}) \sum_{k=0}^{N-1} F_k(zW_N^{-n}) G_k(z) \tag{8}$$

Terms in (8) for $n \neq 0$ are called alisasing terms. If analysis and synthesis filters are such that (8) results in $V(z) = U(z)z^{-r_0}$, i.e $v(n) = u(n - r_0)$, then we say that the MDFB has the Perfect Reconstruction (**PR**) property. A NSC on the matrix $\mathbf{P}(z) = \mathbf{G}(z)\mathbf{F}(z)$ is given in [9].

4. PROPOSITION OF AN EQUIVALENCE BETWEEN LPTV FILTERS AND MDFB

In [8], it was shown that the cascade of analysis and synthesis filter banks could be exactly described by an LPTV filter. In this part, the objective is different since we aim at separately relating analysis filters and synthesis filters to LPTV filters. Therefore, we introduce two equivalence definitions. Thanks to these definitions, we mathematically relate PR of MDFB and invertibility of LPTV filters. As the proofs of further propositions are quite tedious, they are not provided and emphasis is laid on results. However, these proofs are currently available in the Ph.D dissertation [11] and an extended version of these results with applications in digital communications is under writing for a forthcoming publication.

4.1 Definition of the equivalence

Previous reviews of part II and III can be summed up through the Inputs/Outputs representations of an AB, an SB and an LPTV filter on figure 3. According to this figure, it is obvious that the AB and the SB cannot separately be entirely modeled by an LPTV filter. Nevertheless, we propose here two equivalence definitions.

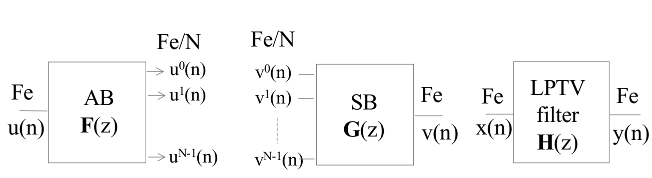

Figure 3: Inputs/Outputs representations of AB, SB and LPTV filter

Definition 1: According to figure 3, we consider an LPTV filter and an AB respectively defined by the MIMO matrix

$\mathbf{H}(z)$ and the polyphase matrix $\mathbf{F}(z)$. We make the assumption that the same signal is processed by the LPTV filter and the AB, namely $x(n) = u(n)$. We denote $y(n)$ as the output of the LPTV filter and $u^i(n)$ as the N outputs of the AB. We say that the LPTV filter and the AB are equivalent if $u^i(n) = y_i(n)$, whatever the input signal $x(n)$, where $y_i(n)$ are the polyphase components (1) of $y(n)$.

We show [11] that an LPTV filter with MIMO matrix $\mathbf{H}(z)$ is equivalent to an AB with polyphase matrix $\mathbf{F}(z)$ if these two matrices are related together according to relation (9). $\mathbf{D}_z$ is the diagonal matrix of powers of z (i.e $[\mathbf{D}_z]_{i,i} = z^i$) and $\mathbf{T}_N(z)$ is defined by (10) where $\mathbf{I}'_N$ is the matrix with opposite diagonal (namely $[\mathbf{I}'_N]_{i,j} = \delta(\underline{N-1-i+j}_N)$).

$$\mathbf{H}(z) = \mathbf{D}_z^{-1}\mathbf{F}(z^N)\mathbf{T}_N(z^N)\mathbf{D}_z \quad (9)$$

$$\mathbf{T}_N(z) = \begin{bmatrix} 1 & \mathbf{0}_{N-1} \\ \mathbf{0}_{N-1}^t & z^{-1}\mathbf{I}'_{N-1} \end{bmatrix} \quad (10)$$

According to this first equivalence definition, if we consider any LPTV filter, there is one and only one equivalent AB. Conversely, given any AB, there is one and only one equivalent LPTV filter.

Definition 2: According to figure 3, we consider an LPTV filter defined by its MIMO matrix $\mathbf{H}(z)$. We denote $x(n)$ and $y(n)$ as the input and output of this filter according to figure 3. We consider $x_i(n)$ the polyphase components of $x(n)$ according to (1). We consider an SB defined by its polyphase matrix $\mathbf{G}(z)$. We make the assumption that the N signals $v^i(n) = x_i(n)$ are passed through the SB and we denote $v(n)$ as the output of the SB. We say that the LPTV filter and the SB are equivalent if $v(n) = y(n)$ whatever the input signal $x(n)$.

We show [11] that an LPTV filter with MIMO matrix $\mathbf{H}(z)$ is equivalent to an SB with polyphase matrix $\mathbf{G}(z)$ if these matrices are related together according to relation (11).

$$\mathbf{H}(z) = \mathbf{D}_z^{-1}\mathbf{I}'_N\mathbf{G}(z^N)\mathbf{D}_z \quad (11)$$

According to this second equivalence definition, if we consider any LPTV filter, there is one and only one equivalent SB. Conversely, given any SB, there is one and only one equivalent LPTV filter.

4.2 Perfect Reconstruction and LPTV filter invertibility

We will now emphasize why the two previously introduced definitions are of interest through the following proposition whose demonstration can also be found in [11].

Proposition: Given an AB defined by its polyphase matrix $\mathbf{F}(z)$, we consider the equivalent LPTV filter defined by an MIMO matrix $\mathbf{H}(z)$. We consider an SB defined by its polyphase matrix $\mathbf{G}(z)$. Then, the cascade of the AB and the SB yields Perfect Reconstruction if and only if the LPTV filter with MIMO matrix $\mathbf{H}(z)$ is invertible according to relation (5) and an inverse LPTV filter is equivalent to the SB.

5. PROPOSITION OF AN INVERTIBLE LPTV FILTERS SET

The previous result is really of interest for the study of LPTV filters. Unlike LPTV filters, MDFB have received considerable attention and chiefly regarding perfect reconstruction. On this topic, [9] proposed a design procedure for Perfect Reconstruction MDFB relying on Lossless transfer matrices. We aim here at using previous results to define a new set of invertible LPTV filters.

5.1 Review of Lossless Filter Banks

A lossless transfer matrix is an $N \times N$, z-transform matrix fullfilling for all complex z: $\mathbf{A}(z)\mathbf{A}_*^t(z^{-1}) = \mathbf{I}_N$. A sufficient condition [9] to get an PR MDFB is that AB polyphase matrix $\mathbf{F}(z)$ is a lossless $z-$transform matrix. Under this condition, SB with polyphase matrix $\mathbf{G}(z) = z^{-r}\mathbf{F}_*^t(z^{-1})$ allows a PR. r is a delay introduced to make the SB causal. In addition, it is known [9] that a FIR z-transform matrix $\mathbf{A}(z)$ with degree p is lossless if and only if it is of the form $\mathbf{A}(z) = \mathbf{A}_p(z)\mathbf{A}_{p-1}...\mathbf{A}_1(z)\mathbf{U}$ where $\mathbf{U}$ is an $N \times N$ constant unitary matrix and $\mathbf{A}_j(z)$ is of the form $\mathbf{A}_j(z) = \left[\mathbf{I}_N - \mathbf{v}_j\mathbf{v}_{j*}^t + z^{-1}\mathbf{v}_j\mathbf{v}_{j*}^t\right]$ where $\mathbf{v}_j$ are column vectors of unit norm.

5.2 Proposition of an invertible LPTV filter set

Using previous equivalence definitions and notations, we call LossLess (**LL**) N periodic LPTV filter with order p an LPTV filter whose MIMO matrix $\mathbf{H}_1(z)$ is of the form (12).

$$\mathbf{H}_1(z) = \mathbf{D}_z^{-1}\prod_{j=1}^{p}\left[\mathbf{I}_N + (z^{-N}-1)\mathbf{v}_j\mathbf{v}_{j*}^t\right]\mathbf{U}\mathbf{T}_N(z^N)\mathbf{D}_z \quad (12)$$

Using previous results, this LPTV filter is invertible with an inverse MIMO matrix given by $\mathbf{H}_2(z)$ (13).

$$\mathbf{H}_2(z) = z^{-Nr}\mathbf{D}_z^{-1}\mathbf{I}'_N\left[\prod_{j=1}^{p}\left(\mathbf{I}_N + (z^N-1)\mathbf{v}_j\mathbf{v}_{j*}^t\right)\mathbf{U}\right]_*^t\mathbf{D}_z \quad (13)$$

An attractive property is that Lossless LPTV filter and its inverse LPTV filter are both FIR LPTV filters. Sensitvity of LL LPTV filter according to a desynchronisation is not discussed here and will be the subject of a forthcoming work.

6. APPLICATION TO A SPREAD SPECTRUM SYSTEM

Because of spreading properties of LPTV filters, it was emphasized that LPTV filters were an original tool to carry out Spread Sprectrum (**SS**) system relying on LPTV filters [3], [4], [5]. Such an application constitutes a solution to achieve a SS Multiuser System. But, until now proposed solutions are trivial LPTV : interleavers or periodical clock changes ([3], [5]). In [4], invertible LPTV filters are more complex but it is possible to show that the use of these LPTV filters for SS results in degradations in terms of BER in case of AWGN channels. LL LPTV filters consitute a large class of LPTV filters and unlike LPTV filters in [4], we will see that the lossless property ensures no BER degradation.

6.1 Presentation of the system

A binary stream (Figure 4) is mapped into a quaternary digital modulation (QPSK). This digital signal is upsampled by a factor L. This factor also representes the spreading factor of the system. Then, the signal is filtered by an LL LPTV filter and passed through the channel. The reception side is composed of the inverse LPTV filter. Then, a hard decision is processed at the output of this inverse LPTV filter to get an estimation of the emitted symbols.

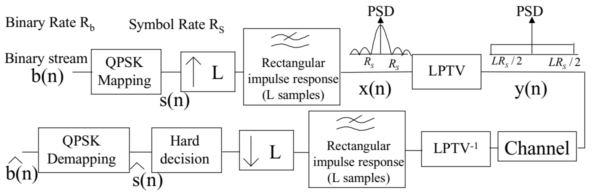

Figure 4: Spread Spectrum System relying on LPTV filter

For simulation, we choose an upsamling factor $L = 8$. Emission LPTV filter has a MIMO matrix according to (12) with a period $N = 40$ and a degree $p = 2$. Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are randomly generated with unit norm and matrix $\mathbf{U}$ is unitary and randomly generated.

6.2 Simulation results

6.2.1 Spreading efficiency of LossLess LPTV filters

Figure 5 illusrates the Power Spectral Density (**PSD**) of the signal before and after emission LPTV filter. This figure points out the good spreading properties of lossless LPTV filters, although lossless filters are a subset of the large class of LPTV filters.

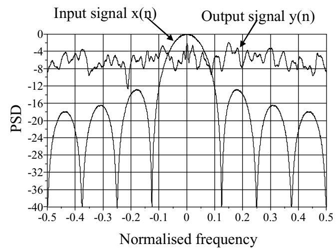

Figure 5: Spreading properties of LL LPTV filters

6.2.2 Performance on an AWGN channel

Let consider a general case where a signal is filtered at the emission side and where the reception side consists in inversing the emission filter, like a Zero Forcing Equalizer (ZFE) technic. If the emission filter is any LTI filter (except a simple delay), inversion operation systematically results in a degradation of BER performance because of amplification of noise. This result still holds if the emission filter is any LPTV filter. However, thanks to lossless property of LL LPTV filters, noise is not amplified by inverse filtering. An attractive consequence is that simulated BER performance of a Spread Spectrum system relying on LL LPTV filters are similar to well known theoretical BER performance of a classical digital system without spreading and relying on an emission filter and a reception matched filter like a Square Root Cosine (**SRC**) filter. Figure 6 illustrates these results.

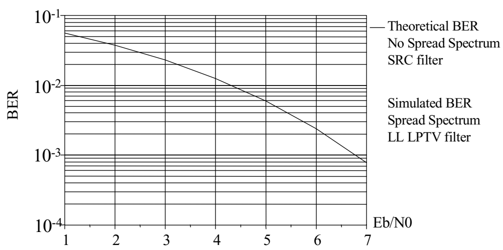

Figure 6: BER for LL LPTV filters

7. CONCLUSION

LPTV filters domain suffers from a lack of theoretical studies and still remains a marginal framework. Conversely, filter banks have received considerable attention because of large applications in digital communications. Previous papers attempted to relate filter banks and LPTV filters showing that a whole maximally decimated filter bank could be modeled by an LPTV filter. However, this elegant conclusion did not really provide interest for LPTV filters. In this paper, we have separately proposed two equivalence definitions between LPTV filters and analysis (and synthesis) banks. This equivalence was of interest because we could relate the Perfect Reconstruction property for a filter bank and the invertibility property for an LPTV filter. Invertible LPTV filter design methods are poor in litterature, and thanks to this proposition, we proposed a new class of invertible LPTV filters: LL LPTV filters. There are many applications, where invertible LPTV filters are required. We chose in this paper to show how we can make the most of the properties of these LL LPTV filters to achieve a Spread Spectrum system. Research of additional orthogonal properties are currently in progress to achieve a multiuser spread spectrum system. The use of LL LPTV filters to reduce Peak to Average Power Ratio reduction in an Orthogonal Frequency Division Multiplexing context is also under study.

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