ON CONVEX VARIABLE STEP-SIZE ALGORITHM IMPLEMENTATION

Corneliu Rusu[1,2] *and Colin F. N. Cowan*[3]

[1]Tampere International Center for Signal Processing, Tampere University of Technology
P.O.BOX 553, FIN-33101, Tampere, Finland
[2]Department of Electronics and Telecommunications, Technical University of Cluj-Napoca,
Str. Baritiu Nr. 26-28 RO-400027, Cluj-Napoca, Romania
[3]Department of Electrical and Electronic Engineering, The Queen's University of Belfast,
Ashby Building, Stranmillis Road, Belfast, BT7 3BY, U.K.

ABSTRACT

In this paper we derive a new design of the Convex Variable Step-Size (CVSS) algorithm, based on measurements obtained with LMS algorithm. Computer simulations are provided to support the proposed approach.

1. INTRODUCTION

In recent years there has been increasing interest in adaptive filters for various signal processing and communications applications. The most widely used adaptive algorithm is the Least-Mean-Square (LMS), originally proposed and developed for engineering applications by Widrow and Hoff [1]. Many researchers have studied the convergence and tracking performance of the LMS algorithm. It has been noticed that a compromise must be done between the convergence speed and the steady-state error. Much effort has been devoted to this issue, and to solve this problem many algorithms have been developed. A popular approach is based on using large step-sizes values when the algorithm is far from optimal solution, and small step-size values near the optimum. As a consequence of this interest there is available a large literature on variable step-size methods.

An overall weakness of the variable step-size algorithms is that the convexity of the cost function cannot be anymore guaranteed, except [2] where the convexity of cost function for such a variable step-size method has been proved. Consequently CVSS (Convex Variable Step-Size) has been recommended for acoustic echo cancelation [3] applications for better tracking capabilities and easier implementation on small and medium finite lengths DSPs. As a possible drawback, the CVSS algorithm can perform worse than some variable step-size algorithms, if the period when LMF (Least Mean Fourth) acts is very long. Also the design presented in [2] depends on the choice of one only parameter , used to differentiate the extra-period LMF acts from the difference between the adaptation periods of the two LMS. As in the case of other algorithms, the tuning of this parameter may be a sensitive issue in implementation. The major contribution of this paper is the direct formulas for tuning the step-size parameters. However, the improvement in convergence and error due to the new design seems not very significant.

The paper is organized as follows. First we recall VSS (Variable Step-Size) algorithm (Section 2.1) as standard variable step-size method [4]. Then we mention main properties of the CVSS algorithm (Section 2.2) to have consistent notations with VSS. Several common aspects of VSS and CVSS algorithms are enlighten, leading to a new design of CVSS algorithm (Section 3). Computer simulations are also provided in Section 4.

2. LMS, VSS AND CVSS ALGORITHMS

We consider an adaptive FIR filter, which is trying to make a copy $\hat{y}$ of the echo-path output y, using the signal x as an input, based upon a measurement of the signal that remains after subtracting $\hat{y}$ from the received signal $y+f$, where f is the far-end signal. We denote by $\hat{\mathbf{h}}$ the estimated filter coefficients and the vectors' length is N. The LMS algorithm results:

$$\hat{\mathbf{h}}(k+1) = \hat{\mathbf{h}}(k) + \quad (k)e(k)\mathbf{x}(k), \tag{1}$$

where here $(k) = $ is the fixed step-size of the LMS algorithm. Here we use LMS recursion without factor 2 in the right-hand side of (1) to be consistent with VSS recursion in [4].

Unlike LMS, in variable step-size methods we have three regions; in two of them LMS acts with different gradients given by step-sizes $_{\max}$ and $_{\min}$, and the middle one is where a certain step-size update formula is applied. This last region will differentiate any algorithm from other variable step-size methods.

2.1 VSS algorithm

The Kwong and Johnston recursion of their variable-step size (VSS) algorithm [4] is (1), with:

$$(k+1) = \begin{cases} \quad_{\max} & '(k+1) > \quad_{\max} \\ \quad_{\min} & '(k+1) < \quad_{\min} \\ '(k+1) & \text{otherwise} \end{cases} \tag{2}$$

where

$$'(k+1) = \quad '(k) + \ e^2(k). \tag{3}$$

The parameter is usually 0.97 and may be chosen in conjunction with to meet the misadjustment requirements. Also $_{\min}$ will be near the value of for the fixed step size LMS. The following approximate expression for the misadjustment, valid for small misadjustment in the stationary case has been provided in [4]:

$$M = \frac{1 - \left[1 - 2\,\frac{(3-\ )\ \ _{\min}}{1-\ ^2}\,\mathrm{tr}(\mathbf{R})\right]^{1/2}}{1 + \left[1 - 2\,\frac{(3-\ )\ \ _{\min}}{1-\ ^2}\,\mathrm{tr}(\mathbf{R})\right]^{1/2}} \tag{4}$$

where $\mathbf{R} = E[\mathbf{x}(k)\mathbf{x}^t(k)]$ and $_{\min}$ is the additive noise variance.

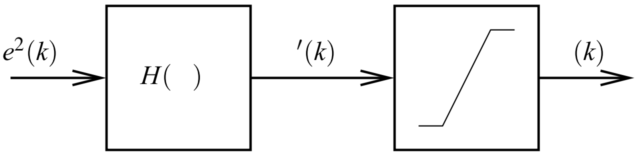

Figure 1: Modeling the step-size computation.

2.2 CVSS algorithm

The convex variable step-size algorithm is resulting from a combination of LMS and LMF (Least Mean Fourth) cost functions. As in other variable step-size methods we have three regions; in two of them LMS acts with different gradients, and the middle one is where the LMF is applied. This last region and the selection of the constants differentiate CVSS algorithm from VSS technique. To comply with (1) recursion, the CVSS update formula for step-size is the following:

$$(k) = \begin{cases} {}_{\max}, & \text{if } e^2(k) > \frac{\max}{}, \\ {}_{\min}, & \text{if } e^2(k) < \frac{\min}{}, \\ e^2(k), & \text{otherwise.} \end{cases}$$

It is easy to see that (k) is a continuous strictly increasing function, which gives a gradient of a convex cost function. It should also be noted that the design rests in only one parameter. Since the parameter of VSS algorithm is usually fixed to 0.97, we have the same situation as in the case of VSS, i.e. one parameter to be determined: for CVSS and for VSS. It is of interest to determine whether we can establish any relationship between these two parameters.

3. JOINT VSS AND CVSS IMPLEMENTATION

Actually what we are looking for is the instant when we can skip from LMS to LMF algorithm; alternatively, we would like to know the values of the error when LMS should act with $_{\max}$ and $_{\min}$, respectively. By substituting $_{\max}$, $_{\min}$ and in (3) we obtain:

$$\begin{aligned} {}_{\max} &= {}_{\max} + \frac{\max}{}; \\ {}_{\min} &= {}_{\min} + \frac{\min}{}, \end{aligned} \tag{5}$$

which both give:

$$= \frac{}{1-}. \tag{6}$$

On the other hand, one can see the step-size computation as it is modeled in Fig. 1, as cascade of linear filter and a threshold system. In both cases the threshold system is given by

$$(k) = \begin{cases} {}_{\max} & \text{if } '(k+1) > {}_{\max} \\ {}_{\min} & \text{if } '(k+1) < {}_{\min} \\ '(k+1) & \text{otherwise} \end{cases}$$

where the linear filters have the following input-output relationships:

- For VSS: $'(k+1) = \ '(k) + \ e^2(k)$;
- For CVSS: $'(k+1) = \ e^2(k)$.

Consequently their system functions are given by:

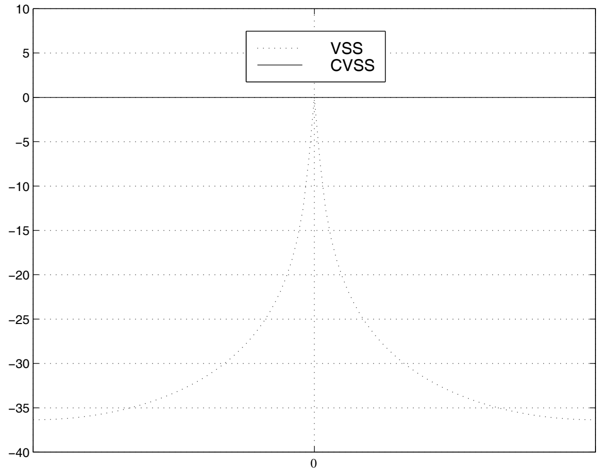

Figure 2: Log-magnitude responses (in dB) for the two linear filters H_{VSS} ($= 1-$) and H_{CVSS} ($= 1$).

- For VSS: $H_{VSS}(z) = \frac{}{z-}$;
- For CVSS: $H_{CVSS}(z) = \ /z$.

  It follows the frequency response magnitudes:
- For VSS: $|H_{VSS}(\ )| = \frac{}{\sqrt{1+\ ^2-2\ \cos}}$;
- For CVSS: $|H_{CVSS}(\ )| = \ .$

  It should be noted that condition (6) leads to

$$|H_{VSS}(0)| = |H_{CVSS}(0)|,$$

which assures that step-size expectation is identical for both algorithms when the error density spectrum is the same. However, their magnitude responses differ (Fig. 2). We can see that corresponding to VSS there is a linear filter H_{VSS} which characteristic is low-pass and narrow band. Consequently the averaging operation on step-size $'(k)$ will provide a smoothed step-size (k), and thus improving the instantaneous behavior of the stochastic gradient. For the CVSS this is not anymore needed, as the convexity of cost function is guaranteed. However, the VSS and CVSS should have different properties according to the spectrum of the signals involved.

We also note that

$$\int_{-} |H_{VSS}(\ )| d \ = \int_{-} \frac{d}{\sqrt{1+\ ^2-2\ \cos}} \tag{7}$$

is equal with 4 $\mathbf{K}(\ )$, where $\mathbf{K}(p)$ is the complete elliptic integral [5]:

$$\mathbf{K}(p) = \int_0^{\ /2} \frac{dx}{\sqrt{1-p^2\sin^2 x}}.$$

In addition, we provide a sample of complete integral values of $\mathbf{K}(p)$ in Table 1 [6]. Although has been selected since it was working well in simulations, we remark also that $\mathbf{K}(0.97)$ fits best of all.

Now let be a uniform $S_e(\ ) = \ ^2_e/(2\ )$ power spectral density of the error. Because the input of the linear filters

p	$\mathbf{K}(p)$	p	$\mathbf{K}$ (p)
1		0.95	2.90833
0.99	3.69563	0.94	2.82078
0.98	3.35414	0.93	2.74707
0.97	3.15587	0.92	2.68355
0.96	3.01611	0.91	2.62777

Table 1: Values of complete elliptic integral $\mathbf{K}(p)$.

$H(\ )$ is the square of the error, then the step-size variance is equal with $\ _e^2$ for CVSS and it can be approximated with $2\ \ _e^2$ for VSS, which is much smaller, whenever (6) is valid.

There is still one sensitive problem, when we want to implement CVSS: How to find parameter , without prior implementation of VSS? Taking into account that is usually very small, we can neglect the term inside the brackets of the denominator of (4). Moreover, we can use the approximation $\sqrt{1-x} \approx 1 - x/2$, for $x \ll 1$. Thus we can get

$$M \approx \frac{(3-\ )\ _{\min}}{1-\ ^2}\,\mathrm{tr}(\mathbf{R}) = \frac{3-}{1+}\ _{\min}\mathrm{tr}(\mathbf{R})\,\frac{}{1-}.$$

If (6) is valid and $\ = 0.97$ we get:

$$M \approx 1.03\ \ _{\min}\ \mathrm{tr}(\mathbf{R}). \qquad (8)$$

On the other hand, in the case of LMS algorithm, when the step-size parameter is small enough, the misadjustment varies linearly with [7]:

$$M \approx \frac{}{2}\mathrm{tr}(\mathbf{R}). \qquad (9)$$

By combining (8) and (9), we can easily compute parameter of CVSS algorithm based on measurements obtained with LMS algorithm:

$$_{cvss} \approx \frac{_{LMS}}{2.06\ \ _{min}}. \qquad (10)$$

Similarly, we get for VSS algorithm:

$$_{vss} \approx 0.0146\frac{_{LMS}}{_{min}}. \qquad (11)$$

4. SIMULATIONS

For the beginning we used the data echo cancelation framework to test our achievements. The near-end sequence is modeled by a non-Gaussian random bipolar sequence from the set $\{1, -1\}$. The far-end signal is generated by an independent random bipolar sequence from the set $\{a, -a\}$, where a is the attenuation of the far-end signal. The performance measure is the normalized form of the tap-error vector:

$$p(k) = \frac{\|\hat{\mathbf{h}}(k) - \mathbf{h}(k)\|}{\|\mathbf{h}(k)\|}.$$

Two types of echo path have been used. The first one consists of channels that are single pole single zero digital filters, with the impulse response series truncated [8]. The impulse response of the echo path is of the form $h(n) = 0.80025^j,\ j = 0, \ldots, 31$. The second type of echo path model is numerically generated as in [8], by sampling a diagram

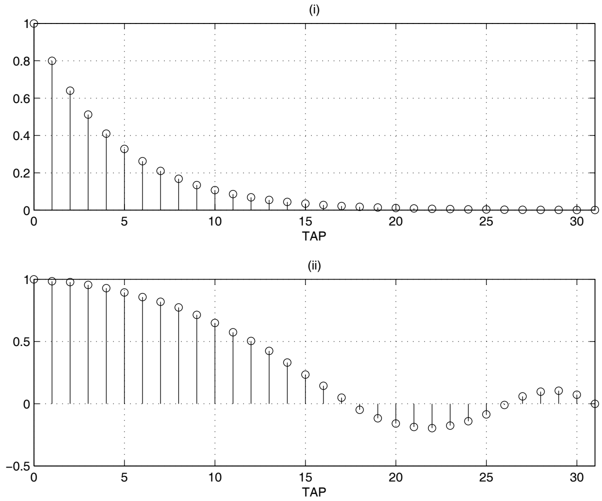

Figure 3: Impulse responses for the two channel models: (i) the single pole single zero digital filter; (ii) the real hybrid.

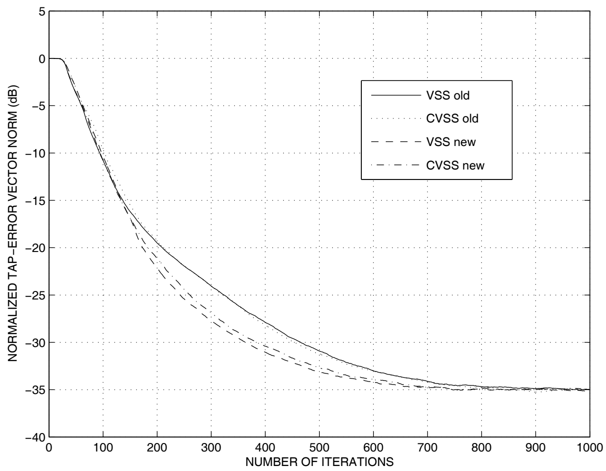

Figure 4: Learning curves for old and new design approaches of VSS and CVSS, when the echo-path is the real hybrid.

of an actual telephone network connection. The impulse responses for the first type channel and for the second echo-path (real hybrid) are shown in Fig. 3. First we reconsider example presented in [2], where for real hybrid two type of simulations have been shown. For $a_{dB} = -15$ with the new notations, the maximum and minimum step-sizes are equal with 0.02 and respectively 0.0055. Now we shall illustrate how CVSS and VSS algorithms work with the former and the new design approach. In the former approach $\ = 48 \cdot 10^{-5}$ as recommended in [4], and $\ = 0.022$. In the new approach $\ = 0.0055/[2.06 \cdot (10^{-3/4})^2] = 0.0844$ and $\ = 0.0146 \cdot 0.0055/(10^{-3/4})^2 = 0.0025$. The outcomes presented in Fig. 4 have been obtained after running the algorithms for 100 times and averaged. It is now clear that for stationary case, for this echo-path and for the signals involved, the new design approach gives better results.

The second set of simulations presents the behavior of the VSS and CVSS algorithms when an unknown systems

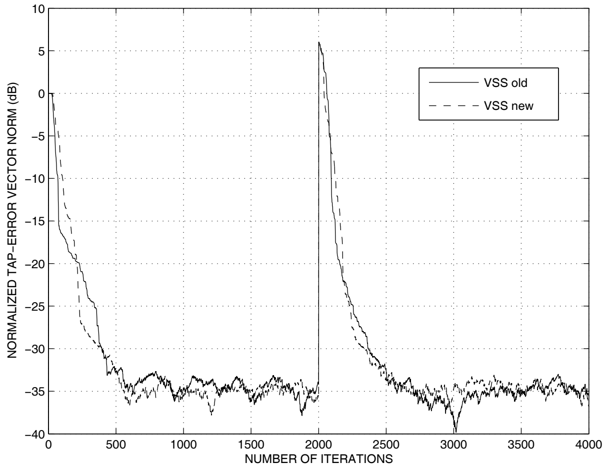

Figure 5: Performance of old and new design approaches of VSS algorithm for a sign change in coefficients.

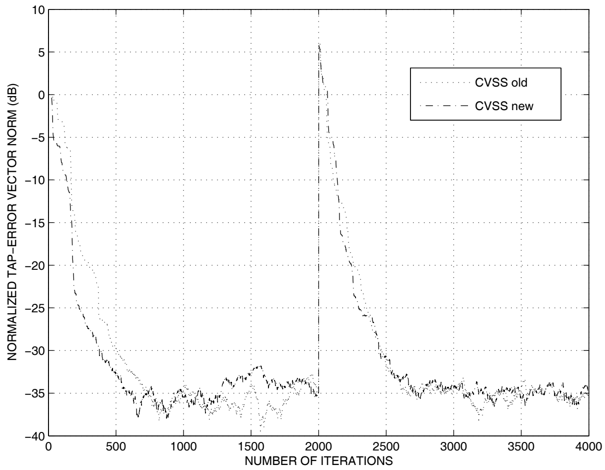

Figure 6: Performance of old and new design approaches of CVSS algorithm for a sign change in coefficients.

coefficients experience a sign change at iteration $k = 2000$. We can see that the new design approach slightly improves the properties of both VSS and CVSS algorithms.

The third set of simulations illustrates how the VSS and CVSS algorithms works with the first echo path filter. As in this case and have close value ($_{CVSS\ old} = 0.025$, $_{CVSS\ new} = 0.016$, $_{VSS\ old} = 7.61 \cdot 10^{-4}$, $_{VSS\ new} = 4.8 \cdot 10^{-4}$), the algorithms behave quite the same.

5. CONCLUSIONS

In this paper we have presented a new design approach for both VSS and CVSS algorithms. As result we have obtained straight relationships for tuning the parameters directly from LMS step-size and additive noise variance. It follows that one can run LMS and then can find the parameters for both VSS and CVSS. Though one may claim that (10) and (11) are rather rough approximations, our simulations show that they give promising results.

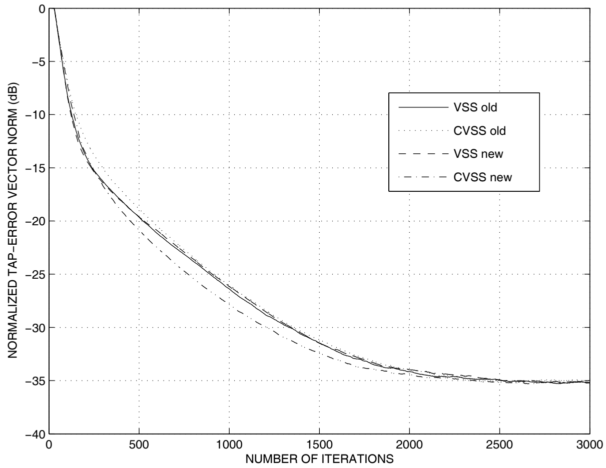

Figure 7: Learning curves for old and new design approaches of VSS and CVSS, when the echo-path is the single pole single zero digital filter.

We recall that in the case of CVSS algorithm the analysis does not arise major difficulties. Moreover, as we have just seen, the CVSS has a simple implementation. The amount of computation load is reduced or is identical in comparison with other variable step-size methods. We do not need steps to be taken to anticipate the step-sizes from exceeding their maximum and minimum limits; the same comparison for error seems effortless. Consequently CVSS is an attractive technique, especially for an easy implementation on small and medium finite lengths DSPs, where VSS (or other variable step-size methods) may have problems, due to the recursive computation of the step-size.

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