GENERALIZATION OF TIME-FREQUENCY SIGNAL REPRESENTATIONS TO JOINT FRACTIONAL FOURIER DOMAINS

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ABSTRACT

The 2-D signal representations of variables rather than time and frequency have been proposed based on either Hermitian or unitary operators. As an alternative to the theoretical derivations based on operators, we propose a joint fractional domain signal representation (JFSR) based on an intuitive understanding from a time-frequency distribution constructing a 2-D function which designates the joint time and frequency content of signals. The JFSR of a signal is so designed that its projections on to the defining joint fractional Fourier domains give the modulus square of the fractional Fourier transform of the signal at the corresponding orders. We derive properties of the JFSR including its relations to quadratic time-frequency representations and fractional Fourier transformations. We present a fast algorithm to compute radial slices of the JFSR.

1. INTRODUCTION

Time-frequency representations are frequently utilized to analyze and process non-stationary signals [1, 2]. One of the major areas of research in time-frequency signal processing is the design of novel time-frequency representations for different applications. Unfortunately, time-frequency distributions do not always convey desirable qualifications in every application. Hence the demand for powerful signal representations has led to a substantial amount of research on the design of 2-D signal representations defined by alternative variables other than time and frequency. Joint time-scale representations, which have attracted much interest especially in the fields of sonar and image processing, constitute one of the earliest examples of this type of representations. Other popular choices of joint variables include higher derivatives of the instantaneous phase of signals for radar and sonar problems [3, 4], dispersive time-shifts for wave propagation problems and analogs of quantum mechanical quantities such as spin, angular momentum, and radial momentum [5], and scale-hyperbolic time, warped time-frequency and warped time-scale.

These signal representations have been mathematically derived by using two alternative approaches which are both based on the operator theory: The variables are associated with either Hermitian operators as in [2] or unitary operators as in [5]. Recently, a joint fractional signal representation has been derived by associating Hermitian fractional operators to fractional Fourier transform (FrFT) variables constituting the joint distribution [6]. The fractional Fourier domains are the set of all domains interpolating between time and frequency. The fractional Fourier domains corresponding to $a = 0$ and $a = 1$ are the time and frequency domains, respectively. The FrFT with order a transforms a signal into the a^{th}-order fractional Fourier domain and the a^{th}-order FrFT of $x(t)$ is given by [7]

$$\begin{aligned} x_a(t) &\equiv \{\mathscr{F}^a x\}(t) \\ &= \int B_a(t,t')x(t')dt' \ , \ -2 < a < 2, \end{aligned} \tag{1}$$

where

$$B_a(t,t') = \frac{e^{-j(\quad \mathrm{sgn}(a)/4+ \quad /2)}}{|\sin \quad|^{1/2}}\, e^{j \quad (t^2 \cot \quad -2tt' \csc \quad +t'^2 \cot \quad)} \tag{2}$$

is the transformation kernel, $\quad = a\frac{}{2}$ and $\mathrm{sgn}(\cdot)$ is the sign function.

In this paper, we derive the joint fractional signal representation (JFSR) which designates the energy contents of signals in fractional Fourier domain variables instead of time and frequency. To this end, rather than using cumbersome mathematical equations based on operator theory, we extend our intuitive understanding from a time-frequency distribution, i.e., a function which designates joint time and frequency contents of signals. Then, we derive some important properties of the JFSR including its relation to quadratic time-frequency representations and fractional Fourier transformations, and present a simple formula for its oblique projections. We also present a fast algorithm to compute radial slices of the JFSR and numerically computed JFSRs of some synthetic signals.

The outline of the paper is as follows. In Section 2, a concise derivation of JFSR is presented as an alternative to the derivation given in [6]. Then, in Section 3, properties of the JFSR are examined. After presenting a fast computation algorithm in Section 4, numerically computed JFSRs of some synthetic signals are shown in Section 5. Finally, conclusions are drawn in Section 6.

2. DISTRIBUTION OF SIGNAL ENERGY ON JOINT FRACTIONAL FOURIER DOMAINS

One of the primary expectations from a time-frequency distribution $D_x(t, f)$ associated with a signal $x(t)$ is that it accurately represents the energy distribution of $x(t)$. It is desired that the signal energy at a frequency f for a time instant t is given by $D_x(t, f)$. Because of the uncertainty relationship between time and frequency domains, it is impossible to satisfy this *point-wise* energy density requirement. Therefore, one is to be usually satisfied with a looser condition on the marginal densities

$$\int D(t,f)df \quad = \quad |x(t)|^2 \tag{3}$$

$$\int D(t,f)dt \quad = \quad |X(f)|^2 \ , \tag{4}$$

and integration of the distribution on the whole time-frequency plane

$$\int\int D(t,f)dt\ df = ||x||^2. \tag{5}$$

where $X(f)$ is the Fourier transform of $x(t)$, and $||\cdot||$ denotes the L_2 norm. One of the prominent *energetic* distributions which satisfy the desired relations (4) and (5) is the Wigner distribution (WD) which is defined as [2]

$$W_x(t,f) = \int x(t+\ /2)x^*(t-\ /2)e^{-j2\ \ f}\ d\ \ . \tag{6}$$

The WD of a signal can be roughly interpreted as an energy density of the signal, since it is real, covariant to time and frequency domain translations and moreover signal energy in any extended time-frequency region can be determined by integrating $W_x(t,f)$ over that region [2]. Another nice property of the WD is that, its oblique projections give the energy distribution with respect to the corresponding fractional Fourier domain [8]. Such properties and its ability to provide high time-frequency domain signal concentration make the WD attractive compared to other representations.

Although the WD and its enhanced versions are useful in time-frequency analysis, in some applications such as signal design and synthesis, it is more useful to have a time-fractional Fourier domain representation. In a mathematical framework, by associating Hermitian operators to individual FrFT domains, a joint fractional representation of signals has been derived in [6].

In this section, the joint fractional domain signal representation (JFSR) is constructed using conditions similar to the ones given in (3)-(5). Let the JFSR of a signal be denoted by $E_x^{\mathbf{a}}(u,v)$ where $\mathbf{a} = (a_1,a_2)$ denotes the orders of fractional Fourier domains u and v, respectively. It is desired that the marginal densities satisfy

$$\int E_x^{\mathbf{a}}(u,v)dv = |x_{a_1}(u)|^2 \tag{7}$$

and

$$\int E_x^{\mathbf{a}}(u,v)du = |x_{a_2}(v)|^2 \tag{8}$$

where $|x_{a_1}(u)|^2$ and $|x_{a_2}(v)|^2$ are the energy contents of the signal at the a_1^{th} and a_2^{th} fractional Fourier domains, respectively. Similar to (5), the overall integral on the $u-v$ plane is desired to be

$$\int\int E_x^{\mathbf{a}}(u,v)du\ dv = ||x||^2. \tag{9}$$

The conditions stated in (7), (8), and (9) make the JFSR a generalization of the WD, since the JFSR reduces to the WD when $(a_1,a_2) = (0,1)$.

To construct the distribution $E_x^{\mathbf{a}}(u,v)$ satisfying the conditions on the marginal densities and the total energy, we make use of the projection property of the WD [8]

$$\int W_x(u\cos\ \ - v\sin\ \ , u\sin\ \ + v\cos\ \ )dv = |x_a(u)|^2\ . \tag{10}$$

In Fig. 1, we observe that the value of the time-frequency distribution at a point P contributes to the energy densities of the fractional Fourier domains u and v at points $u = u_0$ and $v = v_0$, respectively. Therefore, the JFSR can simply be formed by redistributing the WD so that

$$E_x^{\mathbf{a}}(u,v) = C\cdot W_x(P(t(u,v),f(u,v)))\ , \tag{11}$$

where the coordinates (u,v) and (t,f) are related by

$$\begin{bmatrix} \cos\ _1 & \sin\ _1 \\ \cos\ _2 & \sin\ _2 \end{bmatrix} \begin{bmatrix} t \\ f \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \tag{12}$$

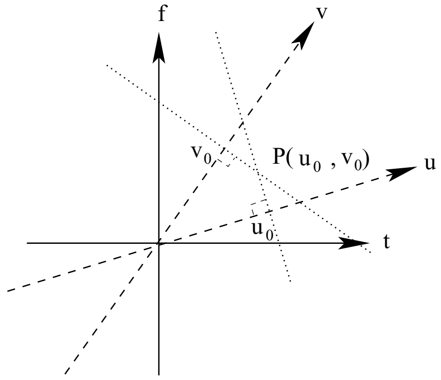

Figure 1: The value of the time-frequency distribution at a point P contributes to the energy densities of the fractional Fourier domains u and v at points $u = u_0$ and $v = v_0$, respectively. The JFSR can simply be formed through redistributing the WD so that, $E_x^{\mathbf{a}}(u,v) = C\cdot W_x(P(t(u,v),f(u,v)))$.

and $_i = a_i\ \ /2$ for $i = 1,2$. By using the total energy constraint (9), the constant C in (11) is determined as

$$C = |\csc(\ _{12})| \tag{13}$$

where $_{12} = \ _2 - \ _1$. Thus, $E_x^{\mathbf{a}}(u,v)$ is explicitly given by

$$|\csc(\ _{12})| \int x(\frac{u\sin\ _2 - v\sin\ _1}{\sin\ _{12}} + \ /2)x^*(\frac{u\sin\ _2 - v\sin\ _1}{\sin\ _{12}} - \ /2) \times e^{-j2\ \ \frac{-u\cos\ _2 + v\cos\ _1}{\sin\ _{12}}}\ d\ \ .$$

An equivalent and more compact form of $E_x^{\mathbf{a}}(u,v)$ can be obtained by using the rotation effect of the FrFT on the WD

$$W_{x_{a_1}}(u,v) = W_x(u\cos\ _{12} - v\sin\ _{12}, u\sin\ _{12} + v\cos\ _{12})\ , \tag{14}$$

which verbally translates that WD of the a_1^{th} order FrFT of a signal $x(t)$ is the same as the WD of the signal $x(t)$ which is rotated by $_1$ radians in the clockwise direction in the time–frequency plane. Consequently, $E_x^{\mathbf{a}}(u,v)$ can be derived in terms of the fractionally Fourier transformed signal $x_{a_1}(t)$ as

$$\begin{aligned} E_x^{\mathbf{a}}(u,v) \quad &= \quad \int x_{a_1}\left(u + \frac{\sin\ _{12}}{2}\right) x_{a_1}^*\left(u - \frac{\sin\ _{12}}{2}\right) \\ &\times \quad e^{-j2\ \ (v - u\cos\ _{12})}\ d \end{aligned} \tag{15}$$

which has the same form as given in [6].

The JFSR can be generalized to define a cross-JFSR distribution of signals $x(t)$ and $y(t)$

$$\begin{aligned} E_{xy}^{\mathbf{a}}(u,v) \quad &= \quad |\csc(\ _{12})|\cdot W_{xy}(P(t(u,v),f(u,v))) \qquad (16) \\ &= \quad \int x_{a_1}\left(u + \frac{\sin\ _{12}}{2}\right) y_{a_1}^*\left(u - \frac{\sin\ _{12}}{2}\right) \\ &\times \quad e^{-j2\ \ (v - u\cos\ _{12})}\ d\ \ . \end{aligned}$$

Defining $E_x^{\mathbf{a}}(u,v)$ through its relation to the WD provides an easily interpretable definition of the JFSR of signals. From the definition given in (15), it follows that JFSR is a quadratic distribution while it is not a time-frequency distribution. Therefore it belongs to a broader class than the familiar Cohen's class. Thus, its introduction to the non-stationary signal processing will bring new insights into the design, filtering, analysis, and synthesis of signals in many applications.

3. PROPERTIES OF THE JFSR

In this section, we investigate the properties of the JFSR. In the properties listed below, the joint fractional Fourier domains have the orders of $\mathbf{a} = (a_1, a_2)$ making angles of $(\ _1, \ _2)$ with respect to the time axis where $\ _i = a_i \frac{}{2}$.

Property 1. The JFSR is a real distribution

$$E_x^{\mathbf{a}}(u,v) = (E_x^{\mathbf{a}})^*(u,v) \ . \tag{17}$$

Property 2. Orthogonal projection of the JFSR of a signal $x(t)$ on to u and v axes give the magnitude square of the FrFTs of the signal at orders associated with these axes: that is,

$$\int E_x^{\mathbf{a}}(u,v)dv = |x_{a_1}(u)|^2 \tag{18}$$

and

$$\int E_x^{\mathbf{a}}(u,v)du = |x_{a_2}(v)|^2 \ . \tag{19}$$

Property 3. The area under the JFSR of a signal $x(t)$ gives the total signal energy

$$\int\int E_x^{\mathbf{a}}(u,v)dudv = \int |x(t)|^2 dt \ , \tag{20}$$

which follows from Property 2 and the unitarity of the FrFT.

Property 4 The JFSR and WD of a signal $x(t)$ are related as

$$\begin{aligned} E_x^{\mathbf{a}}(u,v) &= |\csc \ _{12}| W_{x_{a_1}}\left(u, \frac{v - u\cos \ _{12}}{\sin \ _{12}}\right) \\ &= |\csc \ _{12}| W_x\left(\frac{u\sin \ _2 - v\sin \ _1}{\sin \ _{12}}, \frac{-u\cos \ _2 + v\cos \ _1}{\sin \ _{12}}\right). \end{aligned}$$

Property 5 The JFSR and the FrFT of a signal $x(t)$ are related as

$$E_{x_{a'}}^{\mathbf{a}}(u,v) = E_x^{\mathbf{a}+a'}(u,v) \tag{21}$$

where $\mathbf{a} + a' = (a_1 + a', a_2 + a')$.

Proof: By using (21) in Property 4, the JFSR of $x_a(t)$ can be written as

$$\begin{aligned} E_{x_{a'}}^{\mathbf{a}}(u,v) &= |\csc \ _{12}| \\ &\times \ W_{x_{a'}}\left(\frac{u\sin \ _2 - v\sin \ _1}{\sin \ _{12}}, \frac{-u\cos \ _2 + v\cos \ _1}{\sin \ _{12}}\right). \end{aligned}$$

Then, by using the rotation property of the WD given in (14), right hand side of this expression is simplified to

$$\begin{aligned} E_{x_{a'}}^{\mathbf{a}}(u,v) &= |\csc \ _{12}| W_x(u', v') \\ u' &= \frac{u\sin(\ ' + \ _2) - v\sin(\ ' + \ _1)}{\sin \ _{12}}, \\ v' &= \frac{-u\cos(\ ' + \ _2) + v\cos(\ ' + \ _1)}{\sin \ _{12}} \end{aligned}$$

which proves the property where $\ ' = a' \frac{}{2}$.

Property 6 Any oblique projection of JFSR of a signal $x(t)$ onto an oblique axis making an angle of is

$$P \ [E_x^{\mathbf{a}}](r) = |x_{(a')}(r/M)|^2 \quad , \quad a' = \frac{2 \ '}{} \ , \tag{22}$$

where

$$\ ' = \text{arctan2}(\cos \ _1 + \cos \ + \cos \ _2 \sin \ , \sin \ _1 \cos \ + \sin \ _2 \sin \ ) \tag{23}$$

and

$$M = \sqrt{1 + \sin 2 \ \cos \ _{12}} \ . \tag{24}$$

Proof: By the projection-slice theorem, the oblique projection of the JFSR of $x(t)$ at an angle is given by

$$P \ [E_x^{\mathbf{a}}](r) = \int F_x(\ \cos \ , \ \sin \ )e^{-j2 \ \ r} d \ , \tag{25}$$

where

$$F_x^{\mathbf{a}}(\ , \ ) = \int\int E_x^{\mathbf{a}}(u,v)e^{j2 \ (\ u + \ v)} dudv \tag{26}$$

is the radial slice of the 2–D inverse Fourier transform of the JFSR of $x(t)$. By using (21), the following expression can be obtained for $F_x^{(a)}(\ , \ )$ in terms of the ambiguity function $A_x(.)$ of $x(t)$

$$F_x^{\mathbf{a}}(\ , \ ) = A_x(\ \cos \ _1 + \ \cos \ _2, \ \sin \ _1 + \ \sin \ _2) \ . \tag{27}$$

Thus, the radial slice $F_x^{\mathbf{a}}(\ \cos \ , \ \sin \ )$ of $F_x^{\mathbf{a}}(\ , \ )$ can be expressed as

$$F_x^{\mathbf{a}}(\ \cos \ , \ \sin \ ) = A_x(\ M\cos \ ', \ M\sin \ ') \ , \tag{28}$$

where $\ '$ and M are as given in (23) and (24), respectively. It is known that the radial slice of the ambiguity function of a signal $x(t)$ at the angle has the following relation to the $(a')^{th}$ FrFT of the signal $x(t)$

$$A_x(\ \cos \ ', \ \sin \ ') = \int |x_{(a')}(r)|^2 e^{j2 \ \ r} dr \ . \tag{29}$$

Then, the relation in (22) can be obtained by combining (25), (28), and (29).

4. FAST COMPUTATION OF THE JFSR

In this section, we provide an efficient computation algorithm of the JFSR of a signal on arbitrary radial slices. Throughout the computations, we assume that the signal $x(t)$ is scaled to $x(t/s)$ before sampling, so that its WD is approximately confined into a circle of radius $\ _x/2$. Here, if the time-width and bandwidth of the signal is approximately $\ _t$ and $\ _f$, respectively, then the scaling parameter s becomes $s = \sqrt{\ _f / \ _t}$ providing a signal which has negligible energy outside the interval $[- \ _x/2, \ _x/2]$.

To compute the radial slice of the JFSR of a signal $x(t)$, we use the relation

$$E_{x_a}^{\mathbf{a}}(r\cos \ , r\sin \ ) = |\csc \ _{12}| W_x(r\cos \ ', r\sin \ ') \tag{30}$$

where

$$\ ' = \arctan(\cos \ \cos \ _2 + \sin \ \cos \ _1, \cos \ \sin \ _2 - \sin \ \cos \ _1) \tag{31}$$

and $\ _1$ and $\ _2$ are the corresponding angles of the fractional Fourier domains u and v with respect to the time axis. It has been shown in [9] that the radial slice of the WD along the line $(r\cos \ ', r\sin \ ')$ is

$$W_x(r\cos \ ', r\sin \ ') = \int x_{(a'-1)}\left(\frac{-}{2}\right) x^*_{(a'-1)}\left(\frac{}{2}\right) e^{-j2 \ r \ } d \ . \tag{32}$$

Therefore, (30) and (32) can be used to construct the radial slice of $E_{x_a}^{\mathbf{a}}(u,v)$ as

$$\begin{aligned} E_{x_a}^{\mathbf{a}}(r\cos \ , r\sin \ ) &= |\csc \ _{12}| \int x_{(a'-1)}\left(\frac{}{2}\right) x^*_{(a'-1)}\left(\frac{-}{2}\right) \\ &\times \ e^{-j2 \ r \ } d \ . \end{aligned} \tag{33}$$

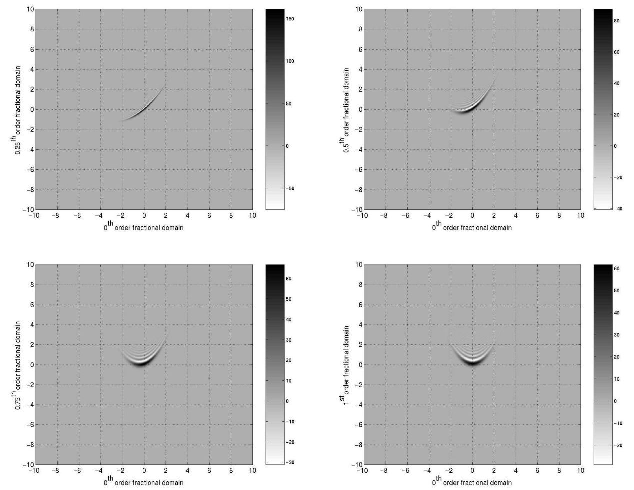

Figure 2: The JFSR of $x(t) = e^{-\ ((t/3)^2+0.3jt^3)}$ at joint fractional Fourier domains with order (a) $(a_1,a_2) = (0,0.25)$, (b) $(a_1,a_2) = (0,0.5)$, (c) $(a_1,a_2) = (0,0.75)$, and (d) $(a_1,a_2) = (0,1)$. The distribution given in (d) is the same as the WD of $x(t)$ since $a_1 = 0$, $a_2 = 1$.

If the double-sided bandwidth of $x_{(a'-1)}$ is $\ _x$ and the time-bandwidth product is N, then the integral in (33) can be discretized by

$$E^{\mathbf{a}}_{x_a}(r\ _x\cos\ ,r\ _x\sin\ ) = \frac{|\csc\ _{12}|}{\ _x} \ _{k=-N}^{N-1} q[k]e^{-\frac{j2\ rk}{\ _x}} \tag{34}$$

where $q[k] = x_{(a'-1)}[k]x^*_{(a'-1)}[-k]$ and $x_{(a'-1)}[k] = x_{(a'-1)}(k/(2\ _x))$ is computed using the algorithm given in [10] with $O(N\log N)$ computational complexity.

As the relationship (33) depends on the FrFTs of the signal $x(t)$, computation of any M uniformly spaced samples on the line segment $r \in [r_i, r_f]$ along the radial slice of $E^{\mathbf{a}}_{x_a}(r\ _x\cos\ ,r\ _x\sin\ )$ can be performed through the chirp-z transform algorithm in $O((N+M)\log(N+M))$ computational complexity [11]. In the following section, the results of the algorithm are presented for a synthetic signal on various joint fractional Fourier planes.

5. SIMULATIONS

In this section, the JFSR of a quadratic frequency modulated (FM) signal which has a non-convex time-frequency support on the time-frequency plane are evaluated for four different joint fractional Fourier order pairs of $\mathbf{a} = (0,0.25),\ (0,0.5),\ (0,0.75)$, and $(0,1)$.

The JFSRs of a quadratic FM signal $x(t) = e^{-\ ((t/3)^2+0.3jt^3)}$ which has a non-convex time-frequency support on the time-frequency plane are presented in Fig. 2. The distribution given in (d) with fractional Fourier order pair $(0,1)$ is the same as the WD of $x(t)$. It is easier to observe the localization of the signal component that depends on the order of the joint fractional Fourier domains. Because the uncertainty relation of the fractional Fourier domains has a tighter lower-bound when compared to the time and frequency domains [7].

6. CONCLUSIONS

A joint fractional domain signal representation is developed using the energy density interpretation of the WD on the time-frequency plane. The (u,v) axes defining the joint representation are chosen as the $\mathbf{a} = (a_1,a_2)$-th order fractional Fourier domains. The distribution is designed so that its projection of the JFSR on to the u and v axes gives the modulus square of the fractional Fourier transform of signals at the corresponding orders a_1 and a_2 as $|x_{a_1}(t)|^2$ and $|x_{a_2}(t)|^2$, respectively. It is shown that the distribution $E^{\mathbf{a}}_x(u,v)$ depends on the WD through a coordinate transformation. Therefore, the JFSR is a real-valued distribution, too. The overall integral of the JFSR on the (u,v) plane gives the total energy of the signal. In this paper, as part of the novel results, oblique projections of the JFSR is also derived and a fast computation algorithm designed for the computation of arbitrary radial slices of the WD in [9] is extended to the computation of the JFSR. The JFSRs of various signals at various fractionally-ordered domains are presented and the localization of the signal components are compared.

The JFSR cannot be analyzed in the framework of the familiar Cohen's class. However, its introduction to the non-stationary signal processing will bring new insights into the design, filtering, analysis, and synthesis of signals.

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