# Adaptive Resolution by Matching Pursuit Algorithms

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## Abstract

*The family of matching pursuit algorithms are greedy heuristic methods for basis selection problem. In this paper, we propose an adaptive structure for matching pursuit algorithms that uses a progressive refinement structure. The proposed adaptive algorithm allows the detector to adjust the tradeoff between the computational complexity and the resolution performance. The novel algorithm is applied to direction of arrival detection problem and it is shown that the algorithm results in a lower computational complexity than the direct implementation of matching pursuit algorithms.*

## 1. Introduction

The matching pursuit (MP) algorithms are suitable for selection of basis for signal decomposition by determining a small, possibly the smallest, subset of vectors chosen from a large redundant set of vectors to match the given data. This problem has various applications such as time/frequency representations [1], speech coding [2], and spectral estimation [3]. There are several types of MP algorithms. These include basic matching pursuit (BMP) [1], orthogonal matching pursuit (OMP) [4] and several other derivatives.

The main advantage of the MP algorithms is the decreased computational complexity. However as will be shown later, there is still a tradeoff between complexity and resolution.

In this paper we propose a novel adaptive structure for MP algorithms in order to decrease the overall complexity and adjust the resolution property of the algorithm. The proposed structure can be applied to several types of applications. In this paper, we consider the direction of arrival (DOA) estimation problem, the idea corresponds to integration of an adaptive structure to MP algorithms in order to provide a tradeoff between computational complexity and the precision of the detected directions. This high precision is achieved by a progressive refinement approach. Furthermore, it will be shown by an example that the computational complexity of the adaptive structures are much lower than that of the direct implementation. In terms of performance, the simulation results show that the direct implementation and adaptive implementation have quite close detection performances with a complexity reduction of more than 50%. It should also be noted that, the adaptive structure for the basis selection algorithms is a general framework and can be applied to several other detection problems.

This paper is organized as follows. The problem statement for DOA detection problem is given in Section 2. In Section 3, the necessary background on the BMP and OMP algorithms is summarized. The adaptive resolution structure for the MP algorithms are introduced in Section 4. Simulation results are presented in section 5. Finally conclusions are given in Section 6.

## 2. Problem Statement

In our system model for DOA estimation, we consider an adaptive antenna array of $N$ elements. The input signal is assumed to be a plane wave. We consider the narrow-band, far-field estimation problem, and hence we assume that the information sources are point sources, and the incoming waves are plane waves. The amplitude of the response of the $l^{th}$ sensor to the $i^{th}$ source is represented by $b_{i,l}(t)$. The output at the $l^{th}$ sensor can then be written as [5]

$$x_l(t) = \sum_{i=1}^{M} b_{i,l}(t) e^{jw_0\tau(\theta_i)} + n(t), \quad l = 1,2,\ldots,N, \tag{1}$$

where $j$ is the complex exponent, $M$ is the number of distinct point sources, $n(t)$ is the AWGN component with mean 0 and variance $\sigma^2$. The center frequency is $w_0$, and $\tau(\theta_i)$ is the time delay between the reference sensor (first sensor) and the $l^{th}$ sensor. The DOA of $i^{th}$ source is denoted by $\theta_i$, where $0^o \leq \theta_i \leq 180^o$.

For a uniform linear array, we can construct the corresponding dictionary $\mathscr{D}$ as

$$\mathscr{D} = \left\{ \begin{array}{cccc} 1 & 1 & \cdots & 1 \\ e^{j\psi_1} & e^{j\psi_2} & \cdots & e^{j\psi_P} \\ \vdots & \vdots & \vdots & \vdots \\ e^{j(N-1)\psi_1} & e^{j(N-1)\psi_2} & \cdots & e^{j(N-1)\psi_P} \end{array} \right\},$$

where $\psi_i$ is the phase difference between elements of the antenna array when the signal arrives from direction $\theta_i$, and hence $\psi_i = \frac{2\pi\mu}{\lambda}\cos(\theta_i)$, where $\mu$ is the separation between antenna elements [5].

As explained above, there is a particular phase shift pattern for distinct DOAs in the antenna array output. In practice, only a few directions are received with substantial amplitudes. In the DOA estimation problem, we exploit two facts: the bijective relation between phases and DOAs, and the sparsity of the DOAs. For the case in (2), the possible range of DOAs is divided into $M$ parts forming the dictionary $\mathscr{D}$. Representing the $i^{th}$ column of $\mathscr{D}$, by $\mathbf{d}_i$, the vector matrix model for this system can be simplified to

$$\mathbf{x} = \sum_{i=1}^{r} c_i \mathbf{d}_{k_i} + \mathbf{n}, \quad k_i \in \{1, 2, \ldots, M\}, \qquad (2)$$

where $c_i$ is the received signal amplitude from arriving direction $\theta_i$, and $\mathbf{n}$ is the AWGN vector. Assuming the noise term is negligible and we have $\mathbf{x} \approx \sum_{i=1}^{r} c_i \mathbf{d}_{k_i}$. The MP algorithms that are briefly reviewed in the following section can be applied directly to the received vector $\mathbf{x}$ in order to estimate the $\mathbf{d}_{k_i}$'s and hence the DOAs due to the bijective relation.

## 3. Matching Pursuit Algorithms

Matching pursuit (MP) algorithms are adaptive approximations that select the approximation vector with no orthogonality constraint. These greedy algorithms are sufficiently good to build compact representations for signals such as speech, music or image data [6]. The greedy method is used when the set of approximating functions is a linear combination of the basis functions. Initially only the first term is optimized. Optimization corresponds to minimizing the discrepancy between the input signal that can be observed as the training data and the current model. This term is kept fixed and then the next term is optimized. This process continues until all $M$ terms are evaluated. This approach is termed as greedy since at any point only a single term is added to the model in order to get a closer approximation. In the neural network literature greedy algorithms are known as network growing algorithms or constructive procedures [7]. Greedy algorithms are frequently used in many statistical methods. Implementation leads to very fast learning methods however quality of optimization can be suboptimal.

For MP algorithms, since the problem is pursuing the goal of determining a small subset of vectors in the dictionary $\mathscr{D}$, that best match the vector $\mathbf{x}$, the algorithms proposed for solution are termed as matching pursuit algorithms. These algorithms are adaptive due to the fact that the basis functions are selected adaptively for best matching a given data from a fixed dictionary.

### 3.1 Basic Matching Pursuit Algorithm

The basic matching pursuit (BMP) algorithm is proposed in [1]. The algorithm is closely related to projection pursuit algorithm [8] that is used frequently by statisticians, and the shape-gain vector quantizer [9].

Consider the space generated by signals of size $N$. Let $\mathscr{D} = \{\varphi_\lambda\}_{\lambda\in\Gamma}$ be a set of redundant vectors with $P$ number of vectors where $P > N$ and with $N$ linearly independent vectors that define $\mathbb{C}^N$ of signals length $N$. We can also assume that $\|\varphi_\lambda\| = 1$ without loss of generality $\forall\lambda$. In MP algorithms, basis selection is performed sequentially, i.e. one at a time. This forces an index requirement for the residual and selected vectors at each iteration.

This iterative procedure can implemented with a fast algorithm by exploiting the recursive structure as follows.

1. **Initialization** Set $m = 0$, $\mathbf{e}_0 = \mathbf{x}$ and compute $\langle \mathbf{e}_0, \varphi_\lambda \rangle_{\lambda\in\Gamma}$.
2. **Best Match** Find $\varphi_{\lambda_m} \in \mathscr{D}$ such that
$$|\langle \mathbf{e}_m, \varphi_{\lambda_m} \rangle| \geq \sup_{\lambda\in\Gamma} |\langle \mathbf{e}_m, \varphi_\lambda \rangle|. \qquad (3)$$
3. **Update** Update the new residue vector as
$$\mathbf{e}_{m+1} = \mathbf{e}_m - \langle \mathbf{e}_m, \varphi_{\lambda_m} \rangle \varphi_{\lambda_m}. \qquad (4)$$
4. **Terminate** If
$$\|\mathbf{x} - \mathbf{e}_{m+1}\| < \varepsilon \qquad (5)$$
or $m = M$ stop. Otherwise $m = m + 1$, and go to step 2.

This approximation is suboptimal due to the fact that it is not performed with all dictionary components but one at a time. If the loop of the BMP algorithm is executed $M$ times, the computational cost of the algorithm is at most $O(PMN)$ [10].

It should also be noted that the output of MP algorithms are not only the coefficient set $\{c_{\lambda_0}, c_{\lambda_1}, \ldots, c_{\lambda_{M-1}}\}$ but also the index set $\{\lambda_0, \lambda_1, \ldots \lambda_{M-1}\}$ which indicate the selected atoms from the dictionary.

### 3.2 Orthogonal Matching Pursuit Algorithm

The orthogonal matching pursuit (OMP) algorithm is proposed in [4] and in [11] independently. OMP is also termed as modified matching pursuit algorithm in the literature [12]. Similar to BMP, the aim of OMP is to obtain an approximate of the input signal $\mathbf{x}$, by sequentially selecting vectors from the dictionary. However, OMP algorithm gives a better approximation performance by orthogonalizing the directions of the projection. This guarantees the convergence of OMP with a finite number of iterations. In BMP the convergence was guaranteed with infinite number of iterations. However, the computational cost of the OMP algorithm is increased due to the employed Gram-Schmidt orthogonalization procedure.

The indices of the $m$ atoms are selected and stored in the index vector $\Lambda_m = [\lambda_0, \lambda_1, \ldots, \lambda_{m-1}]$ with $\Lambda_0 = [\,]$. Similar to the BMP case, for OMP we can assume that $\|\varphi_\lambda\| = 1$ without loss of generality $\forall\lambda$. The OMP algorithm selects the next atom $\varphi_{\lambda_m}$ by finding the vector best aligned with the residual obtained by projecting $\mathbf{e}_m$ onto the dictionary components, that is

$$\lambda_m = \arg\max_{\lambda\in\Gamma} |\langle \varphi_\lambda, \mathbf{e}_m \rangle|, \quad l \notin \Lambda_m. \qquad (6)$$

Then the selected vector component $\varphi_{\lambda_m}$ is orthogonalized by the Gram-Schmidt algorithm as

$$u_m = \varphi_{\lambda_m} - \sum_{l=0}^{m-1} \frac{\langle \varphi_{\lambda_m}, u_l \rangle}{\|u_l\|^2} u_l, \tag{7}$$

with $u_0 = \varphi_{\lambda_0}$.The residue vector $\mathbf{e}_m$ is updated as

$$\mathbf{e}_{m+1} = \mathbf{e}_m - \frac{\langle \mathbf{e}_m, u_m \rangle}{\|u_m\|^2} u_m. \tag{8}$$

The coefficient set $\{c_{\lambda_0}, c_{\lambda_1}, \ldots, c_{\lambda_{m-1}}\}$ changes with each iteration and can be evaluated by taking the orthogonal projection of $\mathbf{x}$ onto the selected atoms. Similar to the BMP algorithm, OMP terminates when either $m = M$, or $\|\mathbf{e}_m\| \leq \varepsilon$. The computational cost of the algorithm is at most $O(NM(M+P))$ [10].

## 4. Adaptive Resolution Structure for Matching Pursuit Algorithms

The precision of the directions that can be detected through an MP algorithm of choice is determined by the number of distinct columns of the dictionary (i.e. $P$). Hence in order to get very precise direction estimations, the value of $P$ must be as high as possible. However such a selection will increase the computational complexity of both BMP and OMP algorithms.

For the DOA estimation problem the idea corresponds to integration of an adaptive structure to MP algorithms in order to provide a tradeoff between computational complexity and the precision of the detected directions. This high precision is achieved by a progressive refinement approach.

In order to make the DOA detection algorithm compatible with low power implementation applications, the computational complexity can be decreased through a progressive refinement technique. Here, a novel adaptive structure for MP algorithms is proposed in order to decrease the overall complexity and adjust the resolution property of the algorithm. The idea is to keep $P = N$ for the first implementation of an MP algorithm of choice and roughly detect regions of DOA. An example of 4 symmetric regions is shown in Fig. 1. This partition can be implemented with uniform or non-uniform intervals depending on the channel characteristics. Once the region is detected for one or more DOAs then the MP algorithm of choice can be applied once more in order to increase the precision of the DOAs. The process can be repeated for a few iterations until the desired level of precision is achieved.

The process can be clarified with a numerical example. Consider an DOA estimation problem with $N = 10$ element uniform linear array as shown in Fig. 2. Assume that there are $M = 3$ stationary point sources. In order to detect within $1.8^o$ the dictionary length is $P = 100$. For detecting the DOAs, direct application the BMP and the OMP algorithms corresponds to a computational complexity of $O(3000)$ and $O(3090)$, respectively.

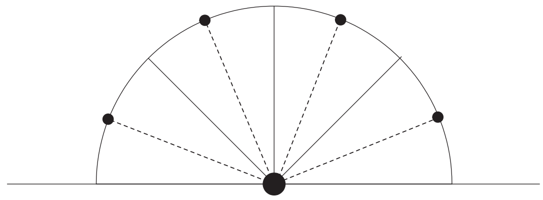

Figure 1: Division of the direction plane by 4 symmetric regions.

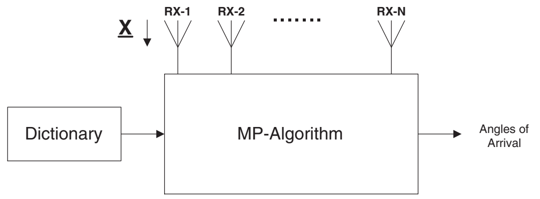


Figure 2: MP detection of direction of arrivals.

Now let us consider a two stage progressive refinement model for adaptive resolution MP structure. At the first stage the input dictionary $\mathscr{D}_{1,1}$ is defined according to $N = 10$ and $P = 10$. Hence BMP or OMP algorithm will have estimates within $18^o$ precision. This process requires $O(300)$ and $O(390)$ operations for BMP and OMP algorithms, respectively. In order to detect the DOAs more precisely finer dictionaries are defined for each interval composed of $1.8^o$ precision. That is, for the second stage, the first dictionary $\mathscr{D}_{2,1}$ has $P = 10$ columns and defines the direction from $0^o$ to $18^o$. All possible DOAs are covered with $\mathscr{D}_{2,i}$ for $i = 1, \ldots, 10$. Regardless of any constraints of being different, let $k$, $l$, and $m$ denote the selected directions after the first stage. A new dictionary can be constructed as $\{\mathscr{D}_{2,k}, \mathscr{D}_{2,l}, \mathscr{D}_{2,m}\}$. Hence the length of this second dictionary is at most 30. The corresponding complexity of the second stage is $O(900)$ operations for BMP and $O(990)$ operations for OMP. The total number of operations are $O(1200)$ and $O(1380)$ for BMP and OMP algorithms, respectively. Hence for this specific example, the computational complexity is reduced by more than 56% and the precision is kept the same as a direct implementation. For further precision, a third stage of MP algorithm can be added to the detector in a similar fashion. In order to verify these results, simulation examples are presented in the following section.

## 5. Simulation Results

In our system, we consider an adaptive antenna array of $N$ elements as in Fig. 2. The input signal is assumed to be a plane wave or equivalently it can be decomposed into plane waves.

In simulations, only the OMP algorithm is considered due to its high resolution properties as shown in [13]. System parameters that are defined by the example above are considered for simula-

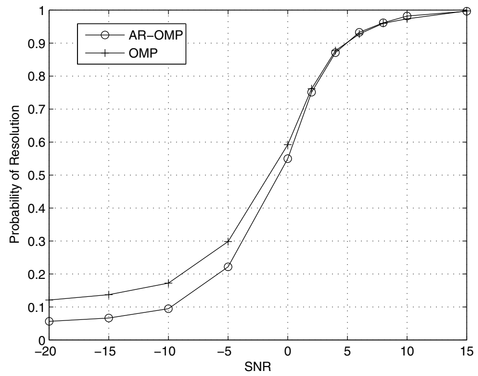


Figure 3: Probability of resolution vs. SNR.

tions. In our work, instead of equally partitioning the direction space $(0^o, 180^o)$, we equally divide the space that spans the cosine of the direction space $(-1, 1)$. Then take the inverse cosine and acquire the partitions in direction space. This is optimum due to the dictionary structure of (2). However the comparison of different partitioning scenarios are not included and beyond the scope of this work. The legends AR-OMP represents the OMP algorithm with two stage implementation. In Fig. 3, probability of detection performances of the algorithms mentioned above are shown. As can be seen in Fig. 3, AR-OMP as well as the OMP algorithm for high SNR values. There is a small discrepancy for low SNR region. This is due to the fact that when SNR is not sufficiently high enough, the coarse detection process in the first stage may give erroneous results, and hence resulting in error propagation.

In Fig. 4, root mean square error (RMSE) in the estimated directions is shown. RMSE is normalized by the null-to-null beamwidth ($BW_{nn}$) of the 10 element antenna array. As it is seen in Fig. 4, at low SNR the performance of the OMP is slightly better, but as for SNR$>$ 0 dB both algorithms have the same performance. Here, we should emphasize that the number of operations of AR-OMP is less than half of the OMP algorithm.

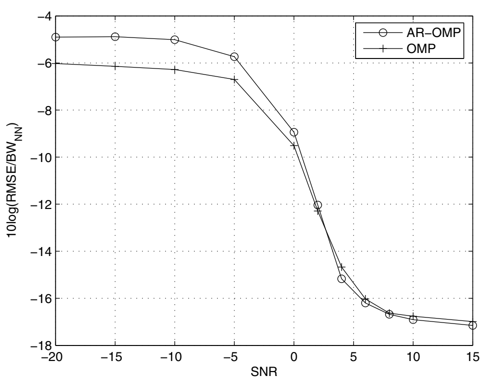


Figure 4: RMSE of DOA normalized by null-to-null beamwidth.

## 6. Conclusions

In this paper we have presented a general framework for reducing the computation complexity of the MP algorithms by dividing the dictionary into partitions and applying the MP algorithm of choice progressively. The proposed structure is applied to the DOA detection problem. It is verified by using simulation results that the performances of the proposed BMP and OMP algorithms in adaptive nature are similar to those of their direct implementation with a major reduction in computational complexity.

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