ORTHONORMAL NON-UNIFORM B-SPLINE SCALING AND WAVELET BASES ON NON-EQUALLY SPACED KNOT SEQUENCE FOR MULTIRESOLUTION SIGNAL APPROXIMATIONS

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ABSTRACT

This paper investigates the mathematical framework of multiresolution analysis based on irregularly spaced knots sequence. Our presentation is based on the construction of nested non-uniform B-spline multiresolution spaces. From these spaces, we present the construction of orthonormal scaling and wavelet basis functions on bounded intervals. For any arbitrary degree of the spline function, we provide an explicit generalization allowing the construction of the scaling and wavelet bases on the non-traditional sequences. We show that the orthogonal decomposition is implemented using filter bank coefficients of which depend on the location of the knots on the sequence. Examples of orthonormal spline scaling and wavelet bases are provided.

1. INTRODUCTION

Since many years, the multiresolution analysis method has been intensively studied see e.g. ([1], [2], [3], [4]). A multiresolution analysis is known as a decomposition of a function space into mutually orthogonal subspaces. The construction of the scaling and wavelet bases is closely related to the multiresolution analysis. The traditional wavelet basis is defined as a set of translations and dilations of one particular function, the mother wavelet. The scaling and wavelet bases, provided in the literature, are constructed under the assumptions that the knots of the infinite sequence are regularly spaced. This paper, takes into account other working hypotheses such as the non-equally spaced data on a bounded interval thus resulting in a more general definition of the scaling and wavelet functions. More specifically, this paper deals with spline scaling and wavelet functions based on non-uniform B-spline functions. Indeed, these functions are widely used to represent curves and surfaces ([5]). Moreover, they are well adapted to a bounded interval when imposing multiplicities at each end of the non-uniform B-spline function definition support ([6]). There is an extensive bibliography on spline scaling and wavelet functions with a uniform spacing knot ([7]). While relatively little works have been published about these functions on arbitrary non-uniform spacing knots ([9]). The construction of the wavelet basis on irregular spacing knots is more complicated than the traditional case (equally spaced knots). On a non-equally spaced knots sequence, the spline wavelets cannot be obtained by translations or dilatations of the mother wavelet. The main objective of this paper is to provide, for this non-traditional configuration of knots sequence, a generalisation of the underlying scaling and wavelet functions, yielding therefore an easy multiresolution structure.

The paper is organized as follows. Section 2 summarizes some necessary background material concerning the non-uniform B-spline functions allowing therefore the design of orthonormal B-spline basis. Section 3 introduces multiresolution spaces on bounded intervals. The construction of spline scaling functions is then described. Section 4 introduces the wavelet spaces and gives the required conditions to design a non-uniform B-spline wavelet basis on bounded intervals. Section 5 is concerned with the two-scale difference equation. Explicit generalizations of the scaling and wavelet functions are provided for any arbitrary degree of the spline function. Some examples are presented.

2. ORTHORMAL NON-UNIFORM B-SPLINE BASIS

Before developing the multiresolution analysis on bounded intervals, we briefly recall the orthonormal non-uniform B-spline basis.

The definition of a non-uniform B-spline function has been proposed initially by Curry and Schoenberg ([6]). Given a set of $d+2$ samples, located at arbitrary known knots. The knots sequence is organized according to an increasing order $t_i < ... < t_{i+d+1}$. For $t \in R$, the ith non-uniform B-spline function of degree d, denoted $B^d_{i,[t_i,t_{i+d+1}]}(t)$, is given by the following equation:

$$B^d_{i,[t_i,t_{i+d+1}]}(t) = (t_{i+d+1} - t_i)[t_i,...,t_{i+d+1}](.-t)^d_+$$

This equation is based on the $(d+1)th$ divided difference applied to the function $(.-t)^d_+$. The definition of the divided difference is as follows:

$$[t_i,...,t_{i+d+1}](.-t)^d_+ = (t_{i+d+1} - t_i)^{-1}\left([t_{i+1},...,t_{i+d+1}](.-t)^d_+ - [t_i,...,t_{i+d}](.-t)^d_+\right)$$

where $(x-t)_+ = \max(x-t,0)$ is the truncation function. The non-uniform B-spline function is represented by a piecewise polynomial of degree d. It has a finite support. If a knot in the sequence $t_i < ... < t_{i+d+1}$ has a multiplicity of order $\mu+1$, i.e. the knot occurs $\mu+1$ times, then the definition of the divided difference applied to the function $g = (.-t)^d_+$ becomes:

$$[t_0,...,t_\mu]g = g^{(\mu)}(t_0)/\mu! \quad \text{if} \quad t_0 = ... = t_\mu .$$

The n non-uniform B-spline functions $\left\{B^d_{i,[t_i,t_{i+d+1}]},\ldots,B^d_{i+n-1,[t_{i+n-1},t_{i+n+d}]}\right\}$, defined on the knots sequence $a=t_i<t_{i+1}<\ldots<t_{i+d+n}=b$, generates a basis for the spline space. This space is spanned by polynomials of degree d. The linear combination of these n B-spline functions defines the spline function. The dimension n of the basis depends on the multiplicities imposed on each knot ([6]) of the sequence.

In this paper, we impose a multiplicity, on each knot, of order $d+1$ ([8]). Therefore, the spline function is defined on a sequence composed only of two consecutive knots, as follows:

$$t_i=t_{i+1}=\ldots=t_{i+d}<t_{i+d+1}=\ldots=t_{i+d+n}$$

This configuration of knots provides the smallest spline space dimension equal to $d+1$. Whatever the degree of the spline, the construction of the basis elements has been generalized [1]. For any degree d of the spline function, the non-uniform B-spline elements are generalized by the following equation:

$$B^d_{i,[t_i,t_{i+1}]}(t)=C^i_d\left(1-\frac{t-t_i}{t_{i+1}-t_i}\right)^{d-i}\left(\frac{t-t_i}{t_{i+1}-t_i}\right)^i \text{ for } t_i\le t\le t_{i+1}$$

and $0\le i\le d$. where $C^i_d=d!/(i!(d-i)!)$ is the binomial coefficient. The orthonormalization of the non-uniform B-spline basis is easily carried out by the Gram-Schmidt method. The elements of the basis are therefore denoted as follows:

$$\left\{\underline{B}^d_{i,[t_i,t_{i+1}]}(t),\ \text{for } i=0,\ldots,d\right\}$$

Figure 1 presents the $d+1$ non-uniform B-spline functions of the orthonormal spline basis for different degrees. The left graph concerns the orthonormal basis for $d=0$. The right graphs correspond to the orthonormal basis for $d=1$ (linear). The layouts are presented on the interval $[0,2]$.

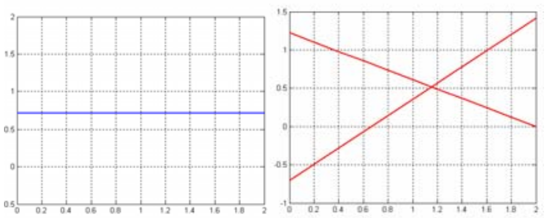

Figure 1 Non uniform orthonormal spline basis for $d=0$ (left graph) and $d=1$ (right graphs)

3. SCALING BASIS FUNCTIONS ON BOUNDED INTERVALS

In a partition of non-equally spaced knots, the underlying concept of translating and dilating one prototype function is not possible anymore. Thus, the construction of the orthonormal non-uniform B-spline scaling and wavelet basis functions on bounded intervals starts with the specification of the underlying multiresolution spaces.

Let us consider an infinite sequence of samples located at non-equally spaced knots. The initial corresponding knots sequence is denoted S_0 according to an increasing order:

$$t_0<t_1<\ldots<t_i<t_{i+1}<\ldots$$

Each knot of this sequence has a multiplicity of order $d+1$. S_0 is considered as the finest partition.

The multiresolution analysis consists in approximating a given function $f(t)$ at various resolution levels j using orthogonal projections on the corresponding approximation subspaces. The space corresponding to the highest resolution level $j=0$ is denoted V_0. It is spanned, on each bounded interval, by $d+1$ orthonormal non-uniform B-spline functions as follows:

$$V_0=span\left\{\varphi^d_{0,k,[t_i,t_{i+1}]}(t)=\underline{B}^d_{k,[t_i,t_{i+1}]}(t),\text{for } k=0,\ldots,d;\ \forall i\in Z\right\}$$

Satisfying the orthonomal conditions:

$$\left\langle\varphi^d_{0,k,[t_i,t_{i+1}]}(t),\varphi^d_{0,l,[t_m,t_{m+1}]}(t)\right\rangle=\delta_{kl}\delta_{im}$$

for $\forall k\in[0,d]$, $\forall l\in[0,d]$, $\forall i\in Z$ and $\forall m\in Z$.

where δ_{kl} and δ_{im} represent the Kronecker Delta. At any resolution level j, the approximation of the function $f(t)$, on a bounded interval $\left[t_{2^j i},t_{2^j(i+1)}\right]\in S_j$ (i.e. coarse partition), is denoted $f_{j,\left[t_{2^j i},t_{2^j(i+1)}\right]}(t)$. The sequence S_j is represented by the following knots $t_0<\ldots<t_{2^j i}<t_{2^j(i+1)}<\ldots$. I.e. going to the next coarser scale amounts to approximate the same signal with one knot out of two.

In order to minimize the approximation error $\left\|f(t)-f_{j,\left[t_{2^j i},t_{2^j(i+1)}\right]}(t)\right\|$, the approximation of the function $f_{j,\left[t_{2^j i},t_{2^j(i+1)}\right]}(t)$ is defined as its orthogonal projection on the subspace V_j on which it belongs. These subspaces are known as scaling subspaces. The corresponding approximation subspace V_j is defined as follows:

$$V_j=span\left\{\varphi^d_{j,k,\left[t_{2^j i},t_{2^j(i+1)}\right]}(t)=\underline{B}^d_{k,\left[t_{2^j i},t_{2^j(i+1)}\right]}(t),\text{for } k=0,\ldots,d;\ \forall i\in Z\right\}$$

Therefore, the multiresolution approximation is composed of embedded partitions: $S_0\supset S_1\supset\ldots\supset S_{j-1}\supset S_j\ldots$. The scaling subspaces are obviously nested as follows:

$$V_0\supset \mathrm{V}_1\supset\ldots\supset V_{j-1}\supset V_j\ldots$$

At any resolution level j, the scaling functions constitute an orthonormal non-uniform B-spline scaling basis of the subspace V_j. Indeed, it is easy to check that the following properties are satisfied:

$$\left\langle\varphi^d_{j,k,\left[t_{2^j i},t_{2^j(i+1)}\right]}(t),\varphi^d_{j,l,\left[t_{2^j m},t_{2^j(m+1)}\right]}(t)\right\rangle=\delta_{kl}\delta_{im}$$

for $\forall k\in[0,d]$, $\forall l\in[0,d]$, $\forall i\in Z$ and $\forall m\in Z$.

where δ_{kl} and δ_{im} represent the Kronecker Delta.

Let us present the scaling basis for $d=0$ and $d=1$, on the bounded interval $[t_i,t_{i+1}]$. The simplest orthonormal non-uniform B-spline scaling basis is built for $d=0$. This scaling basis is known as the Haar scaling function in equally spaced knots. The corresponding basis contains only one function (see figure 1). Its expression is:

$$\varphi^0_{0,0,[t_i,t_{i+1}]}(t) = \underline{B}^0_{0,[t_i,t_{i+1}]}(t) = \frac{1}{\sqrt{t_{i+1}-t_i}} \quad \text{for } t_i \le t \le t_{i+1}$$

For $d=1$, the orthonormal scaling basis is composed of two scaling functions:

$$\varphi^1_{0,0,[t_i,t_{i+1}]}(t) = \sqrt{3}\,\frac{t_{i+1}-t}{(t_{i+1}-t_i)^{3/2}} \quad \text{for } t_i \le t \le t_{i+1}$$

$$\varphi^1_{0,1,[t_i,t_{i+1}]}(t) = \frac{3t-t_{i+1}-2t_i}{(t_{i+1}-t_i)^{3/2}} \quad \text{for } t_i \le t \le t_{i+1}$$

4. WAVELET BASIS FUNCTIONS ON BOUNDED INTERVALS

The approximations of a given function $f(t)$, on a bounded interval, at the two following resolutions j and $j-1$ are respectively equal to their orthogonal projections on the scaling subspaces V_j and V_{j-1}. Recall that the subspace V_{j-1} contains the subspace V_j. The orthogonal complement of the subspace V_j in the subspace V_{j-1} is introduced to carry the necessary details improving the approximation of the function in the subspace V_{j-1}. Therefore, the orthogonal projection of the function $f(t)$ on the subspace V_j is decomposed as the sum of orthogonal projections on V_j and the complement subspace denoted W_j. This subspace W_j is known as orthogonal non-uniform B-spline wavelet subspace, at resolution level j. This leads to the following relation: $V_{j-1} = V_j \oplus W_j$. The wavelet space W_j is spanned by the wavelet functions denoted $\psi^d_{j,k,[t_{2^j i},t_{2^j(i+1)}]}(t)$:

$$W_j = span\left\{\psi^d_{j,k,[t_{2^j i},t_{2^j(i+1)}]}(t) \quad \text{for } k=0,\ldots,d;\ \forall i \in Z\right\}.$$

The wavelet functions must satisfy the following conditions:

$$\left\langle \psi^d_{j,k,[t_{2^j i},t_{2^j(i+1)}]}(t), \varphi^d_{j,l,[t_{2^j p},t_{2^j(p+1)}]}(t)\right\rangle = \delta_{kl}\delta_{ip}$$

for $\forall k \in [0,d]$, $\forall l \in [0,d]$, $\forall i \in Z$, $\forall p \in Z$, and $j \ge 1$.

5. TWO-SCALE EQUATIONS ON BOUNDED INTERVALS

This section studies how the traditional two-scale equations must be adapted to an infinite non-equally spaced knots sequence. A more general definition is then required for the multiresolution analysis based on bounded intervals. Indeed, the scaling (respectively wavelet) functions are not dilated or translated versions of a prototype scaling (respectively wavelet) function.

Since the scaling subspace $V_j \subset V_{j-1}$, the kth normalized scaling function $\varphi^d_{j,k,[t_{2^j i},t_{2^j(i+1)}]}(t)$ belongs to V_j. Therefore this function can be expressed as a linear combination of the basis functions from the subspace V_{j-1}. This leads to the following two-scale equation:

$$\varphi^d_{j,k,[t_{2^j i},t_{2^j(i+1)}]}(t) = \sum_{m=0}^{1}\sum_{n=0}^{d} h^{m,n}_{j,k}(t_{2^{j-1}i}, t_{2^{j-1}(i+1)}, t_{2^{j-1}(i+2)})\,\varphi^d_{j-1,n,[t_{2^{j-1}(i+m)},t_{2^{j-1}(i+1+m)}]}(t)$$

where $k \in [0,d]$ and $i \in Z$.

Since $\left\langle \varphi^d_{j,k,[t_{2^j i},t_{2^j(i+1)}]}(t), \varphi^d_{j,l,[t_{2^j p},t_{2^j(p+1)}]}(t)\right\rangle = \delta_{kl}\delta_{ip}$ ($\forall k \in [0,d]$, $\forall l \in [0,d]$, $\forall i \in Z$ and $\forall p \in Z$), it is easy to show, after some manipulations, that the coefficients are given by these two equations:

$$h^{m,n}_{j,k}(t_{2^{j-1}i}, t_{2^{j-1}(i+1)}, t_{2^{j-1}(i+2)}) = \left\langle \varphi^d_{j,k,[t_{2^j i},t_{2^j(i+1)}]}(t), \varphi^d_{j-1,n,[t_{2^{j-1}(i+m)},t_{2^{j-1}(i+1+m)}]}(t)\right\rangle$$

where for $k \in [0,d]$, $n \in [0,d]$, $m = 0,1$, and $i \in Z$. These coefficients are parameterized by two consecutive knots belonging to the sequence S_j. For convenience reasons, these coefficients are gathered in a matrix denoted $\mathbf{H}_j(t_{2^{j-1}i}, t_{2^{j-1}(i+1)}, t_{2^{j-1}(i+2)})$ as follows:

$$\begin{pmatrix} h^{0,0}_{j,0} & \ldots & h^{0,d}_{j,0} & h^{1,0}_{j,0} & \ldots & h^{1,d}_{j,0} \\ \vdots & & \vdots & \vdots & & \vdots \\ h^{0,0}_{j,d} & \ldots & h^{0,d}_{j,d} & h^{1,0}_{j,d} & \ldots & h^{1,d}_{j,d} \end{pmatrix}$$

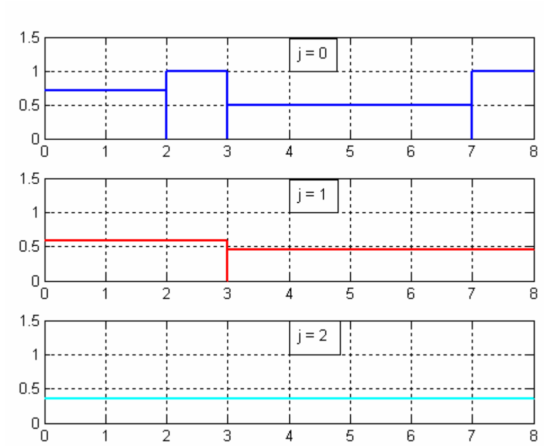

Figure 2 Haar scaling functions at three resolution levels $j = 0,1,2$

Let us complete the Haar scaling basis example. The matrix coefficients $\mathbf{H}_j(t_{2^{j-1}i}, t_{2^{j-1}(i+1)}, t_{2^{j-1}(i+2)})$ becomes:

$$\mathbf{H}_1(t_i, t_{i+1}, t_{i+2}) = \left(h^{0,0}_{1,0}(t_i,t_{i+1},t_{i+2}) \quad h^{1,0}_{1,0}(t_i,t_{i+1},t_{i+2})\right)$$

where

$$h^{0,0}_{1,0}(t_i,t_{i+1},t_{i+2}) = \frac{\sqrt{t_{i+1}-t_i}}{\sqrt{t_{i+2}-t_i}} \quad \text{and} \quad h^{1,0}_{1,0}(t_i,t_{i+1},t_{i+2}) = \frac{\sqrt{t_{i+2}-t_{i+1}}}{\sqrt{t_{i+2}-t_i}}$$

The scaling function at the resolution level $j=1$ becomes

$$\varphi^0_{1,0,[t_{2i},t_{2i+2}]}(t) = h^{0,0}_{1,0}(t_i,t_{i+1},t_{i+2})\varphi^0_{0,0,[t_i,t_{i+1}]}(t) + h^{1,0}_{1,0}(t_i,t_{i+1},t_{i+2})\varphi^0_{0,0,[t_{i+1},t_{i+2}]}(t)$$

Figure 2 presents the orthonormal scaling B-spline basis at three resolution levels ($j = 0,1,2$). We consider a finest sequence S_0 equal to $[0,2,3,7,8]$. The scaling space V_0 is spanned by four scaling functions $\{\varphi^0_{0,0,[0,2]}(t)$, $\varphi^0_{0,1,[2,3]}(t)$, $\varphi^0_{0,2,[3,7]}(t)$, $\varphi^0_{0,3,[7,8]}(t)\}$ represented by the first graph in Figure 2. The scaling space V_1 is spanned by two scaling functions $\{\varphi^0_{1,0,[0,3]}(t)$, $\varphi^0_{1,1,[3,8]}(t)\}$ given by the second graph in Figure 2. Only one scaling function $\varphi^0_{2,0,[0,8]}(t)$, represented by the third graph, generates the scaling space V_2 represented by the third graph in Figure 2.

On the other hand the wavelet subspace $W_j \subset V_{j-1}$. Thus, the kth wavelet function $\psi^d_{j,k,[t_{2^j i},t_{2^j(i+1)}]}(t)$, at resolution level j, can be also expressed as a linear combination of the scaling basis functions from the subspace V_{j-1}. We obtain the following decomposition:

$$\psi^d_{j,k,[t_{2^j i},t_{2^j(i+1)}]}(t)=\sum_{m=0}^{1}\sum_{n=0}^{d} g^{m,n}_{j,k}(t_{2^{j-1}i},t_{2^{j-1}(i+1)},t_{2^{j-1}(i+2)})\varphi^d_{j-1,n,[t_{2^{j-1}(i+m)},t_{2^{j-1}(i+1+m)}]}(t)$$

with $k \in [0,d]$, $m=0,1$, $n \in [0,d]$ and $i \in Z$.

For convenience reasons, these coefficients are also gathered in a matrix denoted $\mathbf{G}_j(t_{2^{j-1}i},t_{2^{j-1}(i+1)},t_{2^{j-1}(i+2)})$. It is enough to replace the set of coefficients $\{h^{m,n}_{j,k}\}$ in the matrix $\mathbf{H}_j(t_{2^{j-1}i},t_{2^{j-1}(i+1)},t_{2^{j-1}(i+2)})$ by the coefficients $\{g^{m,n}_{j,k}\}$.

For computing the wavelet coefficients, we use the fact that (i) the scaling subspace is orthogonal at the wavelet subspace, for any resolution level ($j \ge 1$):

$$\left\langle \psi^d_{j,k,[t_{2^j i},t_{2^j(i+1)}]}(t),\varphi^d_{j,l,[t_{2^j p},t_{2^j(p+1)}]}(t)\right\rangle=0 \text{ for } \forall k \in [0,d],\ \forall l \in [0,d],$$
$$\forall i \in Z \text{ and } \forall p \in Z$$

and (ii) the orthonormality of the wavelet basis:

$$\left\langle \psi^d_{j,k,[t_{2^j i},t_{2^j(i+1)}]}(t),\psi^d_{j,l,[t_{2^j p},t_{2^j(p+1)}]}(t)\right\rangle=\delta_{kl}\delta_{ip} \text{ for } \forall k \in [0,d],$$
$$\forall l \in [0,d],\ \forall i \in Z \text{ and } \forall p \in Z$$

After some manipulations of preceding equations, the coefficients $\{g^{m,n}_{j,k}\}$ are given by solving the following homogeneous system of linear equations:

$$\mathbf{H}_j(t_{2^{j-1}i},t_{2^{j-1}(i+1)},t_{2^{j-1}(i+2)})\mathbf{G}_j(t_{2^{j-1}i},t_{2^{j-1}(i+1)},t_{2^{j-1}(i+2)})^t=\mathbf{0}$$

The solution of this system consists in finding a basis of the null space of $\mathbf{H}_j(t_{2^{j-1}i},t_{2^{j-1}(i+1)},t_{2^{j-1}(i+2)})$ corresponding to the required coefficients $\{g^{m,n}_{j,k}\}$.

Let us construct the Haar wavelet basis. For this, we must find matrix $\mathbf{G}_1(t_i,t_{i+1},t_{i+2})$ as explained above. The matrix becomes:

$$\mathbf{G}_1(t_i,t_{i+1},t_{i+2})=\left(g^{0,0}_{1,0}(t_i,t_{i+1},t_{i+2}) \quad g^{1,0}_{1,0}(t_i,t_{i+1},t_{i+2})\right)$$

where:

$$g^{0,0}_{1,0}(t_i,t_{i+1},t_{i+2})=\frac{\sqrt{t_{i+2}-t_{i+1}}}{\sqrt{t_{i+2}-t_i}} \text{ and } g^{1,0}_{1,0}(t_i,t_{i+1},t_{i+2})=-\frac{\sqrt{t_{i+1}-t_i}}{\sqrt{t_{i+2}-t_i}}$$

Figure 3, presents the Haar wavelet function at the resolution levels $j=1,2$. The first curve, in Figure 3, concerns the two wavelets functions $\{\psi^0_{1,0,[0,3]}(t),\ \psi^0_{1,1,[3,8]}(t)\}$ generating the space W_1. The second curve represents the wavelet function $\psi^0_{2,0,[0,8]}(t)$ spanning the space W_2.

These forms of decomposition are equivalent to the traditional decomposition (knots equally spaced with filter banks) except that the filters ($\mathbf{H}_j$, $\mathbf{G}_j$) depend on the knot location. Let us remember that in the case of equally spaced knots, the matrices $\mathbf{H}_j$ and $\mathbf{G}_j$ are each one modelled by orthogonal filters with a finite impulse response more precisely called conjugate mirror filters. The difference between regular partitions is that each matrix is modelled by a set of different filters because they depend explicitly on the repartition of the knots.

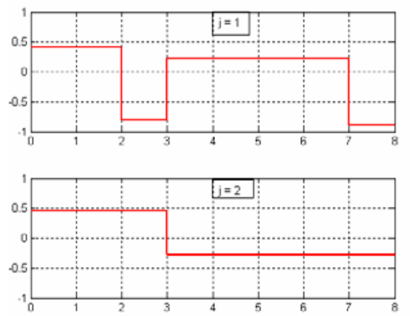

Figure 3 Haar wavelet functions at two resolution levels $j=1,2$

6. CONCLUSION

This paper has explored the underlying mathematical framework of the one-dimensional multiresolution analysis based on non-equally spaced knots sequence. We have shown that the underlying concept of translating and dilating one prototype function is not valid any more. The specifications of the underlying multiresolution spaces involve the construction of orthonormal non-uniform B-spline scaling and wavelet bases on bounded intervals. For any arbitrary degree of the spline function, we have provided an explicit generalization for the construction of the scaling and wavelet bases on non-traditional sequences. We show that the orthogonal decomposition is implemented using different filter banks depending on the knots in the sequence. These first results lead us to investigate, in future work, their application to problems such as image compression.

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