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GRAM PAIR PARAMETERIZATION OF MULTIVARIATE SUM-OF-SQUARES TRIGONOMETRIC POLYNOMIALS

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ABSTRACT

In this paper we propose a parameterization of sum-of-squares trigonometric polynomials with real coefficients, that uses two positive semidefinite matrices twice smaller than the unique matrix in the known Gram (trace) parameterization. Also, we formulate a Bounded Real Lemma for polynomials with support in the positive orthant. We show that the new parameterization is clearly faster and thus can replace the old one in several design problems.

1. INTRODUCTION

Optimization problems with positive univariate trigonometric polynomials have been used lately for the design of FIR filters [1, 2], beamformers [2], adapted wavelets [13], compaction filters [7] and in several other applications. Semidefinite programming (SDP) has been used in these applications via a linear matrix inequality (LMI) parameterization of nonnegative polynomials [1, 8, 9]. This Gram matrix (or trace) parameterization can be generalized in the multivariate case [10], for sum-of-squares polynomials (which are now a proper subset of nonnegative polynomials, while in 1-D the two sets coincide). Some very recent applications are mentioned later, in Section 5; they deal mainly with filter design and discrete-time systems stability.

In this paper, we propose an alternative to the Gram matrix parameterization, valid for trigonometric polynomials with real coefficients. The sum-of-squares polynomial is parameterized as a function of two positive semidefinite matrices (instead of a single one), which are twice smaller than the Gram matrix. We dub *Gram pair* this parameterization, which leads to faster solutions to the SDP problems mentioned above.

For univariate polynomials, a parameterization using two matrices has been proposed in [11]. Besides covering the multivariate case, our parameterization has a different formulation, by using sparse constant matrices, instead of dense matrices related to fast transforms. Our approach uses standard SDP algorithms, while in [11] special algorithms are necessary. Finally, the derivation of our results is self sustained (does not use the theory of real polynomials).

A summary of the paper is as follows. In Section 2, we review the Gram matrix parameterization of sum-of-squares trigonometric polynomials. Section 3 contains the main result, the Gram pair parameterization. In Section 4, a related kind of result is derived, namely a Bounded Real Lemma (BRL) for trigonometric polynomials; in certain cases, such a BRL allows us to circumvent the lack of spectral factorization for multivariate polynomials. Section 5 is dedicated to a simple example, the computation of the minimum value of a polynomial, which illustrates the speed-up given by the Gram pair parameterization (with respect to the Gram matrix one).

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2. SUM-OF-SQUARES POLYNOMIALS

We denote $\mathbf{z} = (z_1, \ldots, z_d)$ the d-dimensional complex variable and $\mathbf{z}^{\mathbf{k}} = z_1^{k_1} \ldots z_d^{k_d}$ a d-variate monomial of degree $\mathbf{k} \in \mathbb{Z}^d$. A symmetric d-variate trigonometric polynomial of degree $\mathbf{n} \in \mathbb{N}^d$, with real coefficients, is

$$R(\mathbf{z}) = \sum_{\mathbf{k}=-\mathbf{n}}^{\mathbf{n}} r_{\mathbf{k}} \mathbf{z}^{-\mathbf{k}}, \quad r_{-\mathbf{k}} = r_{\mathbf{k}}. \tag{1}$$

In the above (multiple) sum, the index $\mathbf{k}$ takes all values such that $-\mathbf{n} \leq \mathbf{k} \leq \mathbf{n}$. On the unit d-circle $\mathbb{T}^d = \{\mathbf{z} \in \mathbb{C}^d \mid |z_i| = 1, \ i = 1 : d\}$, with $\mathbf{z}^{\mathbf{k}} = \exp(j\mathbf{k}^T \omega)$, the polynomial (1) has real values. The support of the polynomial is symmetric with respect to the origin; the support of distinct coefficients can be confined to a halfspace, which is a set $\mathcal{H} \subset \mathbb{Z}^d$ such that $\mathcal{H} \cap (-\mathcal{H}) = \{0\}$, $\mathcal{H} \cup (-\mathcal{H}) = \mathbb{Z}^d$, $\mathcal{H} + \mathcal{H} \subset \mathcal{H}$ (for example, in 2D, the upper halfspace is defined by integer pairs $\mathbf{k} = (k_1, k_2)$, with $k_2 > 0$ or $k_2 = 0$ and $k_1 \geq 0$).

Any trigonometric polynomial (1) that is *positive* on the unit d-circle can be written as a *sum-of-squares* (a proof is e.g. in [3]), namely

$$R(\mathbf{z}) = \textstyle\sum_{l=1}^{\nu} H_l(\mathbf{z}) H_l(\mathbf{z}^{-1}), \tag{2}$$

where the polynomials $H_l(\mathbf{z})$ have real coefficients and contain only monomials with nonnegative degree (their support is in the positive orthant). On $\mathbb{T}^d$, this equality becomes

$$R(\mathrm{e}^{j\omega}) \stackrel{\Delta}{=} R(\omega) = \textstyle\sum_{l=1}^{\nu} |H_l(\omega)|^2. \tag{3}$$

Theoretically, the degrees of the polynomials $H_l(\mathbf{z})$ from (2) can be arbitrarily high. Sum-of-squares trigonometric polynomials can be expressed in terms of positive semidefinite matrices via the Gram (trace) parameterization [10, 5]: the polynomial (1) is sum-of-squares if and only if there exists a matrix $\mathbf{X} \succeq 0$, called Gram matrix associated with $R(\mathbf{z})$, such that

$$r_{\mathbf{k}} = \mathrm{trace}[\boldsymbol{\Theta}_{\mathbf{k}} \cdot \mathbf{X}], \quad \boldsymbol{\Theta}_{\mathbf{k}} = \boldsymbol{\Theta}_{k_d} \otimes \ldots \otimes \boldsymbol{\Theta}_{k_1} \tag{4}$$

where $\mathbf{\Theta}_{k_i}$ are elementary Toeplitz matrices with ones on the k_i-th diagonal and zeros elsewhere ($\otimes$ denotes the Kronecker product).

In a practical implementation, we have to bound $\deg H_l$. In this paper, we analyze the set of sum-of-squares polynomials whose factors satisfy $\deg H_l \leq \mathbf{n}$ (which is a subset of the positive polynomials (1) with degree smaller than $\mathbf{n}$). This amounts to taking the matrices $\mathbf{\Theta}_{k_i}$ of size $(n_i+1)\times(n_i+1)$. Then, the size of the Gram matrix $\mathbf{X}$ is $N \times N$, with

$$N = \prod_{i=1}^{d}(n_i+1). \tag{5}$$

3. GRAM PAIR PARAMETERIZATION

3.1 Basic result

Let the polynomial (with positive orthant support)

$$H(\mathbf{z}) = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} h_{\mathbf{k}} \mathbf{z}^{-\mathbf{k}} \tag{6}$$

be a generic factor in a term of the sum-of-squares representation (2). Let $\mathbf{h} \in \mathbb{R}^N$ be a vector containing the coefficients of $H(\mathbf{z})$; the order of the coefficients in $\mathbf{h}$ is not important; for example, we can enumerate the coefficients in inverse lexicographic order; in 2D, the coefficients of $H(\mathbf{z})$ form a matrix and this order means that the indices $\mathbf{k}$ are enumerated following a column major order: $(0,0)$, $(1,0)$, $\ldots$, $(n_1,0)$, $(0,1)$ etc. We write $\mathbf{n} = 2\tilde{\mathbf{n}} + \mathbf{p}$, where $\mathbf{p} = \mathbf{n} \bmod 2$; the elements of the vector $\mathbf{p}$ indicate the parity (0 means even and 1 means odd) of the elements of $\mathbf{n}$.

We define the (pseudo)-polynomial

$$\tilde{H}(\mathbf{z}) = \mathbf{z}^{\mathbf{n}/2} H(\mathbf{z}) = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} h_{\mathbf{k}} \mathbf{z}^{\mathbf{n}/2-\mathbf{k}}. \tag{7}$$

On the unit circle we have

$$\begin{aligned}\tilde{H}(\omega) &= \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} h_{\mathbf{k}} \left[\cos(\mathbf{k}-\mathbf{n}/2)^T\omega - j\sin(\mathbf{k}-\mathbf{n}/2)^T\omega\right] \\ &\stackrel{\Delta}{=} A(\omega) + jB(\omega).\end{aligned} \tag{8}$$

We perform now two simple operations in (8). First, since $\mathbf{k}-\mathbf{n}/2$ takes values that are symmetric with respect to the origin, we can group the terms in the sum such that $\mathbf{k}-\mathbf{n}/2$ is confined to a halfspace $\mathcal{H}$. We then replace $\mathbf{k}$ with $\mathbf{k}-\tilde{\mathbf{n}}$. It results that

$$A(\omega) = \mathbf{a}^T \chi_c(\omega), \;\; B(\omega) = \mathbf{b}^T \chi_s(\omega), \tag{9}$$

where

$$\begin{aligned}\chi_c(\omega) &= [\,\ldots\, \cos(\mathbf{k}-\mathbf{p}/2)^T\omega \,\ldots\,]^T, \\ \chi_s(\omega) &= [\,\ldots\, \sin(\mathbf{k}-\mathbf{p}/2)^T\omega \,\ldots\,]^T, \\ -\tilde{\mathbf{n}} &\leq \mathbf{k} \leq \tilde{\mathbf{n}}+\mathbf{p}, \;\; \mathbf{k}-\mathbf{p}/2 \in \mathcal{H},\end{aligned} \tag{10}$$

are basis vectors of lengths N_c and N_s, respectively, and $\mathbf{a}$ and $\mathbf{b}$ are vectors of coefficients. The elements of $\mathbf{a}$ and $\mathbf{b}$ depend linearly on the coefficients of $H(\mathbf{z})$; typically, they are a sum or difference of two coefficients. Thus, there exist two matrices $\mathbf{C}_c \in \mathbb{R}^{N_c \times N}$ and $\mathbf{C}_s \in \mathbb{R}^{N_s \times N}$ such that

$$\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} = \begin{bmatrix}\mathbf{C}_c\\ \mathbf{C}_s\end{bmatrix}\mathbf{h}. \tag{11}$$

Moreover, the correspondence between a pair $(\mathbf{a},\mathbf{b})$ and the vector $\mathbf{h}$ is one-to-one; so, the matrix from (11) is nonsingular and $N_c + N_s = N$ (as we will see again later).

Using (9), it results from (8) that

$$\begin{aligned}|H(\omega)|^2 &= |\tilde{H}(\omega)|^2 = A(\omega)^2 + B(\omega)^2 \\ &= \chi_c^T(\omega)\mathbf{a}\mathbf{a}^T\chi_c(\omega) + \chi_s^T(\omega)\mathbf{b}\mathbf{b}^T\chi_s(\omega).\end{aligned} \tag{12}$$

This relation is fundamental in the proof of the following theorem.

Theorem 1 *Let $R(\mathbf{z})$ be a trigonometric polynomial. The polynomial is sum-of-squares, i.e. can be written as in (2), with factors of degree $\mathbf{n}$ if and only if there exist positive semidefinite matrices $\mathbf{Q} \in \mathbb{R}^{N_c \times N_c}$ and $\mathbf{S} \in \mathbb{R}^{N_s \times N_s}$ such that*

$$R(\omega) = \chi_c^T(\omega)\mathbf{Q}\chi_c(\omega) + \chi_s^T(\omega)\mathbf{S}\chi_s(\omega). \tag{13}$$

We name $(\mathbf{Q},\mathbf{S})$ a Gram pair associated with $R(\omega)$ (the name is used also for polynomials $R(\mathbf{z})$ that are not sum-of-squares).

Proof. Let $\mathbf{Q} \succeq 0$, $\mathbf{S} \succeq 0$ be two matrices such that (13) holds. Their eigendecompositions are

$$\mathbf{Q} = \sum_{l=1}^{\nu_c} \alpha_l^2 \mathbf{a}_l \mathbf{a}_l^T, \;\; \mathbf{S} = \sum_{l=1}^{\nu_s} \beta_l^2 \mathbf{b}_l \mathbf{b}_l^T,$$

where α_l^2, β_l^2 are nonzero eigenvalues and $\mathbf{a}_l$, $\mathbf{b}_l$ are eigenvectors. Denoting $\nu = \max(\nu_c, \nu_s)$, it results that

$$R(\omega) = \sum_{l=1}^{\nu}[A_l(\omega)^2 + B_l(\omega)^2],$$

where

$$A_l(\omega) = \alpha_l \mathbf{a}_l^T \chi_c(\omega), \;\; B_l(\omega) = \beta_l \mathbf{b}_l^T \chi_s(\omega).$$

(We consider $\mathbf{a}_l = 0$ if $l > \nu_c$ or $\mathbf{b}_l = 0$ if $l > \nu_s$.) Using the correspondence $(\mathbf{a}_l, \mathbf{b}_l) \to \mathbf{h}_l$ from (11), it follows that $R(\mathbf{z})$ is sum-of-squares.

Reciprocally, if $R(\mathbf{z})$ is sum-of-squares, then each term of the sum-of-squares can be expressed as in (12). It follows that the matrices $\mathbf{Q} \stackrel{\Delta}{=} \sum \mathbf{a}\mathbf{a}^T$ and $\mathbf{S} \stackrel{\Delta}{=} \sum \mathbf{b}\mathbf{b}^T$ (where the sums are taken for all the terms in the sum-of-squares decomposition), satisfy (13). ■

3.2 Parity discussion

The elements and the lengths of the base vectors from (10) depend on the parity of (the elements of) the degree $\mathbf{n}$. For a d-variate polynomial, there are 2^d possible combinations of the parities of the degrees n_i, $i = 1:d$. However, only $d+1$ of them are essentially different, as we can reorder the variables such that the polynomial has, say, even order in the first variables and odd order in the others (and so the vector $\mathbf{p}$ is formed of a sequence of zeros followed by a sequence of ones). Here, we discuss two extreme cases, by means of examples.

If all the degrees n_i are even and so $\mathbf{n} = 2\tilde{\mathbf{n}}$ (i.e. $\mathbf{p} = \mathbf{0}$), then the support of $H(\mathbf{z})$ contains its center of symmetry, as seen in the left of figure 1, for $d = 2$, $n_1 = n_2 = 2$. After the translation of the support implied by (7), the support of $\tilde{H}(\mathbf{z})$ is symmetric with respect to the origin and contains it. The restriction of this support to the upper halfplane contains

$$N_c = \frac{1 + \prod_{i=1}^{d}(2\tilde{n}_i + 1)}{2} = \frac{N+1}{2} \tag{14}$$

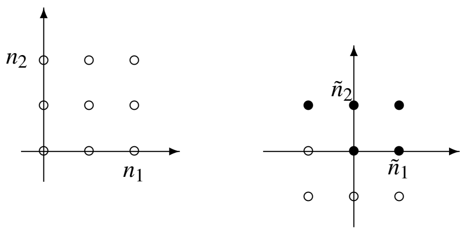

Figure 1: Supports of $H(\mathbf{z})$ (left) and $\tilde{H}(\mathbf{z})$ (right), for even degrees, in the 2D case. The degrees in the upper halfplane are denoted with bullets.

points, where N is defined in (5). In our example, we have $N_c = 5$. Enumerating the points from left to right and upwards, the first base vector from (10) is

$$\chi_c(\omega) = \begin{bmatrix} 1 \\ \cos\omega_1 \\ \cos(-\omega_1+\omega_2) \\ \cos\omega_2 \\ \cos(\omega_1+\omega_2) \end{bmatrix}. \quad (15)$$

Since $\sin 0 = 0$, the second base vector is

$$\chi_s(\omega) = \begin{bmatrix} \sin\omega_1 \\ \sin(-\omega_1+\omega_2) \\ \sin\omega_2 \\ \sin(\omega_1+\omega_2) \end{bmatrix}. \quad (16)$$

Its length is thus

$$N_s = \frac{N-1}{2}. \quad (17)$$

(In our example, $N_s = 4$.) We note that $N_c + N_s = N = 9$, as claimed before. Finally, according to (8), the vectors $\mathbf{a}$ and $\mathbf{b}$ from (9) are

$$\mathbf{a} = \begin{bmatrix} h_{11} \\ h_{01}+h_{21} \\ h_{20}+h_{02} \\ h_{10}+h_{12} \\ h_{00}+h_{22} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} h_{01}-h_{21} \\ h_{20}-h_{02} \\ h_{10}-h_{12} \\ h_{00}-h_{22} \end{bmatrix}. \quad (18)$$

If at least one of the elements of $\mathbf{n}$ is odd, then the support of $H(\mathbf{z})$ no longer contains its center of symmetry, and thus the support of $\tilde{H}(\mathbf{z})$ does not contain the origin. This situation is illustrated by figure 2, with $d = 2$, $n_1 = n_2 = 3$ (and thus $\tilde{n}_1 = \tilde{n}_2 = 1$). The number of points from the support of $\tilde{H}(\mathbf{z})$ that fall in the same halfspace is exactly half the number of points in the support of $H(\mathbf{z})$ and so the base vectors (10) have the (same) length

$$N_c = N_s = \frac{1}{2}\prod_{i=1}^{d}(2\tilde{n}_i + 1) = \frac{N}{2}.$$

In our example, $N_c = N_s = 8$. As all the elements of $\mathbf{n}$ are odd, the first base vector from (10) is

$$\chi_c(\omega) = \begin{bmatrix} \cos(-3\omega_1+\omega_2)/2 \\ \cos(-\omega_1+\omega_2)/2 \\ \cos(\omega_1+\omega_2)/2 \\ \cos(3\omega_1+\omega_2)/2 \\ \cos(-3\omega_1+3\omega_2)/2 \\ \cos(-\omega_1+3\omega_2)/2 \\ \cos(\omega_1+3\omega_2)/2 \\ \cos(3\omega_1+3\omega_2)/2 \end{bmatrix}.$$

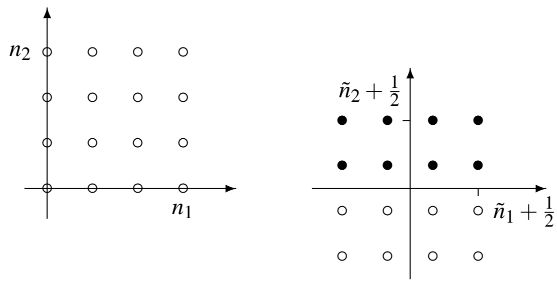

Figure 2: Same as in figure 1, for odd degrees.

The base vector $\chi_s(\omega)$ is obtained by simply replacing cos with sin in the above formula.

3.3 Implementation form

For use in SDP optimization problems, the parameterization (13) has to be expressed in the style of (4). Since in (13) the coefficients of $R(\mathbf{z})$ depend linearly on the elements of the matrices $\mathbf{Q}$ and $\mathbf{S}$, the following theorem is trivial.

Theorem 2 *The trigonometric polynomial $R(\mathbf{z})$ is sum-of-squares with factors of degree $\mathbf{n}$ if and only if there exist positive semidefinite matrices $\mathbf{Q} \in \mathbb{R}^{N_c \times N_c}$ and $\mathbf{S} \in \mathbb{R}^{N_s \times N_s}$ such that*

$$r_{\mathbf{k}} = \text{trace}[\mathbf{\Phi_k Q}] + \text{trace}[\mathbf{\Lambda_k S}], \quad (19)$$

where $\mathbf{\Phi_k} \in \mathbb{R}^{N_c \times N_c}$ and $\mathbf{\Lambda_k} \in \mathbb{R}^{N_s \times N_s}$ are constant matrices.

Of course, for practical use, we need the precise values of the matrices $\mathbf{\Phi_k}$ and $\mathbf{\Lambda_k}$ or an algorithm for building them. We consider here only the case where all elements of the degree $\mathbf{n}$ are even. We assume that the rows and columns of the matrices $\mathbf{Q}$ and $\mathbf{S}$ are numbered via $\mathbf{i}, \mathbf{l} \in \mathbb{Z}^d$; the mapping between these d-dimensional numbers and the usual index range $0 : N_c - 1$ is not unique; the mapping that gives the order of the elements in (15) is used in the sequel. Using the standard trigonometric equalities

$$\begin{aligned} \cos i\omega \cos l\omega &= \tfrac{1}{2}[\cos(i+l)\omega + \cos(i-l)\omega], \\ \sin i\omega \sin l\omega &= \tfrac{1}{2}[-\cos(i+l)\omega + \cos(i-l)\omega], \end{aligned} \quad (20)$$

the relation (13) is equivalent to

$$\begin{aligned} R(\omega) = &\tfrac{1}{2}\textstyle\sum_{\mathbf{i},\mathbf{l}\in\mathcal{H}} q_{\mathbf{il}}[\cos(\mathbf{i}+\mathbf{l})^T\omega + \cos(\mathbf{i}-\mathbf{l})^T\omega] \\ &+ \tfrac{1}{2}\textstyle\sum_{\mathbf{i},\mathbf{l}\in\mathcal{H}^*} s_{\mathbf{il}}[-\cos(\mathbf{i}+\mathbf{l})^T\omega + \cos(\mathbf{i}-\mathbf{l})^T\omega]. \end{aligned} \quad (21)$$

Here, we have denoted $\mathcal{H}^* = \mathcal{H} \setminus \{\mathbf{0}\}$; by $\mathbf{i} \in \mathcal{H}$, we understand implicitly that $|\mathbf{i}| \leq \tilde{\mathbf{n}}$, as in (10) (remind that $\mathbf{p} = \mathbf{0}$). The coefficients of the polynomial are thus given by

$$4r_{\mathbf{k}} = \sum_{\substack{\mathbf{i}+\mathbf{l}=\mathbf{k} \\ \mathbf{i},\mathbf{l}\in\mathcal{H}}} q_{\mathbf{il}} + \sum_{\substack{\mathbf{i}-\mathbf{l}=\pm\mathbf{k} \\ \mathbf{i},\mathbf{l}\in\mathcal{H}}} q_{\mathbf{il}} - \sum_{\substack{\mathbf{i}+\mathbf{l}=\mathbf{k} \\ \mathbf{i},\mathbf{l}\in\mathcal{H}^*}} s_{\mathbf{il}} + \sum_{\substack{\mathbf{i}-\mathbf{l}=\pm\mathbf{k} \\ \mathbf{i},\mathbf{l}\in\mathcal{H}^*}} s_{\mathbf{il}}. \quad (22)$$

We suggest the algorithm to build the matrices $\mathbf{\Phi_k}$ and $\mathbf{\Lambda_k}$ by considering again the example $d = 2$, $n_1 = n_2 = 2$. The upper halfplane support of $R(\mathbf{z})$ is shown in figure 3 (with $\times$); the figure shows also the support of the base vectors (15) and (16) (with circles).

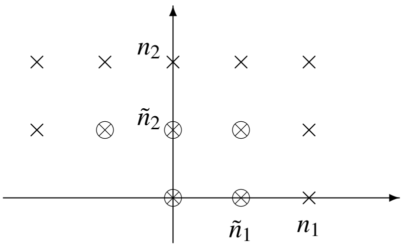

Figure 3: With $\times$, support of $R(\mathbf{z})$. With circles, support of the base vectors (15) and (16).

For building $\mathbf{\Phi_k}$ and $\mathbf{\Lambda_k}$, we look at all the possible results $\mathbf{i}+\mathbf{l}$ and $\mathbf{i}-\mathbf{l}$, with $\mathbf{i}$ and $\mathbf{l}$ in the same halfplane (i.e. the circles from figure 3). For the sum, the table is

$\mathbf{i} \setminus \mathbf{l}$	$(0,0)$	$(1,0)$	$(-1,1)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(-1,1)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$
$(-1,1)$	$(-1,1)$	$(0,1)$	$(-2,2)$	$(-1,2)$	$(0,2)$
$(0,1)$	$(0,1)$	$(1,1)$	$(-1,2)$	$(0,2)$	$(1,2)$
$(1,1)$	$(1,1)$	$(2,1)$	$(0,2)$	$(1,2)$	$(2,2)$

For the difference $\mathbf{i}-\mathbf{l}$, the table is

$\mathbf{i} \setminus \mathbf{l}$	$(0,0)$	$(1,0)$	$(-1,1)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(-1,0)$	$(1,-1)$	$(0,-1)$	$(-1,-1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(2,-1)$	$(1,-1)$	$(0,-1)$
$(-1,1)$	$(-1,1)$	$(-2,1)$	$(0,0)$	$(-1,0)$	$(-2,0)$
$(0,1)$	$(0,1)$	$(-1,1)$	$(1,0)$	$(0,0)$	$(-1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(2,0)$	$(1,0)$	$(0,0)$

When building $\mathbf{\Phi_k}$ and $\mathbf{\Lambda_k}$, we search the values $\mathbf{k}$ and $-\mathbf{k}$ in the tables (for $\mathbf{\Lambda_k}$ we ignore the first rows and columns of the tables) and use the appropriate coefficients and signs as indicated by (22). Here are two pairs of matrices:

$$\mathbf{\Phi}_{(1,0)} = \frac{1}{4}\begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{\Lambda}_{(1,0)} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{\Phi}_{(2,0)} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{\Lambda}_{(2,0)} = \frac{1}{4}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Note that the matrices are sparse. In an efficient algorithm, all the matrices $\mathbf{\Phi_k}$ and $\mathbf{\Lambda_k}$ are built simultaneously, in a double loop in which the indices $\mathbf{i}$ and $\mathbf{l}$ take all admissible values (i.e. scan the sum and difference tables). The resulting program is compact and fast.

4. BOUNDED REAL LEMMA

With the aid of the Gram pair parameterization (19), we formulate now a necessary condition, in the form of an LMI, for the inequality

$$|H(\omega)| \leq \gamma, \tag{23}$$

where $H(\mathbf{z})$ is a polynomial (6) (with positive orthant support) and $\gamma > 0$ is a given scalar. For univariate polynomials, such a condition results trivially by imposing positivity for the polynomial $\gamma^2 - |H(\omega)|^2$; in optimization problems, this means working with $R(\omega) = |H(\omega)|^2$ and then recovering $H(z)$ via spectral factorization. For multivariate polynomials, such an approach is impossible, due to the lack of spectral factorization. We start with some preliminary results.

Lemma 1 *Let $R_1(\mathbf{z})$, $R_2(\mathbf{z})$ be two polynomials of degree $\mathbf{n}$, as in (1). Let $(\mathbf{Q}_2, \mathbf{S}_2)$ be a Gram pair associated with $R_2(\mathbf{z})$. The polynomial $R(\mathbf{z}) = R_1(\mathbf{z}) - R_2(\mathbf{z})$ is sum-of-squares if and only if there exists a Gram pair $(\mathbf{Q}_1, \mathbf{S}_1)$ associated with $R_1(\mathbf{z})$ such that*

$$\mathbf{Q}_1 \succeq \mathbf{Q}_2, \ \ \mathbf{S}_1 \succeq \mathbf{S}_2. \tag{24}$$

Proof. If $R(\mathbf{z})$ is sum-of-squares, then (13) holds for $\mathbf{Q} \succeq 0$, $\mathbf{S} \succeq 0$. Due to the linearity of (19), the matrices $\mathbf{Q}_1 \stackrel{\Delta}{=} \mathbf{Q}_2 + \mathbf{Q}$ and $\mathbf{S}_1 \stackrel{\Delta}{=} \mathbf{S}_2 + \mathbf{S}$ are a Gram pair for the polynomial $R_1 = R_2 + R$. The reverse implication is trivial. ■

As before, let $\mathbf{h}$ be a vector containing the coefficients of $H(\mathbf{z})$; the vectors $\mathbf{a}$ and $\mathbf{b}$ are defined as in (11). Let $R_2(\omega) = |H(\omega)|^2$. From (12), it results that

$$\mathbf{Q}_2 = \mathbf{a}\mathbf{a}^T, \ \ \mathbf{S}_2 = \mathbf{b}\mathbf{b}^T \tag{25}$$

is a Gram pair associated with $R_2(\mathbf{z})$. We can now formulate a Gram pair version of the Bounded Real Lemma for multivariate polynomials.

Theorem 3 *Let $H(\mathbf{z})$ be a polynomial as in (6) and $\gamma > 0$. If there exist positive semidefinite matrices $\mathbf{Q}_1 \in \mathbb{R}^{N_c \times N_c}$ and $\mathbf{S}_1 \in \mathbb{R}^{N_s \times N_s}$ such that*

$$\gamma^2 \delta_{\mathbf{k}} = \mathrm{trace}[\mathbf{\Phi_k}\mathbf{Q}_1] + \mathrm{trace}[\mathbf{\Lambda_k}\mathbf{S}_1] \tag{26}$$

and

$$\begin{bmatrix} \mathbf{Q}_1 & \mathbf{C}_c\mathbf{h} \\ \mathbf{h}^T\mathbf{C}_c^T & 1 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} \mathbf{S}_1 & \mathbf{C}_s\mathbf{h} \\ \mathbf{h}^T\mathbf{C}_s^T & 1 \end{bmatrix} \succeq 0, \tag{27}$$

then the inequality (23) holds. (In (26), $\delta_{\mathbf{0}} = 1$ and $\delta_{\mathbf{k}} = 0$ if $\mathbf{k} \neq 0$.)

Proof. Relation (26) says that the matrices $\mathbf{Q}_1$ and $\mathbf{S}_1$ are a Gram pair associated with $R_1(\mathbf{z}) \stackrel{\Delta}{=} \gamma^2$. Using Schur complements and the notations (11) and (25), the inequalities (27) are equivalent to (24). Hence, according to Lemma 1, the polynomial $R(\omega) \stackrel{\Delta}{=} \gamma^2 - |H(\omega)|^2$ is sum-of-squares and thus nonnegative. It results that (23) holds. ■

5. APPLICATIONS AND RESULTS

Theorems 2 and 3 can be used as sum-of-squares relaxations for solving certain optimization problems. In such relaxations, a positive polynomial is replaced by a sum-of-squares with possibly larger degree. Using the Gram parameterization (4) this kind of relaxations has been used for solving several types of problems:

- multidimensional systems stability test [5]
- design of linear or nonlinear phase 2-D FIR filters [6]
- design of approximately linear phase 2-D IIR filters [4].

Parameterization	Order n						
	4	6	8	10	12	14	16
trace (4)	0.36	1.1	5.4	18	65	270	540
Gram pair (19)	0.34	0.8	2.7	8.2	27	63	110

Table 1: Times, in seconds, for finding the minimum value of 2D trigonometric polynomials using two parameterizations.

The relaxations are SDP problems using (4) (or a BRL resulted from it). All experiments have shown that the solution of the relaxed problem is practically optimal even if the degree of the relaxation is equal to the degree of the initial polynomial. The purpose of the LMIs (19) and (26)–(27) is to replace (4) and its by-products. The advantage is a reduction of complexity, due to the smaller size of the parameter (Gram pair) matrices. For space reasons, we present here the simplest problem with positive polynomials: the computation of the minimum value.

Let $R(\mathbf{z})$ be a trigonometric polynomial (1). We want to compute its minimum value on $\mathbb{T}^d$:

$$\mu^\star = \min_{\omega\in[-\pi,\pi]^d} R(\omega). \tag{28}$$

The problem is NP-hard. Trivially, we express it as a problem with nonnegative polynomials:

$$\begin{array}{rll} \mu^\star = & \max\limits_{\mu} & \mu \\ & \text{subject to} & R(\omega)-\mu \geq 0, \quad \forall\omega\in[-\pi,\pi]^d \end{array} \tag{29}$$

The relaxed form is

$$\begin{array}{rll} \tilde{\mu}^\star = & \max\limits_{\mu} & \mu \\ & \text{s.t.} & R(\mathbf{z})-\mu \ \text{ is sum-of-squares} \end{array} \tag{30}$$

Theoretically, the relaxed solution obeys to $\tilde{\mu}^\star \leq \mu^\star$, but in practice (for random polynomials) the equality is always obtained. For example, the relaxation (30) appears in a multidimensional system stability test [5]. The condition that $R(\mathbf{z})-\mu$ is sum-of-squares can be parameterized either by (4) or by (19). The Gram matrix $\mathbf{X}$ from (4) is $N\times N$, where N is given by (5). The sizes of the Gram pair matrices from (19) are approximately $N/2\times N/2$, see the discussion from section 3.2. An asymptotic complexity analysis (for which there is not enough space here, but which is straightforward), tell that we can expect a speed-up of 2–4 when solving (30) using the Gram pair parameterization (19) instead of (4). However, the analysis does not take sparseness into account. So, experiments are necessary to determine the speed-up.

The SDP programs implementing (30) with the two parameterizations, (4) and (19), have been written with the library SeDuMi [12] and run on a PC with a Pentium IV processor at 1GHz; they are available from the author by e-mail request. Table 1 contains the times needed for obtaining the solution. The orders of the polynomials are $\mathbf{n} = (n,n)$ (i.e. $d=2$). The Gram pair parameterization leads to a twice faster solution for $\mathbf{n} = (8,8)$ (in this case the size of the Gram matrix is $N=81$, while the sizes of the Gram pair matrices are $N_c=41$, $N_s=40$). The speed-up grows to about 5 for $\mathbf{n}=(16,16)$.

6. CONCLUSIONS AND FURTHER WORK

We have presented a parameterization of sum-of-squares trigonometric polynomials using two (a Gram pair) positive semidefinite matrices. This parameterization can be used instead of the Gram parameterization, with faster solutions. It will be interesting to compare the current parameterization with the generalization of the results from [11]. Also, we have given a Bounded Real Lemma for polynomials with positive orthant support. This BRL can be used in applications like the design of 2-D nonlinear phase FIR filters, 2-D IIR filters and 2-D deconvolution.

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