MULTIUSER DETECTION USING RANDOM-SET THEORY

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ABSTRACT
In mobile multiple-access communications, not only the location of active users, but also their number varies with time. In typical analyses, multiuser detection theory has been developed under the assumption that the number of active users is constant and known at the receiver, and coincides with the maximum number of users entitled to access the system. This assumption is often overly pessimistic, since many users might be inactive at any given time, and detection under the assumption of a number of users larger than the real one may impair performance.

This paper assumes a dynamic environment where users are continuously entering and leaving the system, and takes a general approach to the problem of identifying active users and estimating their parameters and data. Our goal is to lay the foundation of multiuser detection theory in an environment where the number and the parameters of active users are unknown at the receiver, and in addition may change from one observation time to the next following a known dynamic model.

Using Random-Set Theory, we develop the tools that are needed for data detection in addition to parameter estimation, whereby a dynamic model for the evolution of parameters has been selected. Application of this theory allows Bayesian-filter equations to be written, which describe the evolution with time of the optimum causal multiuser detector.

We illustrate this theory through a simple example of application, consisting of the detection of the number and identity of active interferers and of the data they carry.

1. INTRODUCTION

We assume signal transmission over a common channel (specifically, we consider CDMA for simplicity’s sake). Let $s(x_i^{(0)})$ denote the signal transmitted by the reference user at discrete time $t$, $t = 1, 2, \ldots$, and $s(x_i^{(i)})$, $i = 1, \ldots, K - 1$, the signals that may be transmitted at the same time by $K - 1$ interferers. Each signal has in it a number of known parameters, reflected by the known function $s(\cdot)$, and a number of random parameters, summarized by $x_i^{(i)}$. The index $i$ reflects the identity of the user, and is typically associated with its signature. The observed signal at time $t$ is a sum of the users active at time $t$, and of a stationary random noise $z_t$, i.e.:

$$y_t = \sum_{x_i^{(i)} \in X_t} s(x_i^{(i)}) + z_t \quad (1)$$

where $X_t$ is a random set.
Once the above densities are made available, they are used in function scribing the receiver, denoted $y$, for example in the form:

$$y_t = s(x_t^{(0)}) + \sum_{x_t^{(1)} \in X_t} s(x_t^{(1)}) + z_t$$

(2)

where now the random set $X_t$ varies in the space $S = \{1, \ldots, K-1\}$ or $S = \{1, \ldots, K-1\} \times \{\pm 1\}$ whether only user identities, or user identities and data are to be detected.

At the basis of RST is the concept of belief function of a random set $X$. This is defined as

$$\beta_X(C) \triangleq P(X \subset C)$$

where $C$ is a subset of an ordinary multiuser state space: $C \subset S$. The density of the belief function is defined as its "set derivative" (this is a generalized Radon-Nikodým derivative). Set derivatives can be computed by using an RST "toolbox," and the resulting densities carry, from a practical viewpoint, the relevant properties of standard density functions of probability theory [3].

The ingredients necessary for application of RST to multiuser detection are the following:

1. The belief density $f(y_t \mid X_t)$. This follows from the channel model and the measuring method.
2. The belief density $f(X_t \mid X_{t-1})$. This follows from the model of the dynamics of the set of active users. The main assumption here is that $\{X_t\}_{t=1}^\infty$ forms a random set sequence with the Markov property, i.e., such that $X_t$ depends on its past only through $X_{t-1}$.

Once the above densities are made available, they are used in the Bayesian filter recursions:

$$f(X_t \mid y_{1:t-1})$$

$$= \frac{f(X_t \mid X_{t-1}) f(X_{t-1} \mid y_{1:t-2}) \delta X_{t-1}}{f(y_t \mid X_{t-1}) f(X_{t-1} \mid y_{1:t-2})}$$

(3)

$$f(y_t \mid X_t) \propto f(y_t \mid X_t) f(X_t \mid y_{1:t-1})$$

which allow one to generate recursively the estimates of $X_t$, for example in the form

$$\tilde{X}_t = \text{arg max}_{X_t} f(X_t \mid y_{1:t})$$

The integral in (3) is a "set integral," the inverse of the set derivative.

With our channel model, the receiver detects only a superposition of interfering signals. Thus, the random set describing the receiver, denoted $y_t$, has conditional density function:

$$f(y_t \mid X_t) = f_s(y_t - \sigma(X_t))$$

(4)

where $f_s(\cdot)$ is the density function of the additive noise, and

$$\sigma(X_t) \triangleq \sum_{x_t^{(1)} \in X_t} s(x_t^{(1)})$$

(5)

Assuming the noise to be Gaussian, we have

$$f_s(y_t - \sigma(X_t)) \propto \exp\{-||y_t - \sigma(X_t)||^2/\sigma_0^2\}$$

Thus, the estimate of $X_t$ is performed by minimizing, over $X_t$, the function

$$m(X_t) \triangleq ||y_t - \sigma(X_t)||^2 - \epsilon(X_t)$$

where $\epsilon(X_t) = N_0 \ln f(X_t \mid y_{1:t-1})$. The first term is the Euclidean distance between the observation and the sum of the interfering signals. This alone would be used in ML detection. The second term in the RHS of the above can be viewed as a correction term, coming from the uppermost step of iterations, and reflecting the influence on $X_t$ of its past history. This plays the role of a priori information to be used in maximum a posteriori decisions, and its consideration and evaluation is the main point of this paper.

2. EXAMPLE OF APPLICATION: DETECTION OF ACTIVE USERS

Assume now the specific situation of a DS-CDMA system with signature sequences of length $L$ and additive white Gaussian noise. At discrete time $t$, we may write, for the sufficient statistics of the received signal,

$$y_t = R A b_t(X_t) + z_t, \quad t = 1, \ldots, T$$

(6)

where $X_t$ is now the random set of all active users, $R$ is the $L \times L$ correlation matrix of the signature sequences (assumed to have unit norm), $A$ is the diagonal matrix of the users’ signal amplitudes, the vector $b_t(X_t)$ has nonzero entries in the locations corresponding to the active-user identities described by the components of $X_t$, and $z_t \sim N(0, N_0/2 I)$ is the noise vector, with $N_0/2$ the power spectral density of the received noise.

Throughout this paper we assume that the only unknown signal quantities may be the identities of the users and their data. Specifically, we may distinguish four cases in our context:

1. **Static channel, unknown identities, known data.** This corresponds to a training phase intended at identifying users, and assumes that the user identities do not change during transmission. In this case we write $X$ in lieu of $X_t$.

2. **Static channel, unknown identities, unknown data.** This may correspond to a tracking phase following 1 above. We write again $X$ in lieu of $X_t$ and assume that $X$ contains the whole transmitted data sequence.

3. **Dynamic channel, unknown identities, known data.** This corresponds to identification of users preliminary to data detection (which, for example, may be based on decorrelation).

4. **Dynamic channel, unknown identities, unknown data.** This corresponds to simultaneous user identification and data detection in a time-varying environment.

If we assume that, at every discrete time instant, only one binary antipodal symbol is transmitted, trained acquisition corresponds to the transmission of known bit streams, and hence to $S = \{0, \ldots, K-1\}$, while in untrained acquisition user identification and data detection should be performed jointly, which corresponds to $S = \{0, \ldots, K-1\} \times \{\pm 1\}$.

Consider now the construction of a dynamic model for $X_t$. We assume that from $t-1$ to $t$ some new users become active and some old users become inactive. We write

$$X_t = S_t \cup N_t$$

(7)
where \( S_t \) is the set of surviving users still active from \( t-1 \), and \( N_t \) is the set of new users becoming active at \( t \). The condition \( X_{t-1} \cap N_t = \emptyset \) is forced, i.e., a user ceasing transmission at time \( t-1 \) cannot re-enter the set of active users at time \( t \).

For the sake of clarity, here we limit ourselves to the construction of a dynamic model for \( X_t \) with trained acquisition. Suppose that there are \( n \) active users at \( t-1 \), with 
\[
X_{t-1} = \{x_{t-1}^{(1)}, \ldots, x_{t-1}^{(n)}\}.
\]
Then we may write, for the set of surviving users,
\[
S_t = \bigcup_{i=0}^{K-1} x_{t}^{(i)}
\]
where \( x_{t}^{(i)} \) denotes either an empty set (a user has become inactive) or the singleton \( \{x_{t}^{(i)}\} \). Let \( \mu \) denote the “persistence” probability, i.e., the probability that a user survives from \( t-1 \) to \( t \). We obtain, for the conditional density of \( S_t \) given that \( X_{t-1} = B \):
\[
f_{S_t|X_{t-1}}(C | B) = \mu^{|C|}(1 - \mu)^{|B| - |C|}, \quad C \subseteq B
\]
while it is 0 otherwise.

For new users, we assume again a binomial birth process with parameter \( \alpha \). Since \( K \) is the maximum user number, we have:
\[
f_{N_t|X_{t-1}}(C | B) = \alpha^{|C|}(1 - \alpha)^{|B| - |C|}, C \cap B = \emptyset
\]
Finally, assuming that births and deaths of users are conditionally independent given \( X_{t-1} \), the generalized convolution operation ruling the pdf of the union of independent random sets [1] becomes, under our assumptions:
\[
f_{X_t|X_{t-1}}(C | B) = \sum_{W \subseteq C} f_{S_t|X_{t-1}}(W | B)f_{N_t|X_{t-1}}(C \setminus W | B)
\]
\[
= f_{S_t|X_{t-1}}(C \cap B)f_{N_t|X_{t-1}}(C \setminus (C \cap B))
\]
Untrained acquisition can be dealt with similarly, provided that the conditional densities in (9)-(10) are multiplied by a factor depending on \( |C| \) to account for the data priors.

Two alternative strategies can at this point be conceived to estimate \( X_t \), whether trained or untrained situations are considered. The former one relies upon implementing the Bayes recursions outlined above, thus defining a causal set-sequence estimator. An alternative approach could be to consider the likelihood of the set sequence \( X_1, \ldots, X_T \), i.e.: 
\[
f(X_1, \ldots, X_T | y_{1:T})
\]
\[
\propto f(y_{1:T} | X_1, \ldots, X_T)f(X_1)\prod_{t=2}^{T} f(X_t | X_{t-1})
\]
where the Markov nature of the model has been exploited. Since under both trained and untrained acquisition the unknown sets may take on a finite number of configurations, maximizing (13) simply amounts to determining a maximum-likelihood path in a trellis of depth \( T \): the state space has cardinality \( 2^K \) for trained acquisition, and \( 3^K \) for untrained acquisition, and the maximization can be undertaken in both cases through a Viterbi algorithm.

\*\*Implicit in the above is the need of assigning a density \( f(X_1) \), which obviously depends on the prior information as to the channel state at the beginning of the transmission.

3. RESULTS

We first illustrate the advantages of joint channel sensing and user demodulation by considering the case of a static CDMA channel with 7 users and processing gain 7; it is further assumed that user “1” is active with probability one, while the other users may be active or not, and the number of active interferers being a uniform discrete random variable. Fig. 1 shows the bit error probability of a random-set-based detector. For comparison purposes, we also show the performance of a classical ML receiver assuming that all of the users are active, and the single-user bound: for all receivers, the spreading codes are \( m \)-sequences. The plots show that joint channel sensing and data demodulation prevents the performance impairment incurred by traditional multiuser systems under unknown channel occupancy.

![Figure 1: Bit error probability of the reference user in a multiuser system with 3 users, independently active with probability \( \alpha \).](image)

Fig. 2 compares again “classic” ML multiuser detection [4], which assumes that all users are simultaneously active and ML detection based on RST, which detects simultaneously the number of active users and the data of the reference user. The ordinate shows the bit error probability of the reference user in a system with 3 users transmitting binary antipodal signals, different active-user probabilities (\( \alpha = 0.1, 0.5, \) and 1), spreading sequences consisting of Kasami sequences with length 15, and perfect power control (and hence equal received powers from all users). The channel is Gaussian and static. The single-user bound is also shown as a reference. This figure was commented upon in Section 1.

We next consider the situation of a dynamic channel where both the user identities and their data are to be estimated. In this new situation no user is active with probability one, and the common persistence probability is \( \mu = 0.8 \), while \( \alpha = 0.2 \). We also assume a frame of \( T = 10 \) signaling intervals, and we consider the following situations: a) Estimation of the set \( X_1 \), under both noncausal (“Viterbi”) and causal (“Bayes”) strategies; b) Estimation of the set \( X_{10} \), under both non-causal and causal strategies. The quantity on the vertical axis is the “bit sequence probability,” i.e., the probability that at some \( t \) the estimated and the true bit stream...
conditions. In addition, we have developed Bayes-filtering elements. We have used a probability theory, called random-parameters to be estimated has a random number of random active interferers is itself a random variable, the set of pa-
signal parameters in a CDMA system. Since the number of
under a dynamic scenario.
trained and untrained systems achieve close performances
as they should. Additionally, some theoretical
developments, not shown here for want of space, show that
RST appears as a most promising tool to achieve fully adaptive receivers.

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REFERENCES

APPENDIX
This appendix describes, mostly in a qualitative fashion, the fundamentals of Random-Set Theory. For a rigorous approach and for additional details, see [1–3, 5]. A finite random set is a mapping \( X : \Omega \to \mathcal{F}(\mathbb{S}) \) from the sample space \( \Omega \) to the collection of closed sets of the space \( \mathbb{S} \), with \( |X(\omega)| < \infty \) for all \( \omega \in \Omega \). Here the space \( \mathbb{S} \) of finite random sets is assumed to be the hybrid space \( \mathbb{S} = \mathbb{R}^d \times U \), the direct product of the \( d \) dimensional Euclidean space \( \mathbb{R}^d \) and a finite discrete space \( U \). The elements of \( \mathbb{S} \) characterize the users’ parameters, which we categorize as continuous (\( d \) real numbers) and discrete (for example, the users’ signatures and their information data). An element of \( \mathbb{S} \) is the pair \((v, u)\), \( v \) a \( d \)-dimensional real vector, and \( u \in U \). The space \( \mathbb{S} \) is endowed with a topology obtained as the product of the Euclidean topology in \( \mathbb{R}^d \) and the discrete topology in \( U \).

The belief function of a finite random set \( X \) is defined as

\[
\beta_X(C) = \mathbb{P}(X \subseteq C)
\]

where \( C \) is a closed subset of \( \mathbb{S} \). The belief function characterizes the probability distribution of a random finite set \( X \), and allows the construction of a density function of \( X \) through the definition of a set integral and a set derivative.
Let \( C(S) \) denote the collection of closed subsets of \( S \). The set derivative of a set function \( F : C(S) \to [0,\infty) \) at a point \( x \in S \) is defined as

\[
\frac{\delta F}{\delta x}(S) \triangleq \lim_{m(\Delta_x) \to 0} \frac{F(S \cup \Delta_x) - F(S)}{m(\Delta_x)}
\]

where \( m(\cdot) \) denotes the hybrid Lebesgue measure, i.e. the product of the ordinary measure in \( \mathbb{R}^d \) and of the counting measure. Thus, the belief density of the random set \( X \) is given by

\[
f_X(X) = \delta \beta_X(\emptyset) (15)
\]

Let \( f \) denote a function defined by

\[
f(X) = \frac{\delta F}{\delta X}(\emptyset)
\]

The set integral of \( f \) over the closed subset \( S \subseteq \mathbb{S} \) is given by

\[
\int_S f(X) \delta X = \sum_{k=1}^{\infty} \frac{1}{k!} \int_S f(\{x_1, \ldots, x_k\}) m(x_1) \cdots m(x_k) (16)
\]

where \( f(\{x_1, \ldots, x_k\}) = 0 \) if \( x_1, \ldots, x_k \) are not distinct (and hence the set has less than \( k \) elements). Since we are dealing with finite random sets, the summation above contains only a finite number of terms.

The special case \( d = 0 \) (which corresponds to making \( S \) a discrete finite set) reduces the set integral to

\[
f(\emptyset) + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\{x_1, \ldots, x_k\} \subseteq \mathbb{S}} f(\{x_1, \ldots, x_k\}) (17)
\]

since in this case the hybrid Lebesgue measure reduces to the counting measure, and the Lebesgue integrals in (16) become summations.

Set derivatives and set integrals turn out to be the inverse of each other. The following generalized fundamental theorem of calculus holds:

\[
f(X) = \frac{\delta F}{\delta X}(\emptyset) \iff F(S) = \int_S f(X) \delta(X) (18)
\]

By using the above result, belief functions and belief densities can be derived from one another.