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LOWER LIMIT CYCLE BOUNDS FOR NARROW TRANSITION BAND DIGITAL FILTERS

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ABSTRACT

A procedure for obtaining tighter bounds on zero-input limit cycles is presented. The determined new bounds are applicable to digital filters of arbitrary order described in state-space formulation and implemented with fixed-point arithmetic. In most filters, we obtain smaller bounds through this new algorithm easy to implement and to execute in a very short computer time. The bounds obtained for narrow transition band digital filters are far lower than those corresponding to classical procedures, yielding enormous computation savings to complete an exhaustive search. Simulation results are presented in different tables that show the validity of the proposed theory.

1. INTRODUCTION

Limit cycle oscillations are a very common finite word-length phenomenon which can strongly degrade the performance of different systems [1]-[9]. As examples of these unwanted oscillations, are those disturbing sigma-delta modulators in audio systems and third-generation (3-G) cellular technology. Limit cycles (defined as a sequence of L output bits which repeats itself indefinitely) are one of the most common sources of instabilities [2]-[4]. Also, in synchronous code division multiple access communication systems, limit cycles have been pointed out as the dominant source of performance degradation [5]. For the above reasons, new procedures for detecting, bounding and suppressing limit cycles are required.

As regards digital filters, several exhaustive search algorithms have been developed to analyse if the resulting filter is globally asymptotically stable [6]-[9]. However, the time required for completing the exhaustive search, testing the convergence to the zero vector of all the initial state vectors under the bound condition, may become extremely long. This time increases rapidly when the bound grows, so it is very important to obtain bounds as tight as possible.

In this work, we present a new formulation which leads to lower bounds for limit cycles, resulting in significant time savings for the exhaustive search. Our approach is different since a new matrix way of bounding on the maximum amplitude for limit cycles is proposed. In this sense, the main contribution of this work is double: (a) The proposed method leads, in most occasions, to lower and tighter bounds, mainly in narrow-band filters; (b) The new bounds can be computed in negligible time and the algorithms for obtaining them are quite easy to program, resulting in significant time savings for the exhaustive search. The paper focus on narrow transition band digital filters and compare our results with those obtained working with the bounds previously published by other authors. However, as the whole formulation is based on the state-space description, ongoing research can be carry out to establish formal links with other applications, such as the sigma-delta limit cycles problem, or oscillations in digital control systems.

The rest of the paper is organized as follows. In Section 2, we discuss the problem description. In Section 3, we deduce the bounds for the limit cycle amplitude. The validity of the theory is illustrated in Section 4 through several examples. We compare the obtained results with several conventional algorithms, and we also show how the proposed bounds are tighter than other classical bounds. Finally, we summarize our conclusions.

Notations and Definitions

a) Bold-typed letters indicate vectors (lower case) and matrices (upper case).

b) $\Re$ denotes the set of real numbers.

c) $\mathbf{x} \in \Re^m$ is a m-dimensional vector in the form $\mathbf{x} = \{x_i\} \quad i = 1,...,m$.

d) $|\mathbf{x}|$ is defined as $\{|x_i|\} \quad i = 1,...,m$, and represents the vector of absolute values of entries of **x**.

e) If $\mathbf{x}, \mathbf{y} \in \Re^m$, the notation $\mathbf{x} < \mathbf{y}$ means $x_i < y_i$ for all $i = 1,...,m$.

f) In the same way, $\mathbf{A} \in \Re^{m\,x\,m}$ is a m x m matrix in the form $\mathbf{A} = \{a_{ij}\} \quad i = 1,...,m$ and $j = 1,...,m$.

g) $|\mathbf{A}|$ is $\{|a_{ij}|\} \quad i = 1,...,m$ and $j = 1,...,m$, representing the matrix of absolute values of entries of **A**.

h) If $\mathbf{A},\mathbf{B}\in\Re^{m\,x\,m}$, the notation $\mathbf{A}<\mathbf{B}$ means $a_{ij}<b_{ij}$ for all $i=1,...,m$ and $j=1,...,m$.

i) $\mathbf{0}$ is the zero matrix

2. PROBLEM DESCRIPTION

The state-space description of discrete-time system is well established [1]. Let us consider a digital filter of the form

$$\mathbf{x}(k+1)=\mathbf{A}\cdot\mathbf{x}(k)+\mathbf{B}\cdot\mathbf{u}(k)$$

$$\mathbf{y}(k)=\mathbf{C}\cdot\mathbf{x}(k)+\mathbf{D}\cdot\mathbf{u}(k)\,.$$

where $\mathbf{u}$ and $\mathbf{y}$ are the input and output of the system, respectively. We only consider a system whose zero input operation under finite word length conditions can be represented in the form

$$\mathbf{x}(k+1)=Q[\mathbf{A}\cdot\mathbf{x}(k)]\,, \qquad (1)$$

where $\mathbf{x}(k)\in\Re^{m}$ is the state vector of the system at k-instant and $\mathbf{A}\in\Re^{m\,x\,m}$ is the constant state-space matrix describing the system under zero-input condition. The function $Q[\cdot]$ represents the non-linear quantization operation which satisfies

$$\left|\,x-Q[x]\,\right|\le\delta\cdot q, \qquad \forall x\in\Re$$

where q is the quantization step size, and δ is the maximum normalized quantization error (δ=1 for two's complement truncation and sign-magnitude truncation, and δ=0.5 for sign-magnitude rounding).

We can rewrite (1) as follows:

$$\mathbf{x}(k+1)=\mathbf{A}\cdot\mathbf{x}(k)+\mathbf{e}(k)\,,$$

where $\mathbf{e}(k)\in\Re^{m}$ is the quantization error vector. Assuming multiplication results can be stored with full precision (double-length accumulator), it holds that

$$\mathbf{e}(k)\le\delta\cdot q\cdot\mathbf{J}_m\,,$$

being $\mathbf{J}_m$ a column vector with all elements equal to 1.

The state $\mathbf{x}(k)$ reached in k steps or iterations from an initial state $\mathbf{x}(0)$ can be expressed in the following form:

$$\mathbf{x}(k)=\mathbf{A}^{k}\cdot\mathbf{x}(0)+\sum_{\ell=0}^{k-1}\mathbf{A}^{\ell}\cdot\mathbf{e}(k-\ell-1)\,.$$

3. BOUNDS ON THE LIMIT CYCLE AMPLITUDE

In this section, we deduce several bounds for the limit cycle amplitude. In this sense, the absolute value of $\mathbf{x}(k)$ can be bounded as follows:

$$|\mathbf{x}(k)|\le\left|\mathbf{A}^{k}\cdot\mathbf{x}(0)\right|+\delta\cdot q\cdot\sum_{\ell=0}^{k-1}\left|\mathbf{A}^{\ell}\right|\cdot\mathbf{J}_m\,.$$

Assuming the considering system is stable, $\mathbf{A}^{k}\to 0$ as $k\to\infty$, and defining

$$\mathbf{S}_A=\sum_{\ell=0}^{\infty}\left|\mathbf{A}^{\ell}\right|, \qquad (2)$$

we have that

$$\lim_{k\to\infty}|\mathbf{x}(k)|\le\delta\cdot q\cdot\mathbf{S}_A\cdot\mathbf{J}_m\,.$$

From the above expressions, we can establish the following upper bound $\mathbf{M}_A$ on the state vector during a limit cycle:

$$\mathbf{M}_A=\delta\cdot q\cdot\mathbf{S}_A\cdot\mathbf{J}_m\,. \qquad (3)$$

The biggest problem for obtaining $\mathbf{M}_A$ lies in calculating $\mathbf{S}_A$ matrix. As we can see in (2), we need to add up to infinite terms but there is no a general way of computing the exact value of $\mathbf{S}_A$.

In order to approximate $\mathbf{S}_A$, and assuming that the system poles are different (this condition holds for Butterworth, Chebyshev, elliptic and Bessel IIR filters, among others), we can diagonalize the matrix $\mathbf{A}$ as

$$\mathbf{A}=\mathbf{T}\cdot\mathbf{D}\cdot\mathbf{T}^{-1},$$

where $\mathbf{D}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ on its main diagonal :

$$\mathbf{D}=diag\{\lambda_1,\lambda_2,...,\lambda_m\}\,,$$

and $\mathbf{T}$ is the matrix of the eigenvectors of $\mathbf{A}$. Now we can express the powers of $\mathbf{A}$ in the following form:

$$\mathbf{A}^{\ell}=\mathbf{T}\cdot\mathbf{D}^{\ell}\cdot\mathbf{T}^{-1},$$

where

$$\mathbf{D}^{\ell}=diag\left\{\lambda_1^{\ell},\lambda_2^{\ell},...,\lambda_m^{\ell}\right\}.$$

Considering that

$$\left|\mathbf{A}^{\ell}\right|\le|\mathbf{T}|\cdot\left|\mathbf{D}^{\ell}\right|\cdot\left|\mathbf{T}^{-1}\right|,$$

and replacing $\left|\mathbf{A}^{\ell}\right|$ in (2), we obtain a sum $\mathbf{S}_D\ge\mathbf{S}_A$ in the form

$$\mathbf{S}_D=\sum_{\ell=0}^{\infty}|\mathbf{T}|\cdot\left|\mathbf{D}^{\ell}\right|\cdot\left|\mathbf{T}^{-1}\right|=|\mathbf{T}|\cdot\sum_{\ell=0}^{\infty}\left|\mathbf{D}^{\ell}\right|\cdot\left|\mathbf{T}^{-1}\right|, \qquad (4)$$

where

$$\left|\mathbf{D}^{\ell}\right| = diag\left\{\left|\lambda_1^{\ell}\right|, \left|\lambda_2^{\ell}\right|, \ldots, \left|\lambda_m^{\ell}\right|\right\},$$

and

$$\sum_{\ell=0}^{\infty}\left|\mathbf{D}^{\ell}\right| = diag\left\{\frac{1}{1-|\lambda_1|}, \frac{1}{1-|\lambda_2|}, \ldots, \frac{1}{1-|\lambda_m|}\right\}.$$

From (4) and replacing $\mathbf{S}_A$ by $\mathbf{S}_D$ in (3) we obtain the bound $\mathbf{M}_D$ on a limit cycle $(\mathbf{M}_D \geq \mathbf{M}_A)$ as:

$$\mathbf{M}_D = \delta \cdot q \cdot \mathbf{S}_D \cdot \mathbf{J}_m.$$

This second bound is equivalent to that of presented in [6] by Yakowitz and can be easily determined once the **A** matrix has been diagonalized, but, as we demonstrate below, it can be substantially tightened.

In order to obtain a bound lower than $\mathbf{M}_D$, we define the sum of the first n terms of $\mathbf{S}_A$ as

$$\mathbf{S}_A(n) = \sum_{\ell=0}^{n-1}\left|\mathbf{A}^{\ell}\right|,$$

and the sum of the terms n, n+1, n+2, … of $\mathbf{S}_D$, in the following form:

$$\mathbf{R}_D(n) = |\mathbf{T}| \cdot \sum_{\ell=n}^{\infty}\left|\mathbf{D}^{\ell}\right| \cdot \left|\mathbf{T}^{-1}\right|. \quad (5)$$

Evaluating the sums in (5), we get

$$\mathbf{R}_D(n) = |\mathbf{T}| \cdot diag\left\{\frac{|\lambda_1|^n}{1-|\lambda_1|}, \frac{|\lambda_2|^n}{1-|\lambda_2|}, \ldots, \frac{|\lambda_m|^n}{1-|\lambda_m|}\right\} \cdot \left|\mathbf{T}^{-1}\right|. \quad (6)$$

The procedure consists of replacing the last terms of (2) by the corresponding ones in (4). We aim to obtain a closer approximation of $\mathbf{S}_A$ (better than $\mathbf{S}_D$), and therefore, a tighter bound. In this sense, we can construct $\mathbf{S}(n)$ as

$$\mathbf{S}(n) = \mathbf{S}_A(n) + \mathbf{R}_D(n), \quad (7)$$

where

$$\mathbf{S}_A \leq \mathbf{S}(n) \leq \mathbf{S}_D,$$

and, obviously, $\mathbf{S}(n) \to \mathbf{S}_A$ as $n \to \infty$.

Now, we can define the new bound $\mathbf{M}(n)$ as

$$\mathbf{M}(n) = \delta \cdot q \cdot \mathbf{S}(n) \cdot \mathbf{J}_m. \quad (8)$$

Clearly, it holds that

$$\mathbf{M}_A \leq \mathbf{M}(n) \leq \mathbf{M}_D.$$

We have that $\mathbf{M}(n) \to \mathbf{M}_A$ as $n \to \infty$.

Theoretically, $\mathbf{M}(n)$ equals $\mathbf{M}_A$ for n infinite, but, in practice, the difference $\mathbf{D}(n)$=$\mathbf{M}(n)$-$\mathbf{M}_A$ becomes insignificant for relatively small values of n. We must take into account that $\mathbf{D}(n)$ satisfies

$$\mathbf{D}(n) < \delta \cdot q \cdot \mathbf{R}_D(n) \cdot \mathbf{J}_m,$$

and that the value of the elements of the diagonal matrix in (6) decrease very fast whenever n increases.

For simplicity, bounds are usually expressed as an integer multiple, $\hat{\mathbf{M}}(n) \in Z^m$, of the quantization step size. That is

$$\hat{\mathbf{M}}(n) = \left\lceil \mathbf{M}(n)\big|_{q=1} \right\rceil,$$

where $\lceil\cdot\rceil$ denotes the rounding to the next larger integer. In this case, we can determine the smallest value of n that makes $\hat{\mathbf{M}}(n)$ minimum as follows. From (7) and (8), we have

$$\hat{\mathbf{M}}(n) = \lceil \boldsymbol{\alpha}(n) + \boldsymbol{\beta}(n) \rceil, \quad (9)$$

where

$$\boldsymbol{\alpha}(n) = \delta \cdot \mathbf{S}_A(n) \cdot \mathbf{J}_m$$

and

$$\boldsymbol{\beta}(n) = \delta \cdot \mathbf{R}_D(n) \cdot \mathbf{J}_m.$$

In order to calculate the final value of $\hat{\mathbf{M}}(n)$, we evaluate expression (9) increasing n. Initially, as n grows, $\hat{\mathbf{M}}(n)$ decreases until a minimum. Once this value is reached, no more reduction is obtained for higher values of n. The smallest value of n that makes $\hat{\mathbf{M}}(n)$ minimum can be determined from the following relation

$$\lceil \boldsymbol{\alpha}(n) \rceil \leq \hat{\mathbf{M}}(n) \leq \lceil \boldsymbol{\alpha}(n) + \boldsymbol{\beta}(n) \rceil.$$

Whenever n increases, $\alpha(n)$ increases too, but the sum $\alpha(n)+\beta(n)$ decreases. Therefore, the smallest value of n, say $n=n_0$, that satisfies

$$\lceil \boldsymbol{\alpha}(n) \rceil = \lceil \boldsymbol{\alpha}(n) + \boldsymbol{\beta}(n) \rceil \quad (10)$$

is the smallest value of n that minimizes $\hat{\mathbf{M}}(n)$. Calling **M** to the minimum value of $\hat{\mathbf{M}}(n)$, finally, we have

$$\mathbf{M} = \hat{\mathbf{M}}(n)\big|_{n=n_0}.$$

To sum up, the final bound **M**, expressed as an integer multiple of the quantization step size, is achieved by evaluating (9) with $n=n_0$, where n_0 is the smallest value of n which satisfies (10). The new bound presented is very simple to compute, tighter than other classical bounds, and can be obtained in a very short time, even for large values of n.

4. NUMERICAL RESULTS AND DISCUSSION

We have designed a set of 7 digital filters to compare the bounds derived along these lines with those from previous studies. Filters have been implemented using MATLAB with the values of bands and ripples given in the first column of the tables 1 and 2.

FILTER	STATE	GMN	YP	NB
1 *Chebyshev Type I* Wp=0.04; Ws=0.05; Rp=0.2; Rs=25;	x1	1673	99	45
	x2	1673	99	64
	x3	7382	778	238
	x4	11335	1052	363
	x5	15418	2536	655
	x6	26455	3682	972
	x7	35931	9679	3048
	x8	47259	11890	3568
2 *Chebyshev Type 2* Wp=0.04; Ws=0.05; Rp=0.2; Rs=25;	x1	4378	24	15
	x2	4378	24	15
	x3	4813	233	145
	x4	4643	229	144
	x5	4858	839	513
	x6	4673	823	510
	x7	4904	3488	2144
	x8	4718	3456	2139
3 *Elliptic* Wp=0.04; Ws=0.05; Rp=0.2; Rs=25;	x1	722	12	12
	x2	839	94	49
	x3	946	96	54
	x4	3078	1904	929
	x5	3372	1938	959

Table 1 *Comparison of Limit Cycle Bounds under two's complement truncation for narrow transition band, low-pass filters*

Table 1 presents the results corresponding to three narrow transition band low-pass filters working with three different methods to calculate the bounds. It shows the zero-input limit cycle bounds, corresponding to each delay element (x_1, x_2, …) of the three filters analysed. The abbreviation YP refers to *Yakowitz and Parker* bounds presented in [6], GMN indicates the *general bounds* described in [7] by Green and Turner, the last column named NB holds the *new bounds* obtained working with the new procedure presented in this work. In this table, we can see that the results corresponding to GMN bounds are always far more pessimistic than NB bounds and can be considerably tighter. Comparing YP and NB bounds, only in the first register of the elliptic filter, the value obtained with YP are equal to the obtained with NB, for the rest of the registers, NB are always smaller. In Table 2, we present the results obtained for 4 narrow transition band band-pass filters, in this case, NB gives better values than GMN and YP for all registers.

FILTER	STATE	GMN	YP	NB

4
Butterworth
`Wp=[0.45 .55];`
`Ws=[0.4 0.6];`
`Rp=0.2;`
`Rs=25;`
	x1	67	6	3

	x2	493	83	7
	x3	493	83	10
	x4	964	224	13
	x5	964	224	18
	x6	1256	396	46
	x7	1256	396	52
	x8	67	6	3
	x9	493	83	7
	x10	493	83	10
	x11	964	224	13
	x12	964	224	18
	x13	1256	396	46
	x14	1256	396	52

5
Chebyshev Type I
`Wp=[0.45 .55];`
`Ws=[0.4 0.6];`
`Rp=0.2;`
`Rs=25;`
	x1	105	39	10

	x2	105	39	13
	x3	191	177	45
	x4	223	196	50
	x5	105	39	10
	x6	105	39	13
	x7	191	177	45
	x8	223	196	50

6
Chebyshev Type 2
`Wp=[0.45 .55];`
`Ws=[0.4 0.6];`
`Rp=0.2;`
`Rs=25;`
	x1	258	30	13

	x2	258	30	13
	x3	277	371	135
	x4	267	366	134
	x5	258	30	13
	x6	258	30	13
	x7	277	371	135
	x8	267	366	134

7
Elliptic
`Wp=[0.45 .55];`
`Ws=[0.4 0.6];`
`Rp=0.2;`
`Rs=25;`
	x1	33	8	4

	x2	49	64	20
	x3	51	65	21
	x4	33	8	4
	x5	49	64	20
	x6	51	65	21

Table 2 *Comparison of Limit Cycle Bounds under rounding for narrow transition band, band-pass filters*

In order to illustrate the computational savings obtained working with the new bounds, we present in Table 3 the number of vectors that is necessary to test to complete the exhaustive search of limit cycles [8].

The exhaustive search consist in testing the convergence to the zero vector of each one of the vectors satisfying the requirements of the bounded absolute value. The set S of initial vectors to test is:

$$S = \left\{ \mathbf{x}(0) \in Z^m \;/\; \left| x_i(0) \right| \le M_i, \;\; i = 1, \ldots, m \right\}.$$

and the exhaustive search requires to test the convergence of N_v initial vectors, where

$$N_v = \prod_{i=1}^{m} \left(2M_i + 1 \right)$$

FILTER	GMN	YP	NB
1	4.1558e+034	.2325e+027	4.5169e+023
2	5.7642e+031	6.8644e+022	1.5553e+021
3	1.9073e+017	1.3467e+013	9.6240e+011
4	9.5507e+042	2.1127e+033	2.0873e+021
5	5.8095e+019	7.5814e+017	2.7158e+013
6	6.2988e+021	4.1068e+018	2.8242e+015
7	4.6676e+011	8.2532e+010	2.5176 e+08

Table 3 *Maximum Number of initial vectors to test in the Exhaustive Search*

Table 3 shows that the new algorithm always yields the tightest bound for the maximum number of initial vectors to test in the exhaustive search. In general, we reduce drastically computational effort, and therefore, the time needed to end the exhaustive search (mainly in high order filters) when we use the new bounds to determine the set S of initial state vectors to test for convergence.

5. CONCLUSIONS

In this work , we have shown that it is possible to obtain a tighter bound for limit cycle that appears in narrow transition band digital filters. The new formulation ensures tighter bounds and also involves the possibility of implementing the exhaustive search algorithm in a really efficient way. The new bounds presented here can obtained in negligible time and with significant time savings comparing with other conventional methods.

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