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A FIXED-POINT SMOOTHING ALGORITHM IN DISCRETE-TIME SYSTEMS WITH CORRELATED SIGNAL AND NOISE

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ABSTRACT

The linear least mean-square fixed-point smoothing problem in discrete-time systems is formulated in the general case where the signal is any nonstationary stochastic process of second order which is observed in the presence of an additive white noise correlated with the signal. Under the only assumption that the correlation functions involved are factorizable kernels, an efficient recursive computational algorithm for the fixed-point smoother is designed. Also, a filtering algorithm is devised.

1. INTRODUCTION

In the last decades, the problem of estimating a random signal observed through a linear mechanism has been a topic widely studied. Specifically, we can find an extensive literature concerning the design of recursive formulas for updating and optimum estimate for some criterion of optimality such as the minimization of the error variance.

In this framework, the (LLMSE) linear least mean-square error estimation problem is reduced to the problem of solving a linear equation, called the Wiener-Hopf equation. The solution to the Wiener-hopf equation, the impulse response function of the optimal estimate, can be determined from the knowledge of the correlation functions of the processes involved. Thus, a method of solution to the LLMSE estimation problems consists in imposing structural assumptions on the properties of the correlation functions involved.

In this line, and under the assumption that the signal is modelled by state-space system, the most known and used LLMSE estimation algorithm is the Kalman filter [1]. Unfortunately, for some applications in many communication problems, it is impossible to specify such state-space models for the signal and then the Kalman filter cannot be applied. Alternatively, a LLMSE estimation algorithm is possible under the only hypothesis that the correlation functions involved are factorizable kernels [see, e.g., [2], [3], [4]]. In fact, since factorizable kernels are suitable for expressing correlation functions of general stationary or nonstationary signal processes, these estimation algorithms will be widely applied.

This paper is concerned with the LLMSE fixed-point smoothing problem using covariance specifications in discrete-time system involving correlated signal and noise. This general estimation problem which includes correlation between the signal and the observation noise is useful in many engineering applications in stochastic control and communications [5]. Then, by assuming that the covariance functions of the signal and noise are factorizable kernels, we get a recursive expression for the LLMSE fixed-point smoother via an imbedding method, where the initial condition is the filter of the signal. The proposed LLMSE fixed-point smoothing algorithm, which includes the recursive formulas for the computation of the filtering estimate, is presented in Theorem 1.

2. PROBLEM FORMULATION

In this section we formulate the discrete LLMSE fixed-point smoothing problem involving correlated signal and observation noise.

Specifically, let $\{x(t_i), t_i \geq t_0\}$ be a signal process which is any real second-order nonstationary stochastic process, with zero-mean and autocorrelation function $R_x(t_i,t_j) = E[x(t_i)x(t_j)]$, for $t_i, t_j \geq t_0$.

We consider that the signal is observed in the presence of an additive white noise and the available observations are given by a stochastic process $\{y(t_i), t_i \geq t_0\}$ through the equation

$$y(t_i) = x(t_i) + v(t_i) \tag{1}$$

where $v(t_i)$ is a zero-mean white noise process with autocorrelation function $E[v(t_i)v(t_j)] = r_i\delta_{ij}$, $r_i > 0$, and correlated with the signal. Let $R_{x_1x_2}(t_i,t_j)$ denote the cross-correlation function between any two processes $x_1(t_i)$ and $x_2(t_j)$.

We assume that the autocorrelation function of the signal and the cross-correlation function between the signal and the observation noise are factorizable kernels of the form

$$\begin{aligned} R_x(t_i,t_j) &= \begin{cases} \mathbf{a}'(t_i)\mathbf{b}(t_j), & t_0 \leq t_j \leq t_i \\ \mathbf{b}'(t_i)\mathbf{a}(t_j), & t_0 \leq t_i \leq t_j \end{cases} \\ R_{xv}(t_i,t_j) &= \begin{cases} \alpha'(t_i)\beta(t_j), & t_0 \leq t_j \leq t_i \\ \gamma'(t_i)\lambda(t_j), & t_0 \leq t_i \leq t_j \end{cases} \end{aligned} \tag{2}$$

with $\mathbf{a}(t_i)$ and $\mathbf{b}(t_i)$ bounded vectors of dimension n, $\alpha(t_i)$ and $\beta(t_i)$ are m-dimensional bounded vectors, and $\gamma(t_i)$ and $\lambda(t_i)$ are l-dimensional bounded vectors.

Then, using the available information from the set of observations $\{y(t_1),\ldots,y(t_N)\}$, we are interested in obtaining the LLMSE estimator of the signal $x(t_k)$ for a fixed instant of time $t_k < t_N$. It is well known that this estimator, $\hat{x}(t_k/t_N)$, is the orthogonal projection of $x(t_k)$ onto the space of linear transformations of the observations set $\{y(t_1),\ldots,y(t_N)\}$.

According to the projection theorem, this element $\hat{x}(t_k/t_N)$ exists, is unique and can be expressed as a linear combination of the observations $y(t_1),\ldots,y(t_N)$ of the form [6]

$$\hat{x}(t_k/t_N) = \sum_{j=1}^{N} h(t_k,t_j,t_N)y(t_j) \tag{3}$$

where $h(t_k,t_i,t_N)$ is an impulse-response function which can be determined from the Wiener-Hopf equation

$$E[x(t_k)y(t_i)] = \sum_{j=1}^{N} h(t_k,t_j,t_N)E[y(t_j)y(t_i)],\ t_0 \le t_i \le t_N \quad (4)$$

From the observation equation (1) and using the hypothesis that the autocorrelation function of the signal and the cross-correlation function between the signal and the observation noise can be respectively expressed in the form (2), we have that the Wiener-Hopf equation (4) can be rewritten as

$$h(t_k,t_i,t_N)r_i = R_{xy}(t_k,t_i) - \sum_{j=1}^{N} h(t_k,t_j,t_N)R(t_j,t_i),\ \ t_0 \le t_i \le t_N \quad (5)$$

where the correlation functions $R_{xy}(t_i,t_j) = R_x(t_i,t_j) + R_{xv}(t_i,t_j)$ and $R(t_i,t_j) = R_x(t_i,t_j) + R_{xv}(t_i,t_j) + R_{vx}(t_i,t_j)$ are factorizable kernels which can be expressed in the form

$$\begin{aligned} R_{xy}(t_i,t_j) &= \begin{cases} \mathbf{c}'(t_i)\eta(t_j), & t_0 \le t_j \le t_i \\ \mathbf{d}'(t_i)\mu(t_j), & t_0 \le t_i \le t_j \end{cases} \\ R(t_i,t_j) &= \begin{cases} \mu'(t_i)\eta(t_j), & t_0 \le t_j \le t_i \\ \eta'(t_i)\mu(t_j), & t_0 \le t_i \le t_j \end{cases} \end{aligned} \quad (6)$$

with $\mathbf{c}'(t_i) = \left[\mathbf{a}'(t_i), \alpha'(t_i), \mathbf{0}'_l\right]$, $\eta'(t_i) = \left[\mathbf{b}'(t_i), \beta'(t_i), \gamma'(t_i)\right]$, $\mathbf{d}'(t_i) = [\mathbf{b}'(t_i), \mathbf{0}'_m, \gamma'(t_i)]$, and $\mu'(t_i) = \left[\mathbf{a}'(t_i), \alpha'(t_i), \lambda'(t_i)\right]$, bounded vectors of dimension $p = n+m+l$, and $\mathbf{0}_l$ denotes the l-dimensional vector whose elements are zero.

In the next section, equation (5) will be used to design an efficient fixed-point smoothing algorithm. Specifically, the proposed algorithm will allow us to compute the fixed-point smoother of the signal $x(t_k)$, $\hat{x}(t_k/t_N)$, for a fixed instant $t_k < t_N$, through a recursive expression where the initial condition is the filter $\hat{x}(t_k/t_k)$.

3. RECURSIVE FIXED-POINT SMOOTHING ALGORITHM

In Theorem 1, with the above assumptions, a recursive least mean-squares algorithm for the fixed-point smoothing estimate of the signal is shown in linear discrete-time systems.

Theorem 1 *Let $\{x(t_i), t_i \ge t_0\}$ be a signal process satisfying the hypothesis established in Section 2. The fixed-point smoothing estimate of the signal $x(t_k)$, given a realization of a sequence of observations $\{y(t_1),\dots,y(t_N)\}$, is recursively computed as follows:*

$$\hat{x}(t_k/t_N) = \hat{x}(t_k/t_{N-1}) + h(t_k,t_N,t_N)\left[y(t_N) - \mu'(t_N)\mathbf{e}(t_{N-1})\right] \quad (7)$$

with the initial condition at $t_N = t_k$, the filter of the signal,

$$\hat{x}(t_k/t_k) = \mathbf{c}'(t_k)\mathbf{e}(t_k) \quad (8)$$

The vector $\mathbf{e}(t_N)$ obeys the recursive expression

$$\mathbf{e}(t_N) = \mathbf{e}(t_{N-1}) + \mathbf{g}(t_N,t_N)\left[y(t_N) - \mu'(t_N)\mathbf{e}(t_{N-1})\right] \quad (9)$$

with the initialization $\mathbf{e}(t_0) = \mathbf{0}_p$ and where

$$\begin{aligned} \mathbf{g}(t_N,t_N) &= [\eta(t_N) - \mathbf{Q}(t_{N-1})\mu(t_N)] \\ &\times \left\{r_N + \left[\eta'(t_N) - \mu'(t_N)\mathbf{Q}(t_{N-1})\right]\mu(t_N)\right\}^{-1} \end{aligned} \quad (10)$$

with the $p \times p$-dimensional matricial function $\mathbf{Q}(t_N)$ given by

$$\begin{aligned} \mathbf{Q}(t_N) &= \mathbf{Q}(t_{N-1}) + \mathbf{g}(t_N,t_N)\left[\eta'(t_N) - \mu'(t_N)\mathbf{Q}(t_{N-1})\right] \\ \mathbf{Q}(t_0) &= \mathbf{0}_{p\times p} \end{aligned} \quad (11)$$

where $\mathbf{0}_{p\times p}$ denotes the $p \times p$ dimensional matrix whose elements are all zero.

Moreover,

$$\begin{aligned} h(t_k,t_N,t_N) &= \left[\mathbf{d}'(t_k) - \mathbf{f}'(t_k,t_{N-1})\right]\mu(t_N) \\ &\times \left\{r_N + \left[\eta'(t_N) - \mu'(t_N)\mathbf{Q}(t_{N-1})\right]\mu(t_N)\right\}^{-1} \end{aligned} \quad (12)$$

where the vector $\mathbf{f}'(t_k,t_N)$ is recursively computed, from the initialization $\mathbf{f}'(t_k,t_k) = \mathbf{c}'(t_k)\mathbf{Q}(t_k)$, through the expression

$$\begin{aligned} \mathbf{f}'(t_k,t_N) &= \mathbf{f}'(t_k,t_{N-1}) \\ &+ h(t_k,t_N,t_N)\left[\eta'(t_N) - \mu'(t_N)\mathbf{Q}(t_{N-1})\right] \end{aligned} \quad (13)$$

Proof.

To start with, we obtain a recursive equation for the impulse response function $h(t_k,t_i,t_N)$. For that, if we subtract the equation (5) for t_N and t_{N-1} and take (6) into account, we can write

$$\begin{aligned} \left[h(t_k,t_i,t_N) - h(t_k,t_i,t_{N-1})\right]r_i &= -h(t_k,t_N,t_N)\mu'(t_N)\eta(t_i) \\ &- \sum_{j=1}^{N-1}\left[h(t_k,t_j,t_N) - h(t_k,t_j,t_{N-1})\right]R(t_j,t_i) \end{aligned}$$

Then, if we introduce a function $\mathbf{g}(t_i,t_N)$ satisfying the equation

$$\mathbf{g}(t_i,t_N)r_i = \eta(t_i) - \sum_{j=1}^{N}\mathbf{g}(t_j,t_N)R(t_j,t_i) \quad (14)$$

we obtain that

$$h(t_k,t_i,t_N) - h(t_k,t_i,t_{N-1}) = -h(t_k,t_N,t_N)\mu'(t_N)\mathbf{g}(t_i,t_{N-1}) \quad (15)$$

Moreover, following a similar reasoning for the equation (14), we obtain the recursive equation for $\mathbf{g}(t_i,t_N)$ as

$$\mathbf{g}(t_i,t_N) - \mathbf{g}(t_i,t_{N-1}) = -\mathbf{g}(t_N,t_N)\mu'(t_N)\mathbf{g}(t_i,t_{N-1}) \quad (16)$$

Now, we proceed to derive the expression (10) for $\mathbf{g}(t_N,t_N)$. By taking $t_i = t_N$ in (14) and using (6), we have

$$\begin{aligned} \mathbf{g}(t_N,t_N)r_N &= \eta(t_N) - \sum_{j=1}^{N}\mathbf{g}(t_j,t_N)\eta'(t_j)\mu(t_N) \\ &= \eta(t_N) - \mathbf{Q}(t_N)\mu(t_N) \end{aligned} \quad (17)$$

where we have introduced the $p \times p$ dimensional matrix

$$\begin{aligned} \mathbf{Q}(t_N) &= \sum_{j=1}^{N} \mathbf{g}(t_j, t_N)\eta'(t_j) \\ \mathbf{Q}(t_0) &= \mathbf{0}_{p\times p} \end{aligned} \tag{18}$$

Hence, if we subtract $\mathbf{Q}(t_{N-1})$ from $\mathbf{Q}(t_N)$, and use (16) and (18), the recursive equation (11) for $\mathbf{Q}(t_N)$ is derived. Finally, using (11) in (17), it is easy to check that $\mathbf{g}(t_N, t_N)$ satisfies the expression (10).

Next, following a similar reasoning, we establish the expression (12) for $h(t_k, t_N, t_N)$. Specifically, if we put $t_i = t_N$ in (5) and use (6) in the resultant equation, we get

$$h(t_k, t_N, t_N) r_N = \mathbf{d}'(t_k)\mu(t_N) - \mathbf{f}(t_k, t_N)\mu(t_N) \tag{19}$$

where the function $\mathbf{f}(t_k, t_N)$ is defined as

$$\mathbf{f}(t_k, t_N) = \sum_{j=1}^{N} h(t_k, t_j, t_N)\eta'(t_j) \tag{20}$$

Then, by subtracting $\mathbf{f}(t_k, t_{N-1})$ from $\mathbf{f}(t_k, t_N)$ and using (15), we have

$$\begin{aligned} \mathbf{f}(t_k, t_N) - \mathbf{f}(t_k, t_{N-1}) &= h(t_k, t_N, t_N)\eta'(t_N) \\ &\quad - h(t_k, t_N, t_N)\mu'(t_N) \sum_{j=1}^{N-1} \mathbf{g}(t_j, t_{N-1})\eta'(t_j) \end{aligned}$$

As a consequence, from the definition (18) for $\mathbf{Q}(t_N)$, we obtain the recursive expression (13) for updating $\mathbf{f}(t_k, t_N)$. Furthermore, substituting (13) in (19), the equation (12) for $h(t_k, t_N, t_N)$ is derived.

Now, in order to determine the initial condition of (13), $\mathbf{f}(t_k, t_k)$, we first put $t_N = t_k$ in the Wiener-Hopf equation (5)

$$\begin{aligned} h(t_k, t_i, t_k) r_i &= R_{xy}(t_k, t_i) - \sum_{j=1}^{k} h(t_k, t_j, t_k) R(t_j, t_i) \\ &= \mathbf{c}'(t_k)\eta(t_i) - \sum_{j=1}^{k} h(t_k, t_j, t_k) R(t_j, t_i) \end{aligned}$$

where we have applied the expression (6) in the last equality. Thus, from (14), it is clear that

$$h(t_k, t_i, t_k) = \mathbf{c}'(t_k)\mathbf{g}(t_i, t_k) \tag{21}$$

Consequently, if we put $t_N = t_k$ in (20) and use (21) and (18) in the resultant equation, we find that the initial condition on the difference equation (13) for $\mathbf{f}(t_k, t_N)$ at $t_N = t_k$ is

$$\mathbf{f}(t_k, t_k) = \mathbf{c}'(t_k)\mathbf{Q}(t_k)$$

Finally, from the definition (3) for the fixed-point smoother of the signal $\hat{x}(t_k/t_N)$ and using (15), we have

$$\begin{aligned} \hat{x}(t_k/t_N) - \hat{x}(t_k/t_{N-1}) &= h(t_k, t_N, t_N) y(t_N) \\ &\quad - h(t_k, t_N, t_N)\mu'(t_N) \sum_{j=1}^{N-1} \mathbf{g}(t_j, t_{N-1}) y(t_j) \end{aligned}$$

Therefore, if we denote

$$\begin{aligned} \mathbf{e}(t_N) &= \sum_{j=1}^{N} \mathbf{g}(t_j, t_N) y(t_j) \\ \mathbf{e}(t_0) &= \mathbf{0}_p \end{aligned} \tag{22}$$

we conclude that the fixed-point smoother of the signal, $\hat{x}(t_k/t_N)$ satisfies the recursive expression (7) with the initialization at $t_N = t_k$, the filter of the signal, $\hat{x}(t_k/t_k)$. From (3) and using (21) and (22), the expression (8) for the filter of the signal is deduced. Moreover, by subtracting $\mathbf{e}(t_{N-1})$ from $\mathbf{e}(t_N)$ and applying (16), the recursive equation (9) for the vector $\mathbf{e}(t_N)$ is immediate.

Remark 1 *Notice that a LLMSE filtering algorithm of the signal in discrete-time systems is given by the equations* (8)-(11) *of Theorem 1.*

4. NUMERICAL EXAMPLE

In this section, the behavior of the fixed-point smoothing estimates obtained from the LLMSE estimation algorithm presented in Theorem 1 is analyzed by means of a numerical example.

In this example, we consider that the signal $\{x(t_i), t_i \geq 0\}$ is a Gaussian process with zero-mean and autocorrelation function

$$R_x(t_i, t_j) = e^{|t_i - t_j|}, \quad t_i, t_j \geq 0 \tag{23}$$

Moreover, the observation noise $v(t_i)$ is assumed to be a centered white Gaussian process and the cross-correlation function between the signal and the observation noise is

$$R_{xv}(t_i, t_j) = 0.1 t_i^2 t_j^3 \tag{24}$$

It is clear that (23) and (24) are factorizable kernels of the form (2) with $\mathbf{a}(t_i) = e^{-t_i}$, $\mathbf{b}(t_i) = e^{t_i}$, $\alpha(t_i) = \gamma(t_i) = t_i^2$, $\beta(t_i) = \lambda(t_i) = 0.1 t_i^3$.

Next, the efficiency of the proposed recursive fixed-point smoothing algorithm is checked for the observation noises with variance parameters $r = 0.25$, $r = 1$ and $r = 3$. For that, we have written a program in MATLAB which simulates, in every case, the observations (1) and provides the fixed-point smoothers of the signal computed through the recursive formulas given in Theorem 1.

In particular, this program has been applied to calculate the optimum fixed-point smoother of the signal $x(t_k)$, at the fixed instant of time $t_k = 0.15$, on the basis of the observations $y(t_i)$, with $t_i = i/100$, $i = 1, \ldots, 100$. Specifically, the first estimation of $x(t_k)$ is based on the observation set $\{y(0.01), y(0.02), \ldots, y(t_k)\}$. In the second estimation, the observation set $\{y(0.01), y(0.02), \ldots, y(t_k), y(t_k + 0.01))\}$ is considered, and so forth, until $\{y(0.01), y(0.02), \ldots, y(1)\}$

In Figure 1, the optimum fixed-point smoothers $\hat{x}(0.15/t_N)$ for $r = 0.25$, $r = 1$ and $r = 3$ are compared. Note that, the simulated value for signal at $t_k = 0.15$ is $x(0.15) = 0.7151$. As could be expected, Figure 1 shows a better behavior of the optimum fixed-point smoother for a smaller noise level.

Moreover, the performance of the above estimates is evaluated through the mean-square values (MSV) of the fixed-point smoothing errors of the signal. This MSV is computed

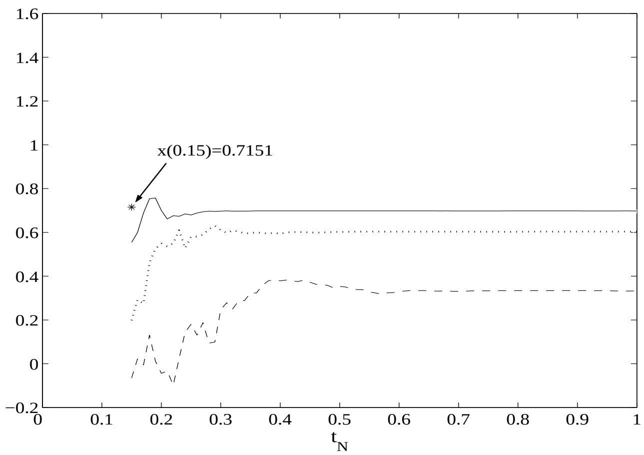

Figure 1: Fixed-point smoothing estimate $\hat{x}(0.15/t_N)$ of the signal for $r = 0.25$ (solid line), $r = 1$ (dotted line), and $r = 3$ (dashed line).

Variance parameter of the observation noise	MSV of the fixed-point smoothing error
$r = 0.25$	8.6333×10^{-4}
$r = 1$	0.0215
$r = 3$	0.1976

Table 1: Mean-Square Values of the fixed-point smoothing algorithm errors for $r = 0.25$, $r = 1$, and $r = 3$.

by the expression

$$\frac{1}{85}\sum_{j=16}^{100}\left(x(0.15) - \hat{x}(0.15/t_j)\right)^2$$

Table 1 summarizes the MSV of the fixed-point smoothing errors of the signal for the different observation noises with variance parameters $r = 0.25$, $r = 1$, and $r = 3$. From these results we find that the MSVs are smaller and, consequently, the estimation accuracy of the fixed-point smoother is improved, as the observation noise variance parameter decreases.

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