THEORY AND LATTICE STRUCTURES FOR OVERSAMPLED LINEAR PHASE PARAUNITARY FILTER BANKS WITH ARBITRARY FILTER LENGTH

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ABSTRACT

This paper presents the theory and lattice structures of a large class of oversampled linear phase paraunitary filter banks. We deal with FIR filter banks with real-valued coefficients in which all analysis filters have the same *arbitrary* filter length and share the same symmetry center. Necessary existence conditions on symmetry polarity of the filter banks are firstly derived. Lattice structures are developed for type-I oversampled linear phase paraunitary filter banks [1]. Furthermore, these lattice structures can be proven to be complete. Finally, several design examples are presented to confirm the validity of the theory and lattice structures.

1. INTRODUCTION

Filter banks (FB) have found many applications in image/audio/video processing, discrete multitome modulation and channel equalization and so on [2, 3]. Most past research works focused on critically sampled FBs which have the same number of channels and subsampling factors. However, oversampled filter banks (OFB) whose number of channels P is greater than their subsampling factor M, have gained great interests recently, because they can provide more degree of design freedom, residual redundancy left in subband signals and improved noise and erasure resistance [4, 5]. In many applications, especially in image and video processing, linear phase (LP) property of filters is always crucial. Moreover, simple symmetric extension methods can be employed for LP filters to accurately handle the boundaries of finite length signals. For various practical purposes, only causal FIR oversampled linear phase filter banks (OLPFB) with real-valued coefficients are in consideration.

A few works have been done in such area. OLPFBs were firstly studied in [6] for the restrictive cosine modulated case with integer oversampling ratio where oversampling ratio $r = P/M$ must be an integer. A more general OFB with integer oversampling ratio is studied in [7]. For rational oversampling ratio case, a lattice structure for oversampled linear phase paraunitary filter banks (OLPPUFB) was reported in [8]. Recently, a more generalized lattice structure has been reported in [1]. Similar lattice structures were also given in [9, 10]. However, all these works were focused on a restrictive class of OLPFBs in which the filter length L must be multiple of the subsampling factor M, i.e., $L = KM$. This restriction on length is actually *not* necessary and leaves a large class of OLPFBs out of their designs. Thus, it is highly desirable to remove this restriction and develop more general structures for OLPFBs which can cover more possible design choices.

In this paper, we investigate the theory and lattice factorizations for a large class of OLPPUFBs, in which all analysis filters have the same *arbitrary* length $L = KM + \beta$ $(0 \leq \beta < M)$, and have the same symmetry center. For this class of OLPPUFBs, we first derive the necessary restrictions on the filter symmetry polarity. Then, lattice factorizations are developed for type-I OPLLUFBs. Compared with the reported work in [8, 10], our result is more general and can cover a larger class of OLPPUFBs than theirs. Furthermore, our lattice structures are proven to be complete. Finally, two design examples are presented to confirm the validity of the proposed structures. They also show the design flexibility with regard to filter length.

The main motivation for this work is to design more flexible OLPPUFBs which may give more possible choices for a given desired application. In most traditional works [8, 1, 11, 10], the filter length must be multiple of M, which greatly limits the possible design of OLPPUFBs for large M. For example, for 8-channel OLPPUFBs with $M = 6$, if the length is constrained to be less than 18 by the constraints of complexity, there are only two possible choices in conventional designs. Although M can be made small for OLPPUFBs, for example $M = 2$, from which conventional design methods have more choices on filter length, the subsampling factor M is usually only a little bit less than the number of channels P, i.e., oversampling ratio $r \approx 1$, to maintain subband sample efficiency. The restriction becomes more severe for larger M. However, this restriction is from the traditional FB design methods, *not* from OLPPUFBs. Our proposed design method can overcome this drawbacks of conventional designs and provide more possible OLPPUFBs satisfying given constraint length and the length increment among FBs can be made as small as possible. Continuing the above example, the proposed design method can give 12 possible OLPPUFBs compared to only 2 choices in traditional method and can offer better trade off between filter length and performance than traditional methods.

Notations: Bold-faced quantities denote matrices and vectors. $\mathbf{I}_M$, $\mathbf{J}_M$ and $\mathbf{0}_{P\times M}$ denote the $M \times M$ identity matrix, the $M \times M$ reversal matrix and $P \times M$ null matrix, respectively. For FIR FBs, the $P \times M$ polyphase matrix can be written as $\mathbf{E}(z) = \sum_{i=0}^{K-1} \mathbf{e}[i]z^{-i}$, where $\mathbf{e}[K-1] \neq \mathbf{0}$. $K-1$ is defined as the order of the polyphase matrix, i.e., the FIR FB. It is related to the maximum possible length L of the analysis filters by $L = KM$. For arbitrary $P \times M$ real-valued constant matrix $\mathbf{A}$ and matrix polynomial $\mathbf{A}(z)$ with real-valued coefficients, we say $\mathbf{A}$ and $\mathbf{A}(z)$ are orthogonal and paraunitary (PU), respectively, if the $M \times P$ matrix $\mathbf{A}^T$ and $\tilde{\mathbf{A}}(z)$ satisfy $\mathbf{A}^T\mathbf{A} = \mathbf{I}_M$ and $\tilde{\mathbf{A}}(z)\mathbf{A}(z) = \mathbf{I}_M$, respectively, where $\tilde{\mathbf{A}}(z) = \mathbf{A}^T(z^{-1})$.

2. NECESSARY CONDITIONS FOR OLPPUFBS

Consider a P-channel OLPPUFB with subsampling factor M and equal filter length $L = KM + \beta$ $(0 \leq \beta < M)$ as shown in Fig. 1. Without loss of generality, we can always arrange the P channel

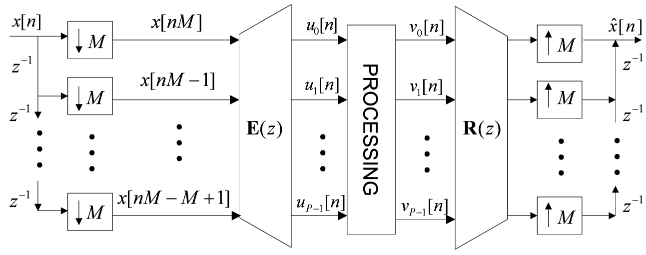


Figure 1: The polyphase form of a P-channel FB with subsampling facotr M

linear phase filters in such an order that the first n_s filters are symmetric, while the other $n_a = P - n_s$ are antisymmetric filters. The associated $P \times M$ analysis polyphase matrix $\mathbf{E}(z)$ should satisfy the LP condition [12],

$$\mathbf{E}(z) = z^{-(K-1)}\mathbf{D}_P\mathbf{E}(z^{-1})\hat{\mathbf{J}}_M(z) \tag{1}$$

where

$$\hat{\mathbf{J}}_M(z) = \begin{bmatrix} z^{-1}\mathbf{J}_\beta & \mathbf{0}_{\beta\times(M-\beta)} \\ \mathbf{0}_{(M-\beta)\times\beta} & \mathbf{J}_{M-\beta} \end{bmatrix} \tag{2}$$

and $\mathbf{D}_P = \mathrm{diag}(\mathbf{I}_{n_s}, -\mathbf{I}_{n_a})$. With the LP and PU properties, the necessary existence conditions on symmetry polarity of OLPPUFBs can be obtained which are stated in the following theorem.

Theorem 1. *For the class of OLPPUFBs stated above, n_s and n_a have the following bounds:*

1. *When M and β are both even, and K is arbitrary $(K \geq 1)$, $M/2 \leq n_s \leq P - M/2$ and $M/2 \leq n_a \leq P - M/2$.*
2. *When M is even and β is odd, and K is arbitrary $(K \geq 1)$, $M/2 + 1 \leq n_s \leq P - M/2$ and $M/2 \leq n_a \leq P - (M/2 + 1)$.*
3. *When M is odd and β is even,*
   - *(a) If K is even, $(M+1)/2 \leq n_s \leq P - (M+1)/2$ and $(M+1)/2 \leq n_a \leq P - (M+1)/2$.*
   - *(b) If K is odd, $(M+1)/2 \leq n_s \leq P - (M-1)/2$ and $(M-1)/2 \leq n_a \leq P - (M+1)/2$.*
4. *When M is odd and β is odd,*
   - *(a) If K is even, $(M+1)/2 \leq n_s \leq P - (M-1)/2$ and $(M-1)/2 \leq n_a \leq P - (M+1)/2$.*
   - *(b) If K is odd, $(M+1)/2 \leq n_s \leq P - (M+1)/2$ and $(M+1)/2 \leq n_a \leq P - (M+1)/2$.*

Proof. For case (1) assume $M = 2m$ and $\beta = 2l$, $\mathbf{E}(z)$ must have the following form according to the LP condition (1),

$$\begin{bmatrix} \mathbf{S}_0(z) & z^{-K}\mathbf{S}_0(z^{-1})\mathbf{J}_l & \mathbf{S}_1(z) & z^{-(K-1)}\mathbf{S}_1(z^{-1})\mathbf{J}_{m-l} \\ \mathbf{A}_0(z) & -z^{-K}\mathbf{A}_0(z^{-1})\mathbf{J}_l & \mathbf{A}_1(z) & -z^{-(K-1)}\mathbf{A}_1(z^{-1})\mathbf{J}_{m-l} \end{bmatrix} \tag{3}$$

where matrices $\mathbf{S}_0(z)$, $\mathbf{A}_0(z)$, $\mathbf{S}_1(z)$ and $\mathbf{A}_1(z)$ have size $n_s \times l$, $n_a \times l$, $n_s \times (m-l)$ and $n_a \times (m-l)$, respectively. Then, from the PU condition $\tilde{\mathbf{E}}(z)\mathbf{E}(z) = \mathbf{I}_{2m}$, we can obtain a system of matrix polynomial equations.

$$\begin{aligned}
\mathbf{S}_0^T(z^{-1})\mathbf{S}_0(z) + \mathbf{A}_0^T(z^{-1})\mathbf{A}_0(z) &= \mathbf{I}_l \\
\mathbf{S}_0^T(z^{-1})\mathbf{S}_0(z^{-1}) - \mathbf{A}_0^T(z^{-1})\mathbf{A}_0(z^{-1}) &= \mathbf{0} \\
\mathbf{S}_1^T(z^{-1})\mathbf{S}_1(z) + \mathbf{A}_1^T(z^{-1})\mathbf{A}_1(z) &= \mathbf{I}_{m-l} \\
\mathbf{S}_1^T(z^{-1})\mathbf{S}_1(z^{-1}) - \mathbf{A}_1^T(z^{-1})\mathbf{A}_1(z^{-1}) &= \mathbf{0} \\
\mathbf{S}_0^T(z^{-1})\mathbf{S}_1(z) + \mathbf{A}_0^T(z^{-1})\mathbf{A}_1(z) &= \mathbf{0} \\
\mathbf{S}_0^T(z^{-1})\mathbf{S}_1(z^{-1}) - \mathbf{A}_0^T(z^{-1})\mathbf{A}_1(z^{-1}) &= \mathbf{0}
\end{aligned}$$

By evaluating these equations at $z = 1$, the following equations can be obtained: $\mathbf{S}_0^T(1)\mathbf{S}_0(1) = \mathbf{I}_l = \mathbf{A}_0^T(1)\mathbf{A}_0(1)$, $\mathbf{S}_1^T(1)\mathbf{S}_1(1) = \mathbf{I}_{m-l} = \mathbf{A}_1^T(1)\mathbf{A}_1(1)$ and $\mathbf{S}_0^T(1)\mathbf{S}_1(1) = \mathbf{0} = \mathbf{A}_0^T(1)\mathbf{A}_1(1)$. Finally, by defining matrix $\mathbf{T}_s \triangleq [\mathbf{S}_0(1)\ \mathbf{S}_1(1)]$ which has size $n_s \times m$, it can be seen that $\mathrm{rank}(\mathbf{T}_s) = \mathrm{rank}(\mathbf{T}_s^T\mathbf{T}_s) = \mathrm{rank}(\mathbf{I}_m) = m$. However, $m = \mathrm{rank}(\mathbf{T}_s) \leq \min\{n_s, m\} \leq n_s$. Similar argument is also applicable to n_a by defining $\mathbf{T}_a \triangleq [\mathbf{A}_0(1)\ \mathbf{A}_1(1)]$. Using the condition of $n_s + n_a = P$, we can obtain the bounds in case (1). Similar proof can also be applicable to other cases in Theorem 1. For some cases, we need to evaluate PU conditions at both $z = \pm 1$ to get tighter bounds. □

Note that Theorem 1 is applicable to both critically sampled and oversampled systems. These necessary conditions on symmetry polarity of filters are a very useful guideline for FB design. It permits the FB designer to narrow down the solution space and to avoid impossible design specifications. It also helps to explain why only some solutions exist and plays a important role in the development of design. To our knowledge, it is the most general form which can cover previous works [12, 8] as special cases. With $P = M$, one can check it is consistent with the results in [12]. Imposing $\beta = 0$, it is also consistent with the results in [8, 1]. However, the cases of (2), (3b) and (4b) in Theorem 1 are new results. This theorem also reveals the increased design freedom of oversampled systems with length not constrained to be multiple of M. Recall that in critically sampled systems, even-channel LPFB with equal length cannot have odd filter length. However, the cases (2), (3b) and (4a) of Theorem 1 indicate that it can exist in oversampled systems. Actually, compared to the length conditions in [12], there is no length constraint for OLPFB. Compared to reported works [6, 8, 1, 9, 10], our scheme can yield real *arbitrary* filter length for type-I OLPPUFB, which in turn can provide more possible choices to be selected in desired applications than before.

3. LATTICE STRUCTURES FOR OLPPUFBS WITH ARBITRARY FILTER LENGTH

In this section, we deal with lattice factorizations for analysis polyphase matrix $\mathbf{E}(z)$ of OLPPUFBs. It can be shown easily that the general lattice construction approach can be applied here [12]-[15]:

$$\mathbf{E}(z) = \mathbf{G}_K(z)\mathbf{G}_{K-1}(z)\mathbf{G}_{K-2}(z)\cdots\mathbf{G}_1(z)\mathbf{E}_0(z) \tag{4}$$

where the starting block $\mathbf{E}_0(z)$ with order N_0 and length $N_0M + \beta$ (with order $N_0 - 1$ and length N_0M if $\beta = 0$) has both LP and PU properties, and each block $\mathbf{G}_i(z)$ with order N_1 can propagate both LP and PU properties and increase filter length N_1M. Such a cascade form would finally generate an OLPPUFB with filter length $L = (KN_1 + N_0)M + \beta$. The LP propagating block in [1] can be used here for $\mathbf{G}_i(z)$. The difference between our factorization and previous ones is the starting block $\mathbf{E}_0(z)$. Contrary to [8, 1], $\mathbf{E}_0(z)$ cannot be made order zero if $\beta \neq 0$, i.e., constant matrix, which is treated as a trivial case by us because it would impose multiple of $(M - \beta)$ zero filter coefficients at fixed positions. Thus, $\mathbf{E}_0(z)$ has at least order one for the case of $\beta \neq 0$, i.e., $N_0 \geq 1$.

A valuable classification of OLPFBs was given in [1] according to the order of $\mathbf{G}_i(z)$. We adopt their classification here, namely type-I system with $n_s = n_a$ and type-II system with $n_s \neq n_a$. In this paper, we study type-I OLPPUFBs with arbitrary filter length in detail. For type-I system, $\mathbf{G}_i(z)$ has the following form,

$$\begin{aligned}
\mathbf{G}_i(z) &= \frac{1}{2}\mathbf{\Phi}_i\mathbf{W}\mathbf{\Lambda}(z)\mathbf{W} \qquad (5) \\
&= \frac{1}{2}\begin{bmatrix} \mathbf{U}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & z^{-1}\mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}
\end{aligned}$$

where each submatrix has size $p \times p$ and $\mathbf{U}_i$ is an square orthogonal matrix. After each stage $\mathbf{G}_i(z)$, the filter length can be increased by M. In order to have filter length $L = KM + \beta$ for $\mathbf{E}(z)$, the staring block $\mathbf{E}_0(z)$ should be a LPPU system with length $M + \beta$, i.e., an order one matrix polynomial with linear phase. We study them for even and odd length, respectively.

3.1 OLPPUFBs with Even Length

In this subsection, we investigate the lattice structure for type-I OLPPUFBs with even length $L = KM + \beta$. They consist of three possible cases. One is the case of even M and even β, the other two are even K and even β, or odd K and odd β both with odd M. We firstly study the case of even $M = 2m$ and even $\beta = 2l$ $(0 \leq l < m)$. By LP condition (1), it can be seen easily that $\mathbf{E}_0(z)$ should have

$$\mathbf{E}_0^e(z) = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{U}_{00}\mathbf{\Gamma}_p + z^{-1}\mathbf{U}_{00}\mathbf{\Gamma}_m\mathbf{J}_{f+\alpha} & \frac{1}{\sqrt{2}}(\mathbf{U}_{02}+z^{-1}\mathbf{U}_{02}\mathbf{J}_{2g-2\alpha}) & \mathbf{U}_{00}\mathbf{\Gamma}_m + z^{-1}\mathbf{U}_{00}\mathbf{\Gamma}_p\mathbf{J}_{f+\alpha} & \mathbf{U}_{01} & \mathbf{U}_{01}\mathbf{J}_{m-l} \\ \mathbf{V}_{00}\mathbf{\Gamma}_p - z^{-1}\mathbf{V}_{00}\mathbf{\Gamma}_m\mathbf{J}_{f+\alpha} & \frac{1}{\sqrt{2}}(\mathbf{V}_{02}-z^{-1}\mathbf{V}_{02}\mathbf{J}_{2g-2\alpha}) & \mathbf{V}_{00}\mathbf{\Gamma}_m - z^{-1}\mathbf{V}_{00}\mathbf{\Gamma}_p\mathbf{J}_{f+\alpha} & \mathbf{V}_{01} & -\mathbf{V}_{01}\mathbf{J}_{m-l} \end{bmatrix} \quad (6)$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{U}_{00} & \mathbf{U}_{02} & \mathbf{U}_{01} & \mathbf{U}_{01}\mathbf{J}_{m-l} & z^{-1}\mathbf{U}_{02}\mathbf{J}_{2g-2\alpha} & z^{-1}\mathbf{U}_{00}\mathbf{J}_{f+\alpha} \\ \mathbf{V}_{00} & \mathbf{V}_{02} & \mathbf{V}_{01} & -\mathbf{V}_{01}\mathbf{J}_{m-l} & -z^{-1}\mathbf{V}_{02}\mathbf{J}_{2g-2\alpha} & -z^{-1}\mathbf{V}_{00}\mathbf{J}_{f+\alpha} \end{bmatrix} \begin{bmatrix} \mathbf{\Gamma}_p & \mathbf{0} & \mathbf{\Gamma}_m & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{2}}\mathbf{I}_{2g-2\alpha} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{2m-2l} \\ \mathbf{0} & \frac{1}{\sqrt{2}}\mathbf{I}_{2g-2\alpha} & \mathbf{0} & \mathbf{0} \\ \mathbf{J}_{f+\alpha}\mathbf{\Gamma}_m\mathbf{J}_{f+\alpha} & \mathbf{0} & \mathbf{J}_{f+\alpha}\mathbf{\Gamma}_p\mathbf{J}_{f+\alpha} & \mathbf{0} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{U}_{00} & \mathbf{U}_{02} & \mathbf{U}_{01} & \mathbf{U}_{01}\mathbf{J}_{m-l} & \mathbf{U}_{02}\mathbf{J}_{2g-2\alpha} & \mathbf{U}_{00}\mathbf{J}_{f+\alpha} \\ \mathbf{V}_{00} & \mathbf{V}_{02} & \mathbf{V}_{01} & -\mathbf{V}_{01}\mathbf{J}_{m-l} & -\mathbf{V}_{02}\mathbf{J}_{2g-2\alpha} & -\mathbf{V}_{00}\mathbf{J}_{f+\alpha} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{f+2g-\alpha+2m-2l} & \mathbf{0} \\ \mathbf{0} & z^{-1}\mathbf{I}_{f+2g-\alpha} \end{bmatrix} \mathbf{\Gamma}_e = \frac{1}{\sqrt{2}}\mathbf{\Phi}_0^e\mathbf{\Lambda}_e(z)\mathbf{\Gamma}_e \quad (7)$$

the following form,

$$\mathbf{E}_0^e(z) = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{S}_{00}+z^{-1}\mathbf{S}_{00}\mathbf{J}_{2l} & \mathbf{S}_{01} & \mathbf{S}_{01}\mathbf{J}_{m-l} \\ \mathbf{A}_{00}-z^{-1}\mathbf{A}_{00}\mathbf{J}_{2l} & \mathbf{A}_{01} & -\mathbf{A}_{01}\mathbf{J}_{m-l} \end{bmatrix} \quad (8)$$

where $\mathbf{S}_{00}$ and $\mathbf{A}_{00}$ have size $p\times 2l$, $\mathbf{S}_{01}$ and $\mathbf{A}_{01}$ have size $p\times(m-l)$. By further imposing the PU condition, i.e., $\tilde{\mathbf{E}}_0^e(z)\mathbf{E}_0^e(z) = \mathbf{I}_M$, we can obtain a system of matrix equations,

$$\mathbf{S}_{00}^T\mathbf{S}_{00} + \mathbf{J}_{2l}\mathbf{S}_{00}^T\mathbf{S}_{00}\mathbf{J}_{2l} = \mathbf{I}_{2l} = \mathbf{A}_{00}^T\mathbf{A}_{00} + \mathbf{J}_{2l}\mathbf{A}_{00}^T\mathbf{A}_{00}\mathbf{J}_{2l} \quad (9)$$

$$\mathbf{S}_{01}^T\mathbf{S}_{01} = \mathbf{I}_{m-l} = \mathbf{A}_{01}^T\mathbf{A}_{01} \quad (10)$$

$$\mathbf{S}_{00}^T\mathbf{S}_{01} = \mathbf{0} = \mathbf{A}_{00}^T\mathbf{A}_{01} \quad (11)$$

From these equations, we can establish a very useful algebraic property of matrices $\mathbf{S}_{00}$ and $\mathbf{A}_{00}$.

Theorem 2. *The matrices $\mathbf{S}_{00}$ and $\mathbf{A}_{00}$ have the same bounds on their ranks: $l \le rank(\mathbf{S}_{00}), rank(\mathbf{A}_{00}) \le l+\min\{l, p-m\}$.*

Proof. From the PU equation of (10), we know that $\mathbf{S}_{01}$ is an orthogonal matrix with $\text{rank}(\mathbf{S}_{01}) = m-l$. Define $p\times(m+l)$ matrix $\mathbf{T}_s \triangleq [\mathbf{S}_{00}\ \mathbf{S}_{01}]$. Along with PU equation of (10) and (11), it can be shown easily that

$$\mathbf{T}_s^T\mathbf{T}_s = \begin{bmatrix} \mathbf{S}_{00}^T\mathbf{S}_{00} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m-l} \end{bmatrix}$$

Because of the rank equality, i.e., $\text{rank}(\mathbf{S}_{00}^T\mathbf{S}_{00}) = \text{rank}(\mathbf{S}_{00})$, we have $\text{rank}(\mathbf{T}_s) = \text{rank}(\mathbf{T}_s^T\mathbf{T}_s) = \text{rank}(\mathbf{S}_{00}) + m - l \le \min\{p, m+l\}$, i.e., $\text{rank}(\mathbf{S}_{00}) \le l - m + \min\{p, m+l\} = l + \min\{l, p-m\}$. However, by the PU equation (9) and Sylvester's rank inequality [16], we can obtain $2l = \text{rank}(\mathbf{S}_{00}^T\mathbf{S}_{00} + \mathbf{J}_{2l}\mathbf{S}_{00}^T\mathbf{S}_{00}\mathbf{J}_{2l}) \le 2\text{rank}(\mathbf{S}_{00}^T\mathbf{S}_{00}) = 2\text{rank}(\mathbf{S}_{00})$ which means the inequality, $\text{rank}(\mathbf{S}_{00}) \ge l$. Combining these two inequalities, the rank condition for $\mathbf{S}_{00}$ can be obtained. Similar proof is also applicable to matrix $\mathbf{A}_{00}$. □

From the above analysis on the rank of $\mathbf{S}_{00}$, we propose a parameterized form for $\mathbf{S}_{00}$ to help factorization,

$$\mathbf{S}_{00} = [\mathbf{U}_{00}\Gamma_p\ \frac{1}{\sqrt{2}}\mathbf{U}_{02}\ \mathbf{U}_{00}\Gamma_m] \quad (12)$$

In addition, define two variables for indicating dimension of various matrices,

$$f \triangleq \max\{l-(p-m), 0\},\ g \triangleq l - f \quad (13)$$

Thus $\mathbf{U}_{00}$ has size $p\times(f+\alpha)$, $\mathbf{U}_{02}$ has size $p\times(2g-2\alpha)$ and $0\le\alpha\le g$. Matrices $\mathbf{\Gamma}_p = (\mathbf{\Gamma}_1+\mathbf{\Gamma}_2)/2$ and $\mathbf{\Gamma}_m = (\mathbf{\Gamma}_1-\mathbf{\Gamma}_2)\mathbf{J}_{f+\alpha}/2$, where $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are two *arbitrary* square orthogonal matrices with size $f+\alpha$. Apply similar parameterized form for $\mathbf{A}_{00}$ with matrices $\mathbf{V}_{00}$ and $\mathbf{V}_{02}$ with the same dimension as $\mathbf{U}_{00}$ and $\mathbf{U}_{02}$. By such parameterization, it can be shown that $\mathbf{E}_0^e(z)$ has form in (6) at the top of this page, where $\mathbf{\Phi}_0^e$ can be further factorized in the form (14)

$$\mathbf{\Phi}_0^e = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{U}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_0 \end{bmatrix}\begin{bmatrix} \mathbf{I}_{m+g-\alpha} & \mathbf{I}_{m+g-\alpha} \\ \mathbf{I}_{m+g-\alpha} & \mathbf{I}_{m+g-\alpha} \end{bmatrix}\begin{bmatrix} \mathbf{I}_{m+g-\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{m+g-\alpha} \end{bmatrix} \quad (14)$$

where matrices $\mathbf{U}_0 = [\mathbf{U}_{00}\ \mathbf{U}_{02}\ \mathbf{U}_{01}]$ and $\mathbf{V}_0 = [\mathbf{V}_{00}\ \mathbf{V}_{02}\ \mathbf{V}_{01}]$. It can be shown easily that such factorization can ensure PU property of $\mathbf{E}_0^e(z)$ as long as $\mathbf{U}_0$ and $\mathbf{V}_0$ are square or rectangular orthogonal matrices with orthogonal columns because $\mathbf{\Gamma}_e$ is an rectangular orthogonal matrix with orthogonal columns. It can also be shown easily that our proposed parameterized form for $\mathbf{S}_{00}$ in (12) is a solution of Eq. (9). Note that this factorization is also applicable to critically sampled systems, i.e., $P = M$, covering it as a special case of ours, in which it degenerates into structure in [12] with $\alpha = 0$ due to $\text{rank}(\mathbf{S}_{00}) = l$. This also shows the design flexibility of OLPPUFBs with more possible values of α, i.e., $0\le\alpha\le g$. Furthermore, if $\beta = 0$, i.e., filter length $L = KM$, it degenerates into structure in [8] for oversampled systems, or structure in [13] for critically sampled systems. The remaining two cases with odd M are discussed in section 3.3.

3.2 OLPPUFBs with Odd Length

In this subsection, we investigate a lattice structure for type-I OLPPUFBs with odd length. They also consist of three possible cases. One is the case of even M and odd β, the other two are even K and odd β, or odd K and even β both with odd M. We firstly study the case of even $M = 2m$ and odd $\beta = 2l-1$ $(1\le l\le m)$. By LP condition (1), it can be seen that $\mathbf{E}_0(z)$ should have the following form,

$$\mathbf{E}_0^o(z) = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{S}_{00}+z^{-1}\mathbf{S}_{00}\mathbf{J}_{2l-1} & \mathbf{S}_{01} & \mathbf{q} & \mathbf{S}_{01}\mathbf{J}_{m-l} \\ \mathbf{A}_{00}-z^{-1}\mathbf{A}_{00}\mathbf{J}_{2l-1} & \mathbf{A}_{01} & \mathbf{0} & -\mathbf{A}_{01}\mathbf{J}_{m-l} \end{bmatrix} \quad (15)$$

where $\mathbf{S}_{00}$ and $\mathbf{A}_{00}$ have size $p\times(2l-1)$, $\mathbf{S}_{01}$ and $\mathbf{A}_{01}$ have size $p\times(m-l)$, $\mathbf{q}$ and $\mathbf{0}$ are $p\times 1$ column vectors. By further imposing the PU condition, i.e., $\tilde{\mathbf{E}}_0^o(z)\mathbf{E}_0^o(z) = \mathbf{I}_M$, we can obtain a system of matrix equations,

$$\mathbf{S}_{00}^T\mathbf{S}_{00} + \mathbf{J}_{2l-1}\mathbf{S}_{00}^T\mathbf{S}_{00}\mathbf{J}_{2l-1} = \mathbf{I}_{2l-1} = \mathbf{A}_{00}^T\mathbf{A}_{00} + \mathbf{J}\mathbf{A}_{00}^T\mathbf{A}_{00}\mathbf{J}$$

$$\mathbf{S}_{01}^T\mathbf{S}_{01} = \mathbf{I}_{m-l} = \mathbf{A}_{01}^T\mathbf{A}_{01}$$

$$\mathbf{S}_{00}^T\mathbf{S}_{01} = \mathbf{0} = \mathbf{S}_{00}^T\mathbf{S}_{01}$$

$$\mathbf{S}_{00}^T\mathbf{q} = 0,\ \ \mathbf{S}_{01}^T\mathbf{q} = 0,\ \ \mathbf{q}^T\mathbf{q} = 2$$

From these equations, we can also establish the algebraic property of matrices $\mathbf{S}_{00}$ and $\mathbf{A}_{00}$.

Theorem 3. *The matrices $\mathbf{S}_{00}$ and $\mathbf{A}_{00}$ have same bounds on their ranks: $l\le rank(\mathbf{S}_{00}), rank(\mathbf{A}_{00}) \le l + \min\{l-1, p-m-1\}$.*

The proof is similar to Theorem 2 and omitted here. From the above analysis on the rank of $\mathbf{S}_{00}$, we also propose a parameterized form for $\mathbf{S}_{00} = [\mathbf{U}_{00}\Gamma_p\ \frac{1}{\sqrt{2}}\mathbf{U}_{02}\ \mathbf{U}_{00}\Gamma_m]$, where $\mathbf{U}_{00}$ has size

$$\mathbf{E}_0^o(z) = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{U}_{00}\boldsymbol{\Gamma}_p + z^{-1}\mathbf{U}_{00}\boldsymbol{\Gamma}_m\mathbf{J}_{f+\alpha} & \frac{1}{\sqrt{2}}(\mathbf{U}_{02}+z^{-1}\mathbf{U}_{02}\mathbf{J}_{2g-2\alpha}) & \mathbf{U}_{00}\boldsymbol{\Gamma}_m + z^{-1}\mathbf{U}_{00}\boldsymbol{\Gamma}_p\mathbf{J}_{f+\alpha} & \mathbf{U}_{01} & \sqrt{2}\mathbf{u} & \mathbf{U}_{01}\mathbf{J}_{m-l} \\ \mathbf{V}_{00}\boldsymbol{\Gamma}_p - z^{-1}\mathbf{V}_{00}\boldsymbol{\Gamma}_m\mathbf{J}_{f+\alpha} & \frac{1}{\sqrt{2}}(\mathbf{V}_{02}-z^{-1}\mathbf{V}_{02}\mathbf{J}_{2g-2\alpha}) & \mathbf{V}_{00}\boldsymbol{\Gamma}_m - z^{-1}\mathbf{V}_{00}\boldsymbol{\Gamma}_p\mathbf{J}_{f+\alpha} & \mathbf{V}_{01} & \mathbf{0} & -\mathbf{V}_{01}\mathbf{J}_{m-l} \end{bmatrix} \quad (16)$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{U}_{00} & \mathbf{U}_{02} & \mathbf{U}_{01} & \sqrt{2}\mathbf{u} & \mathbf{U}_{01}\mathbf{J}_{m-l} & z^{-1}\mathbf{U}_{02}\mathbf{J}_{2g-2\alpha} & z^{-1}\mathbf{U}_{00}\mathbf{J}_{f+\alpha} \\ \mathbf{V}_{00} & \mathbf{V}_{02} & \mathbf{V}_{01} & \mathbf{0} & -\mathbf{V}_{01}\mathbf{J}_{m-l} & -z^{-1}\mathbf{V}_{02}\mathbf{J}_{2g-2\alpha} & -z^{-1}\mathbf{V}_{00}\mathbf{J}_{f+\alpha} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma}_p & \mathbf{0} & \boldsymbol{\Gamma}_m & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{2}}\mathbf{I}_{2g-2\alpha} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{2m-2l+1} \\ \mathbf{0} & \frac{1}{\sqrt{2}}\mathbf{I}_{2g-2\alpha} & \mathbf{0} & \mathbf{0} \\ \mathbf{J}_{f+\alpha}\boldsymbol{\Gamma}_m\mathbf{J}_{f+\alpha} & \mathbf{0} & \mathbf{J}_{f+\alpha}\boldsymbol{\Gamma}_p\mathbf{J}_{f+\alpha} & \mathbf{0} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{U}_{00} & \mathbf{U}_{02} & \mathbf{U}_{01} & \sqrt{2}\mathbf{u} & \mathbf{U}_{01}\mathbf{J}_{m-l} & \mathbf{U}_{02}\mathbf{J}_{2g-2\alpha} & \mathbf{U}_{00}\mathbf{J}_{f+\alpha} \\ \mathbf{V}_{00} & \mathbf{V}_{02} & \mathbf{V}_{01} & \mathbf{0} & -\mathbf{V}_{01}\mathbf{J}_{m-l} & -\mathbf{V}_{02}\mathbf{J}_{2g-2\alpha} & -\mathbf{V}_{00}\mathbf{J}_{f+\alpha} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{f+2g-\alpha+2m-2l+1} & \mathbf{0} \\ \mathbf{0} & z^{-1}\mathbf{I}_{f+2g-\alpha} \end{bmatrix} \boldsymbol{\Gamma}_o = \frac{1}{\sqrt{2}}\boldsymbol{\Phi}_0^o\boldsymbol{\Lambda}_o(z)\boldsymbol{\Gamma}_o \quad (17)$$

$p \times (f+\alpha)$, $\mathbf{U}_{02}$ has size $p \times (2g-1-2\alpha)$ and $0 \le \alpha \le g-1$. Matrices $\boldsymbol{\Gamma}_p$ and $\boldsymbol{\Gamma}_m$ are defined in the same way as before. Apply similar parameterized form for $\mathbf{A}_{00}$ with matrices $\mathbf{V}_{00}$ and $\mathbf{V}_{02}$ with the same dimension as $\mathbf{U}_{00}$ and $\mathbf{U}_{02}$, respectively. By such parameterization, it can be shown that $\mathbf{E}_0^o(z)$ has form in (16) at the top of this page, where $\boldsymbol{\Phi}_0^o$ can be further factorized in the form (18), where

$$\boldsymbol{\Phi}_0^o = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{U}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_0 \end{bmatrix} \begin{bmatrix} \mathbf{I}_{m+g-\alpha} & \mathbf{0} & \mathbf{I}_{m+g-\alpha} \\ \mathbf{0} & \sqrt{2} & \mathbf{0} \\ \mathbf{I}_{m+g-\alpha} & \mathbf{0} & \mathbf{I}_{m+g-\alpha} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{m+g-\alpha+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{m+g-\alpha} \end{bmatrix} \quad (18)$$

where matrices $\mathbf{U}_0 = [\mathbf{U}_{00}\ \mathbf{U}_{02}\ \mathbf{U}_{01}\ \mathbf{u}]$ and $\mathbf{V}_0 = [\mathbf{V}_{00}\ \mathbf{V}_{02}\ \mathbf{V}_{01}]$. It can be shown easily that such factorization can guarantee PU property of $\mathbf{E}_0^o(z)$ if $\mathbf{U}_0$ and $\mathbf{V}_0$ are orthogonal matrices with orthogonal columns because $\boldsymbol{\Gamma}_o$ is an orthogonal rectangular matrix with orthogonal columns. It should be pointed out that this factorization is a *new* result. It is not applicable to critically sampled systems because even-channel critically sampled LPFBs with equal filter length cannot have odd length. On the contrary, oversampled systems don't have this constraint. This reveals the higher design freedom of OLPPUFBs. The above result is for even M. The remaining two cases with odd M are discussed in the next section.

3.3 Alternative Structures

In this section, we present alternative structures of odd M for arbitrary filter length to the above two even M cases. Firstly, note that although M is constrained to be even, we can still obtain *arbitrary* filter length for OLPPUFBs in consideration. For the case of odd M, one simple solution is to double M so as to let it be an even number and correspondingly to double the number of channels P to maintain the oversampling ratio r. However, this would increase the dimension severely for large odd M, like 8-channel FB with $M = 7$.

An alternative and better solution presented here is to develop a specific lattice structure for odd $M = 2m+1$. If β is even, i.e., $\beta = 2l$ $(0 \le l \le m)$, the starting block $\mathbf{E}_0(z)$ should have the following form by the LP condition (1),

$$\mathbf{E}_0(z) = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{S}_{00}+z^{-1}\mathbf{S}_{00}\mathbf{J}_{2l} & \mathbf{S}_{01} & \mathbf{q} & \mathbf{S}_{01}\mathbf{J}_{m-l} \\ \mathbf{A}_{00}-z^{-1}\mathbf{A}_{00}\mathbf{J}_{2l} & \mathbf{A}_{01} & \mathbf{0} & -\mathbf{A}_{01}\mathbf{J}_{m-l} \end{bmatrix} \quad (19)$$

On the other hand, if β is odd, i.e., $\beta = 2l+1$ $(0 \le l \le m)$, $\mathbf{E}_0(z)$ should have the following form,

$$\mathbf{E}_0(z) = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{S}_{00}+z^{-1}\mathbf{S}_{00}\mathbf{J}_{2l+1} & \mathbf{S}_{01} & \mathbf{S}_{01}\mathbf{J}_{m-l} \\ \mathbf{A}_{00}-z^{-1}\mathbf{A}_{00}\mathbf{J}_{2l+1} & \mathbf{A}_{01} & -\mathbf{A}_{01}\mathbf{J}_{m-l} \end{bmatrix} \quad (20)$$

It can be seen that the $\mathbf{E}_0(z)$ of these two odd M cases have the same structural form as before in (15) and (8), just with different dimensions for some submatrices. The differences are denoted by the subscripts in the reversal matrices. Thus, we can also obtain the similar rank conditions and lattice factorizations for them along with previous analysis. Due to limited space, the details are omitted here.

It is clear that our proposed lattice structures for type-I OLPPUFBs with equal filter length $L = KM+\beta$ exist for all possible even or odd K, M and β. Therefore, the filter length L can be made really *arbitrary*, i.e., removing the constraint of $L = KM$. Note that, this cannot be achieved in critically sampled systems because of their length constraints, for example, even-channel critically sampled LPPUFBs can only have equal even filter length.

3.4 Completeness of the Structure

It is clear that our proposed lattice structures for type-I OLPPUFBs with equal filter length $L = KM+\beta$ exist for all possible K, M and β. Furthermore, a strong result is that the converse is also true, i.e., our proposed structures can cover all such FBs in consideration. We conclude this section with the following theorem on the completeness issue.

Theorem 4. *The polyphase matrix $\mathbf{E}(z)$ of a type-I OLPPUFB with equal filter length $L = KM+\beta$ can always be factorized as in (4), where its factors are given by (5) and (7) or (17).*

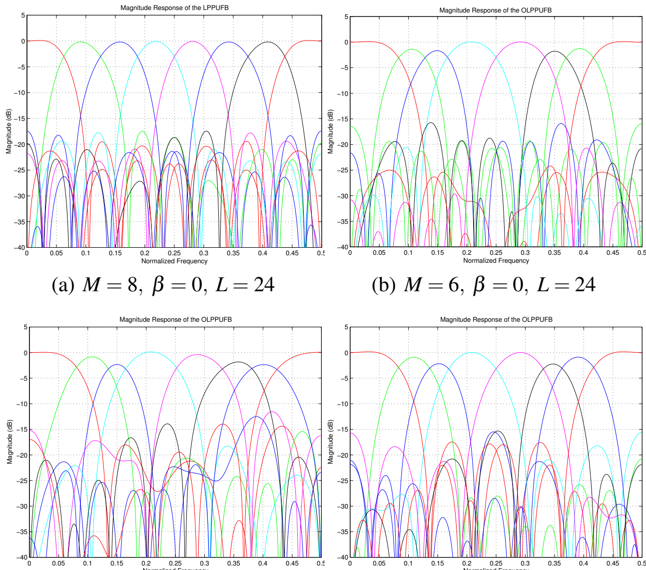
Proof. This theorem can be rephrased as follows: if there exists an analysis polyphase matrix $\mathbf{E}(z)$ of an OLPPUFB with equal filter length $L = KM+\beta$, which must satisfy LP condition (1), then $\mathbf{E}(z)$ can always be factorized as in (4), whereas $\mathbf{E}_0(z)$ can always be factorized as in (6) or (16). The former part is achieved by performing the *order reduction* process as presented in [17, 12, 8]. According to [15, 1], the design space is still complete with the simplification of $\mathbf{G}_i(z)$ in (5). After $K-1$ order reduction steps performed by $\mathbf{G}_i(z)$, the remainder is an order one OLPPUFB with length $M+\beta$ which takes the form in (8) or (15) for different settings of M and β. Thus, we need only prove that $\mathbf{E}_0(z)$ can always be factorized as in (6) or (16) with different dimensions for different settings of M and β. A similar proof for only even-channel critically sampled FBs with even subsampling factor M was given in [12]. For oversampled FBs, due to orthogonality of matrices $\mathbf{S}_{01}$ and $\mathbf{A}_{01}$, we should just show that our proposed parameterized form for matrix $\mathbf{S}_{00}$ can span the space of all possible $\mathbf{S}_{00}$ for both even and odd values of M and β, respectively. The essential idea for its proof is the rank condition of matrix $\mathbf{S}_{00}$ stated in Theorems 2 and 3. For the case of even M and β, the rank of our parameterized form of $\mathbf{S}_{00}$ is $\text{rank}(\mathbf{S}_{00}) = \text{rank}([\mathbf{U}_{00}, \mathbf{U}_{02}]) = l+g-\alpha$ due to orthogonality of matrix $\mathbf{U}_0$. Within all possible values of α, our parameterized form of $\mathbf{S}_{00}$ satisfies the rank condition stated in Theorem 2 and can achieve the lower and upper bounds by $\alpha = g$ and $\alpha = 0$, respectively. Thus, it spans the space of all possible $\mathbf{S}_{00}$. The proof is also applicable to matrix $\mathbf{A}_{00}$. This completes the proof for the case of even M and β. Similar proof can be done for the other cases. □

4. DESIGN EXAMPLES

Two preliminary design examples of our proposed factorization are provided in this section. The chosen optimization criterion is minimization of the stopband attenuation which is a classical cost function in FB design and optimization. Denote the passband of the ith filter $H_i(z)$ as $[\omega_{i,L}, \omega_{i,H}]$ and the transition bandwidth as ε. The cost function to be optimized is,

$$C_{\text{stopband}} = \sum_{i=0}^{P-1} \left(\int_0^{\omega_{i,L}-\varepsilon} |H_i(e^{j\omega})|^2 \mathrm{d}\omega + \int_{\omega_{i,H}+\varepsilon}^{\pi} |H_i(e^{j\omega})|^2 \mathrm{d}\omega \right)$$

ε is selected to be 0.1 for all the following examples.



(a) $M = 8,\ \beta = 0,\ L = 24$ (b) $M = 6,\ \beta = 0,\ L = 24$

(c) $M = 6,\ \beta = 3,\ L = 21$ (d) $M = 6,\ \beta = 4,\ L = 22$

Figure 2: Design examples of 8-Channel LPPUFBs with different subsampling factors M and different β.

Magnitude responses of several design examples on 8-channel LPPUFBs with different subsampling factors M and different β, i.e., different filter lengths $L = KM + \beta$, are shown in Fig. 2. A critically sampled LPPUFB with $\beta = 0$ and $K = 3$, i.e., length $L = 24$ is shown in Fig. 2(a). An OLPPUFB with $M = 6$, $\beta = 0$ and $K = 4$, i.e., length $L = 24$ is shown in Fig. 2(b). These two cases are two special cases of our proposed OLPPUFBs, which were studied in [13] and [8]. The other two design examples are new ones. The third case with $M = 6$, $\beta = 3$ and $K = 3$, i.e., filter length $L = 21$ is shown in Fig. 2(c). Another case with $M = 6$, $\beta = 4$ and $K = 3$, i.e., $L = 22$ is shown in Fig. 2(d). It is clear that these two cases have filter length L which is not multiple of subsampling factor M. Note that these two cases cannot be obtained by previous design methods in [6, 8, 1, 9, 10]. These two preliminary design examples confirm the validity of our proposed lattice structures. They also show the higher design flexibility on filter length L, which can be chosen freely. To our knowledge, the lattice structures and designs for OLPPUFBs with arbitrary filter length is firstly addressed by our proposed methods. In particular, they are more general so that it can cover the class of LPFBs in [13, 12, 8] as special cases.

5. CONCLUSIONS

This paper has presented the theory and lattice structures for a large class of OLPPUFBs, where all analysis filters have the same length $L = KM + \beta (0 \le \beta < M)$ and symmetry center. Necessary existence conditions on the filter symmetry polarity are investigated. Lattice structures for type-I OLPPUFBs with arbitrary filter length are developed. Compared with the reported works [8, 10], our result is more general and can cover a larger class of OLPPUFBs than theirs. Furthermore, our lattice structures are proven to be complete, i.e., all type-I OLPPUFBs can always be factorized by our lattice structures. To our knowledge, this is so far the most general type-I OLPPUFBs in the literature. Finally, several design examples are presented to demonstrate the validity of our new lattice structures. They show the design flexibility for arbitrary filter length L.

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