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Basis orthonormalization procedure impacts of the basic quadratic non-uniform spline space on the scaling and wavelet functions

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ABSTRACT

This paper investigates the mathematical framework of the multiresolution approach under the assumption that the sequence knots are irregularly spaced. The study is based on the construction of nested non-uniform quadratic spline multiresolution spaces. We focus on the construction of suitable quadratic orthonormal spline scaling and wavelet bases. If no more additional conditions than multiresolution ones are imposed, the orthonormal basis of the quadratic spline space is represented, on each bounded interval of the sequence, by three discontinuous scaling functions. Therefore, the quadratic spline wavelet basis, closely related to the scaling basis, is also defined by a set of discontinuous wavelet functions on each bounded interval of the sequence. We show that a judicious orthonormalization procedure of the basic quadratic spline space basis allows to (i) satisfying the continuity conditions of the scaling and wavelet functions, (ii) reducing the number of the wavelet functions to only one function, and (iii) reducing the complexity of the filter bank.

1. INTRODUCTION

Since many years, the multiresolution analysis method has been intensively studied see e.g. ([1], [2], [3], [4]). The scaling and wavelet bases, provided in the literature, are constructed under the assumptions that the knots of the infinite sequence are regularly spaced. However for reasons related to the experiment (e.g. presence of clouds during the measurement of the brightness of a star) or for treatment facilities, signals can not be sampled at regularly spaced moments. Hence the traditional approaches such as the multiresolution analysis, based on regularly sampled data, cannot be used any more just as they are. Our presentation focuses on the multiresolution analysis adapted to non-equally spaced data, located at known knots, thus resulting in a more general definition of the scaling and wavelet functions. Relatively little works have been published about these functions on arbitrary non-uniform spacing knots ([8]). The scaling functions presented in this paper are constructed from the quadratic non-uniform B-spline functions. This choice is directly related to the adaptation of the B-splines (whatever the spline degree), to any bounded interval when imposing multiplicities at each knot of the sequence ([5]). On a non-equally spaced knots sequence, we show that the spline scaling and wavelet functions cannot be obtained as traditionally by translations or dilatations of one prototype function. The main objective of this paper is to construct suitable orthonormal quadratic scaling and wavelet bases within the framework of irregularly spaced data yielding therefore to an easy multiresolution structure.

The outline of this paper is as follows. Section 2 summarizes some necessary background material concerning the quadratic non-uniform basic spline space. Section 3 introduces the multiresolution analysis concepts yielding therefore to the construction of the approximation and detail subspaces. The new orthonormalization procedure of the scaling and wavelet bases is presented in section 4. Section 5 provides the orthogonal decomposition algorithm. Some implementation results are then provided.

2. BASIC QUADRATIC SPLINE SPACE

Before presenting the multiresolution analysis adapted to irregularly spaced data, we briefly introduce the basic definitions necessary for our later developments. We assume that the continuous given signal is represented by its samples located at irregularity positions. These known positions are represented by the following sequence of knots, $t_0 < t_1 < ... < t_i < t_{i+1} < ...$ organized in an increasing order. The quadratic non-uniform B-spline function is represented by a piecewise polynomial of degree two. In this paper, a multiplicity of order 3 is imposed on each knot of the sequence (). It has been shown that, for this particular configuration of knots, only three non-uniform B-spline functions $\{B^2_{0,[t_i,t_{i+1}]}, B^2_{1,[t_i,t_{i+1}]}, B^2_{2,[t_i,t_{i+1}]}\}$, defined on the bounded interval $[t_i, t_{i+1}]$, generates a basis for the basic quadratic spline space ([5]). Thus, the quadratic spline function, denoted $f(t)$, is defined as a linear combination of three quadratic non-uniform B-spline functions:

$$f(t) = \sum_{k=0}^{2} a_{k,[t_i,t_{i+1}]} B^2_{k,[t_i,t_{i+1}]}(t)$$

for $t_i \le t \le t_{i+1}$ and all $i \in N$ (1)

where the expressions of the non-uniform B-spline functions are given as follows:

$B^2_{0,[t_i,t_{i+1}]}(t) = (t_{i+1}-t)^2 / (t_{i+1}-t_i)^2$; $B^2_{2,[t_i,t_{i+1}]}(t) = (t-t_i)^2 / (t_{i+1}-t_i)^2$;

and $B^2_{1,[t_i,t_{i+1}]}(t) = 2(t-t_i)(t_{i+1}-t) / (t_{i+1}-t_i)^2$ (2)

The basic quadratic spline space, denoted V_0, is spanned by the piecewise polynomials of degree two:

$$V_0 = \left\{ f : f(t) = \sum_{k=0}^{2} a_{k,[t_i,t_{i+1}]} B^2_{k,[t_i,t_{i+1}]}(t);\ \text{for all}\, i \in N,\, a_{k,[t_i,t_{i+1}]} \in L_2 \right\}$$

3. MULTIRESOLUTION ANALYSIS CONCEPTS

The construction of orthonormal quadratic spline scaling and wavelet basis starts with the specification of the underlying multiresolution spaces. Let us first consider an infinite sequence S_0 of non-equally spaced knots organized in an increasing order ($t_0 < t_1 < ... < t_i < t_{i+1} < ...$). This sequence is considered as the finest sequence. At a given resolution level j, we define the bounded interval $I_{j,i}$,:as follows:

$$I_{j,i} = \left\lfloor t_{2^j i}, t_{2^j (i+1)} \right\lfloor \qquad (3)$$

At resolution level j, the corresponding sequence S_j is thus built from the union of bounded subintervals $I_{j,i}$ as defined below:

$$S_j = \bigcup_{i=0}^{\infty} I_{j,i} \ \text{ with } i \in N \qquad (4)$$

Going from the resolution level $j-1$ to the resolution level j (coarse resolution) consists in removing one knot out of two in the sequence S_{j-1}. Hence, we obtain obviously a set of embedded sub-sequences as follows:

$$S_0 \supset S_1 \supset ... \supset S_{j-1} \supset S_j .... \qquad (5)$$

A multiresolution analysis consists in approximating a given signal, denoted $s(t)$, at different resolution levels j. Calssicaly, in order to minimize the approximation error ($\left\| s(t) - s_j(t) \right\|$), the approximation of the signal $s(t)$ at resolution level j (i.e. $s_j(t)$) is defined as its orthogonal projection on the subspace on which it belongs. This subspace is denoted V_j and known also as quadratic spline scaling subspace. Remember that functions belonging to the basic spline space V_0 are piecewise polynomials of degree two over bounded intervals $I_{0,i}$. Therefore, functions defined on $I_{j,i}$ are obviously quadratic piecewise polynomials. Moreover, the embedded sub-sequence structure imposes imbrications of the quadratic spline scaling subspaces as follows:

$$V_0 \supset V_1 \supset ... \supset V_{j-1} \supset V_j .... \qquad (6)$$

The orthogonal complement of the subspace V_j in the subspace V_{j-1} is introduced to carry the necessary details improving the signal approximation in the subspace V_{j-1}. Therefore, the orthogonal projection of the signal $s(t)$ on the subspace V_j is decomposed as the sum of orthogonal projections of $s(t)$ on V_j and on the complement subspace denoted W_j. This subspace W_j is known as the quadratic spline wavelet subspace, at resolution level j. This leads to the following traditional relation:

$$V_{j-1} = V_j \oplus W_j \qquad (7)$$

4. ORTHONORMAL QUADRATIC SPLINE SCALING AND WAVELET BASES

In order to satisfy the significant conditions imposed by the multiresolution approach, we propose to study the orthonormalization of the quadratic non-uniform spline basis of the basic spline space V_0. Hence the basic spline space V_0 can also be defined as :

$$V_0 = \left\{ f : f(t) = \sum_k c_k \underline{B}^2_k(t) \quad c_k \in L_2 \right\} \qquad (8)$$

where the set $\left\{ \underline{B}^2_k(t) \right\}$ represents the functions of the orthonormal spline scaling basis which will be constructed by a new orthonormalization procedure improving the features of the previous procedure described in [7].

Let us recall that in the previous procedure, the classical Gram-Schmidt method has been applied to orthonormalize the quadratic spline basis on each bounded interval without considering any relationship between the adjacent intervals. Therefore, the scaling functions carried out on each interval are not continuous at each end-point of consecutive intervals. Moreover the spline wavelet functions are neither continuous, on the bounded interval on which they belong, nor at the end-point of adjacent intervals. However the continuity feature of the functions represents an important problem for some applications.

In this paper, we focus on the construction of scaling and wavelet functions satisfying continuity conditions between consecutive bounded intervals. The proposed orthonormalization procedure is quite different from the previous one. The new procedure is described in such away that its generalization to any degree of the spline function remains obvious.

4.1. Quadratic orthonormal spline scaling basis

We propose to divide the problem into two parts. Initially, we start by dealing the problem of the spline basis orthogonalization. Then, we study the normalization problem under the same constraints of continuity. The proposed approach is closely related to the features of the basic spline basis elements defined in section 1. In fact, the two spline basis functions ($B^2_{0,I_{j,i}}(t), B^2_{2,I_{j,i}}(t)$), defined on any bounded interval $[t_i, t_{i+1}]$, are naturally symmetrical compared to the point $t_{s_i} = (t_i + t_{i+1})/2$. While the third function $B^2_{1,I_{j,i}}(t)$ is its self symmetric compared to the point t_{s_i}. Based on these features, we propose to shift vertically each non-uniform B-spline function belonging to the same bounded interval, using shifting parameters denoted $\left\{ s^2_{k,I_{j,i}} \right\}$ as follows.

$$b^2_{0,I_{j,i}}(t) = s^2_{0,I_{j,i}} + B^2_{0,I_{j,i}}(t)\ ;\ b^2_{1,I_{j,i}}(t) = s^2_{1,I_{j,i}} + B^2_{1,I_{j,i}}(t)\ ;$$

and $b^2_{2,I_{j,i}}(t) = s^2_{2,I_{j,i}} + B^2_{2,I_{j,i}}(t)$ (9)

The parameters $s^2_{k,I_{j,i}}$ ($k = 0,1,2$) are computed in order to guarantee, on any bounded interval $I_{j,i}$ the orthogonality conditions of the functions as given bellow:

$$\left\langle b^2_{k,I_{j,i}}(t), b^2_{l,I_{j,i}}(t) \right\rangle = 0$$

for $k \neq l$; $k = 0,1,2$, $l = 0,1,2$ and all $i \in N$ (10)

We show that the shifting parameters computed according to equation (10) are constant values on each bounded intervals and at any resolution level j. Moreover $s^2_{0,I_{j,i}} = s^2_{2,I_{j,i}}$. So, the parameters are rename as follows:

$$s^2_{k,I_{j,i}} = s^2_k \text{ for } k = 0,1 \text{ and all } i \in N$$

It is easy to show that the function $\underline{b}^2_{1,I_{j,i}}(t)$ ensures naturally the continuity between consecutive intervals since:

$$b^2_{1,I_{j,i}}(t_{2^i i}) = b^2_{1,I_{j,i}}(t_{2^i i+1}) = s^2_1 \text{ for all } i \in N \quad (11)$$

If we swap between the two functions $b^2_{0,I_{j,i}}(t)$ and $b^2_{2,I_{j,i}}(t)$ from one interval to another one, we thus construct two new continuous functions on the entire sequence. Indeed, at the end-points of the intervals $I_{j,i}$ and $I_{j,i+1}$, one can easily check that:

$$b^2_{0,I_{j,i}}(t_{2^i i+1}) = b^2_{2,I_{j,i+1}}(t_{2^i i+1}) = 1 + s^2_0 \text{ for all } i \in N \quad (12)$$

The proposed orthogonal quadratic spline scaling basis of the approximation space V_j is therefore given as follows:

$$\begin{cases} \varphi^2_{j,0}(t) = \sum_{i=0}^{\infty} b^2_{0,I_{j,2i}}(t) + \sum_{i=0}^{\infty} b^2_{2,I_{j,2i+1}}(t) \\ \varphi^2_{j,1}(t) = \sum_{i=0}^{\infty} b^2_{1,I_{j,2i}}(t) + \sum_{i=0}^{\infty} b^2_{1,I_{j,2i+1}}(t) \\ \varphi^2_{j,2}(t) = \sum_{i=0}^{\infty} b^2_{2,I_{j,2i}}(t) + \sum_{i=0}^{\infty} b^2_{0,I_{j,2i+1}}(t) \end{cases} \text{ for } i \in N,\ j \in N \quad (13)$$

Since the orthogonal conditions are satisfied on disjoint intervals of the knots sequence, it easy to check that the orthogonal conditions of the scaling functions $\varphi^2_{j,k}(t)$ are ensured on the global sequence S_j.

The second step consists in normalizing each scaling function $\varphi^2_{j,k}(t)$ on the global knots sequence S_j maintaining therefore the continuity of each scaling function. The orthonormal spline scaling basis, of the approximation space V_j, is then represented by the following three scaling functions:

$$\underline{\varphi}^2_{j,0}(t) = \varphi^2_{j,0}(t) / N_{j,0} ; \underline{\varphi}^2_{j,1}(t) = \varphi^2_{j,1}(t) / N_{j,1} ; \underline{\varphi}^2_{j,2}(t) = \varphi^2_{j,2}(t) / N_{j,2}$$

where $N_{j,k}$ are the normalization parameters at resolution level j.

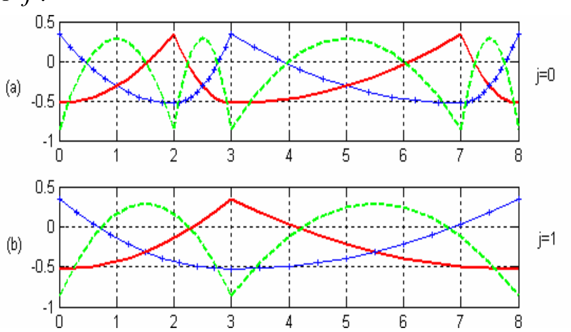

Figure 1: Orthonormal quadratic spline scaling basis at

$$j = 0,1$$

For writing convenience reasons in our later developments, we introduce new normalization factors $n_{j,k}$. The basis functions are renamed as follows:

$$\underline{B}^2_{0,I_{j,i}}(t) = b^2_{0,I_{j,i}}(t) / n_{j,0} ;\ \underline{B}^2_{1,I_{j,i}}(t) = b^2_{1,I_{j,i}}(t) / n_{j,1} \text{, and}$$

$$\underline{B}^2_{2,I_{j,i}}(t) = b^2_{2,I_{j,i}}(t) / n_{j,2}$$

The new orthonormalization factors are introduced according to the parity index i of the bounded interval $I_{j,i}$ as follows:

if i is even: $n^i_{j,0} = N_{j,0}$, $n^i_{j,1} = N_{j,1}$, $n^i_{j,2} = N_{j,2}$

if i is odd: $n^i_{j,0} = N_{j,2}$, $n^i_{j,1} = N_{j,1}$, $n^i_{j,2} = N_{j,0}$.

Figure 1 presents the quadratic spline bases of the respective scaling spaces V_0 and V_1 on the initial finest knots sequence $S_0 = [t_0 = 0, t_1 = 2, t_2 = 3, t_3 = 7, t_4 = 8]$ with the following shifting parameters $s^2_0 = -0.612$ and $s^2_1 = -0.373$ as previously described. The dashed lines correspond to the function $\underline{\varphi}^2_{j,1}(t)$, the solid lines concern the function $\underline{\varphi}^2_{j,2}(t)$, and the marked lines (+) represent the function $\underline{\varphi}_{j,0}(t)$ at each resolution levels.

4.2. Quadratic orthonormal spline wavelet basis

We concentrate now on the construction of the spline wavelet basis. Since the approximation subspace V_0 contains W_1, any wavelet function $\psi^2_{k,I_{1,i}}(t) \in W_1$, can be decomposed, on each bounded interval $I_{1,i}$, using the basis of the approximation space V_0 as follows:

$$\psi^2_{k,I_{1,i}}(t) = \sum_{m=2i}^{2i+1} \sum_{n=0}^{2} g^{m,n}_{1,k} \underline{B}^2_{n,I_{0,i}}(t) \text{, for all } \forall i \in N \quad (14)$$

The wavelet function is parameterized by six coefficients $\{g^{m,n}_{1,k}\}$ which must be computed. In previous work ([7]). the proposed solution suggests to build an orthonormal basis composed of 3 wavelet functions yielding the computation of 18 coefficients. However, the computational procedure becomes too heavy when the degree of the spline function increases.

In this present paper, we propose to process on only one wavelet function $\{\psi^2_{1,I_{1,i}}(t), i \in N\}$, on each bounded interval $I_{1,i}$. According to the multiresolution analysis, the wavelet functions must satisfy the following traditional conditions:

(i) the spline scaling subspace is orthogonal to the wavelet subspace, for any resolution level ($j \geq 1$):

$$\left\langle \psi^2_{1,I_{j,i}}(t), \underline{B}^2_{k,I_{j,i}}(t) \right\rangle = 0 \text{ for } k = 0,1,2,\ \forall i \in N \quad (15)$$

(ii) the orthogonality of the wavelet basis at all and cross scales:

$$\left\langle \psi^1_{1,I_{j,i}}(t), \psi^1_{1,I_{l,p}}(t) \right\rangle = \delta_{jl}\delta_{ip}$$

for $i \in N$, $p \in N$, $j \geq 1$, $l \geq 1$ (16)

Some additional conditions are imposed to the wavelet functions in order to:

(iii) ensuring the C^0 wavelet function regularity inside each bounded interval summarized by the following relation:

$$\sum_{n=0}^{2} g^{2i,n}_{j,1} \underline{B}^2_{n,I_{j-1,i}}(t_{2^{j-1}i}) = \sum_{n=0}^{2} g^{2i+1,n}_{j,1} \underline{B}^2_{n,I_{j-1,i}}(t_{2^{j-1}i})$$

for all $k \in N$, $i \in N$, and $j \geq 1$ (17)

(iv) guaranteeing the continuity of each wavelet function between adjacent bounded intervals, results in:

$$\psi^2_{1,I_{j,i}}(t_{2^j i}) = \psi^2_{1,I_{j,i+1}}(t_{2^j(i+1)}) \text{ for all } i \in N \text{ and all } j \geq 1 \quad (18)$$

The continuity between two adjacent intervals is ensured (condition (iv)), by means of one selected parameter (e.g. the coefficient $g^{m_i,n_i}_{j,1}$) which is updated when going from one bounded interval to the adjacent one. At any resolution level j , the initial value of the selected coefficient is set to be equal to a constant value which is evaluated in such away that the wavelet function is normalized on its all definition domain S_j . The constructed spline wavelet function, belonging to the detail subspace W_j , is therefore given as follows:

$$\psi^2_j(t) = \sum_{i=0}^{\infty} \psi^2_{1,I_{j,i}}(t) \text{ for } j \geq 1 \quad (19)$$

Figure 2 presents the quadratic spline wavelet functions at two resolution levels $j=1$ (graph (a)) and $j=2$ (graph (b)) on the previous initial finest knots sequence S_0 .

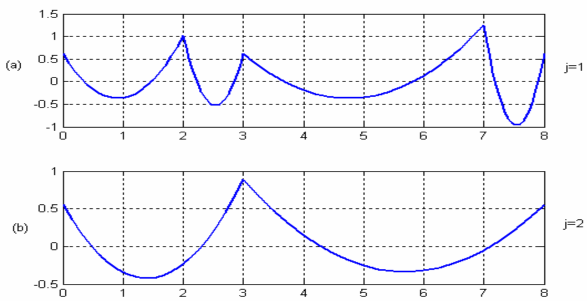

Figure 2 Quadratic spline wavelet functions at $j = 1,2$

5. ORTHOGONAL DECOMPOSITION ALGORITHM

This section concerns the orthogonal decomposition of any signal $s(t)$ represented by its discrete samples irregularly spaced. We assume that this signal belongs to the basic spline space V_0 . The approximation of the signal $s(t)$ at resolution level j, on each bounded interval $I_{j,i}$, is denoted $s_{I_{j,i}}(t)$. It belongs to the spline scaling space V_j . According to the mutiresolution concepts presented earlier, one can decompose any signal belonging to the spline space V_{j-1} , according to the relation $V_{j-1} = V_j \oplus W_j$ as follows:

$$\sum_{n=0}^{1} s_{I_{j-1,i+n}}(t) = s_{I_{j,i}}(t) + r_{I_{j,i}}(t) \text{ for all } i \in N, \ j \geq 1 \quad (20)$$

Where the detail signal ($r_{I_{j,i}}(t) \in W_j$) can be expressed using the spline wavelet basis, on each bounded interval, as follows:

$$r_{I_{j-1,i}}(t) = \sum_{m=2i}^{2i+1} d^i_{j,1} \psi^2_{1,I_{j,m}}(t) \text{ for all } i \in N \text{ and all } j \geq 1 \quad (21)$$

Since the signal $s_{I_{j,i}}(t)$ belongs to the spline scaling space V_j , one can decompose it using the orthonormal scaling basis of the corresponding space V_j as follows:

$$s_{I_{j,i}}(t) = \sum_{k=0}^{2} c^i_{j,k} \underline{B}^2_{k,I_{j,i}}(t) \text{ for all } i \in N \text{, and } j \in N \quad (22)$$

Thus, the approximated signal $s_{I_{j-1,i}}(t) \in V_{j-1}$ is given by the following relation:

$$\sum_{m=2i}^{2i+1} \sum_{k=0}^{2} c^m_{j-1,k} \underline{B}^2_{k,I_{j-1,m}}(t) = \sum_{k=0}^{2} c^i_{j,k} \underline{B}^2_{k,I_{j,i}}(t) + \sum_{m=2i}^{2i+1} d^i_{j,1} \psi^2_{1,I_{j,m}}(t)$$

for all $i \in N$ and all $j \geq 1$ (23)

where the weighted coefficients $\{c^m_{j,k}\}$ (respectively $\{d^i_{j,1}\}$) are given by the orthogonal projection of $s_{I_{j,i}}(t)$ (respectively $r_{I_{j,i}}(t)$) on the spline scaling subspace V_j (respectively W_j). After some manipulations, we show that at any resolution level j, the set of coefficients $\{c^m_{j,k}\}$ are closely related to the coefficients $\{c^m_{j-1,k}\}$ by means of the coefficients set $\{h^{l,k}_{j,n}(t_{2^j i}, t_{2^j(i+1)})\}$ gather into the matrix $\mathbf{H}_{I_{j,i}}$, on each bounded interval $I_{j,i}$, as follows:

$$\underline{\mathbf{c}}_{I_{j,i}} = \mathbf{H}_{I_{j,i}} \underline{\mathbf{c}}_{I_{j-1,i}} \quad (24)$$

The computation of the detail coefficients at any resolution level j, on the interval $I_{j,i}$, are also given by the following relation:

$$\underline{\mathbf{d}}_{I_{j,i}} = \mathbf{G}_{I_{j,i}} \underline{\mathbf{c}}_{I_{j-1,i}} \quad (25)$$

For lack of place, we don't give the expressions of $\mathbf{H}_{I_{j,i}}$ and $\mathbf{G}_{I_{j,i}}$. We present now the implementation of the multiresolution approximations, in the context of irregularly spaced data. The orthogonal decomposition, as seen earlier, processes on the approximation coefficients $\underline{\mathbf{c}}_{I_{j,i}}$ at each resolution level. Therefore an initialization step is required to find the first coefficients set $\underline{\mathbf{c}}_{I_{0,i}}$. Remember that the signal to be decomposed is represented only by its irregularly spaced samples $s(t_i)$ with $t_i \in S_0$. The computation of these first coefficients needs to have the continuous signal $s(t)$. We assume that this signal belongs to the basic spline space V_0 as introduced in section 1:

$$V_0 = \left\{ f : f(t) = \sum_{i=0}^{2} a_{k,I_{0,i}} B^2_{k,I_{0,i}}(t); \text{for } t_i \leq t \leq t_{i+1}, \text{ all } i \in N, a_{k,I_{0,i}} \in L_2 \right\}$$

Consequently, we propose to interpolate the discrete signal, on each bounded interval $I_{0,i}$, using the non-uniform spline basis $\{B^2_{k,I_{0,i}}\}$ of the basic space V_0 according to the following relation:

$$s_{I_{0,i}}(t) = \sum_{k=0}^{2} a_{k,I_{0,i}} B^2_{k,I_{0,i}}(t) \text{ for all } i \in N$$

Recall that the basic spline space V_0 is also represented by the orthonormal scaling spline basis $\{\varphi^2_{k,I_{0,i}}(t)\}$. The orthogonal projection of the signal $s_{I_{0,i}}(t)$, on the basic spline space V_0 allows the computation of the approximation coefficients $\underline{\mathbf{c}}_{I_{0,i}}$:

$$c^i_{0,k} = \left\langle s_{I_{0,i}}(t), \underline{B}^2_{k,I_{0,i}}(t) \right\rangle \Big/ \left\langle \underline{B}^2_{k,I_{0,i}}(t), \underline{B}^2_{k,I_{0,i}}(t) \right\rangle \text{ for } k = 0,1,2$$

and all $i \in N$. The coefficients $\underline{\mathbf{c}}_{I_{0,i}}$ are thus provided as follows:

$$c^i_{0,k} = \sum_{l=0}^{2} a_{l,I_{0,i}} \left\langle B^2_{l,I_{0,i}}(t), \underline{B}^2_{k,I_{0,i}}(t) \right\rangle \Big/ \left\langle B^2_{k,I_{0,i}}(t), \underline{B}^2_{k,I_{0,i}}(t) \right\rangle$$

for $k = 0,1,2$ and all $i \in N$ (26)

The weighted coefficients values $\{a_{k,I_{0,i}}\}$ have been extensively studied in previous works (see [7]). On each bounded $I_{0,i}$, we propose to use the following values:

$$a_{0,I_{0,i}} = s(t_i) \text{ , } a_{1,I_{0,i}} = s(t_i) + s'(t_i)(t_{i+1} - t_i)/2 \text{ , } a_{2,I_{0,i}} = s(t_{i+1})$$

The coefficient $a_{1,I_{0,i}}$ requires the first derivative value of the signal $s(t)$ evaluated at the knot t_i. Let us point out that the quality of the multiresolution approach depends closely on the calculation method from which the first derivative $s'(t_i)$ is evaluated. We propose to construct a quadratic polynomial which passes through three consecutive samples $s(t_{i-1})$, $s(t_i)$, and $s(t_{i+1})$. It is then easy to deduce the first derivative value on the knot t_i. The implementation results are presented by figure 3.

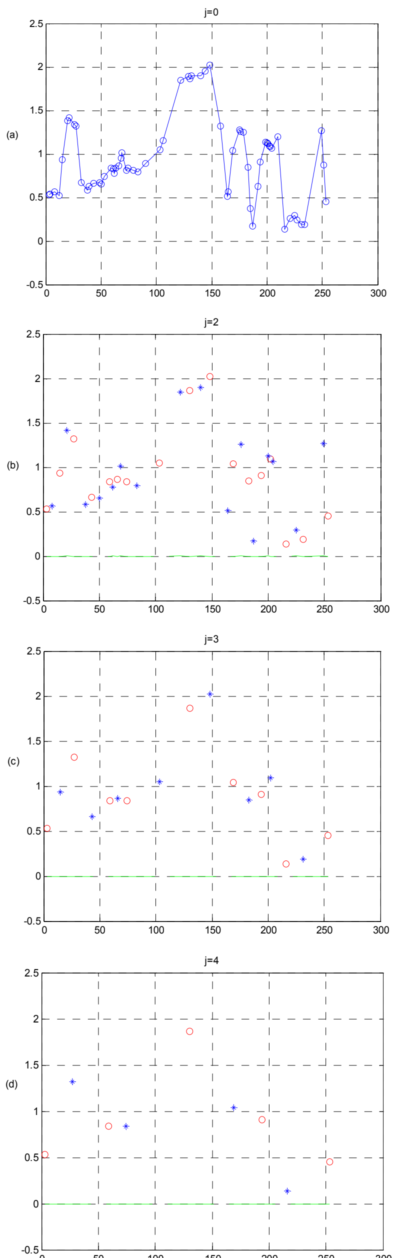

Figure 3 Orthogonal decomposition; 'o' irregularly discrete samples, '*' approximated signals, '-' residual signals

The original signal, given by the graph (a), is represented by its 65 samples irregularly spaced marked by the symbol 'o'. So that we can interpret the results, we present the decomposition at resolution levels $j = 2,3,4$ corresponding respectively to the graphs (b), (c) and (d).The discrete approximated signals are represented by the symbol '*'. The detail signals are represented by solid lines. In this example the residual signals, at each resolution level, turn out to be very small in magnitude. The multiresolution analysis becomes more interesting for spline function degree greater than one (i.e. linear case). Indeed, for the linear case one can imagine easily that the residual signals becomes more important because the approximated signal is given by connecting two consecutive points by a line.

6. CONCLUSION

This paper explored the underlying mathematical framework of the one-dimensional multiresolution analysis based on non-equally spaced knots sequence. The specifications of the multiresolution spaces involve the construction of orthonormal non-uniform quadratic spline scaling and wavelet bases. We have shown that the orthonormalization procedure of the spline basis of the basic spline space affects the scaling and wavelet functions. We have proposed a new orthonormalization procedure which (i) reduces the number of spline wavelet functions to only one wavelet function, (ii) satisfies the continuity conditions of the scaling and wavelet functions on the considered knots sequence. Moreover the orthogonal decomposition is implemented using filter banks less complicated than the filter banks provided by the previous orthonormalization procedure. These first results lead us to investigate, in future work, the construction of spline scaling and wavelet basis for higher degrees of the piecewise polynomials.

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