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QUATERNIONIC APPROACH TO THE ONE-REGULAR EIGHT-BAND LINEAR PHASE PARAUNITARY FILTER BANKS

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ABSTRACT

Besides perfect reconstruction and linear phase, regularity is a desirable essential property of filter banks for image coding as it is associated with the smoothness of the related wavelet basis. This paper shows how to constrain quaternionic factorizations of eight-band linear phase paraunitary filter banks to have the first regularity structurally imposed. The result is not very general but some facts make it notable. Firstly, these systems are a direct extension of the standard eight-point discrete cosine transform (DCT) and this facilitates practical applications. Secondly, the first regularity eliminates the DC leakage which cause visually annoying checkerboard artifact. Finally, our solution offers clear advantages over the known ones as the regularity conditions are formulated directly in terms of quaternionic lattice coefficients. Namely, both regularity and losslessness can be easily preserved regardless of coefficient quantization unavoidable in finite-precision implementations.

1. INTRODUCTION

Filter banks used in image coding must satisfy many requirements besides simple perfect reconstruction (biorthogonality). Paraunitary property (losslessness) is related to a clear relation between the errors in the fullband and subband domains [1]. Thus, the analysis of subband quantization and the design of bit allocation algorithm are easier. In turn, linear phase (LP) responses of filters are necessary to use symmetric extension to efficiently handle the boundaries of the processed finite-length signal [2]. Finally, regular filter banks applied in wavelet-based compression schemes provide smoother signal reconstructions from quantized transform coefficients [3].

The usual way to obtain a filter bank that have desirable properties is to appropriately factorize its polyphase transfer matrix and possibly make some of the factorization constituents imposed by the remaining ones. The parameters of the independent factorization components are related to the degrees of design freedom. Their number should be minimized and unconstrained optimization of their values should be possible — these factors greatly simplify the synthesis of the filter bank .

A lot of factorizations were derived for the mentioned subclasses of filter banks. However, only a few of them maintain their properties after the transition to finite-precision number representation. This is necessary if effective and low-power fixed-point or multiplierless implementations of algorithms are aimed, especially in the case of mobile devices.

A very significant problem is how to obtain paraunitary property together with regularity regardless of coefficient quantization. Lifting schemes promoted recently [4–7] rule out losslessness, though both perfect reconstruction and regularity can be easily obtained in spite of finite precision. On the contrary, the conventional lattice and dyadic-based factorizations, which are lossless in a perfect world, cannot preserve even perfect reconstruction after quantization [8].

With these facts in their minds, the authors have proposed a new family of factorizations which utilize quaternion multiplication as an elementary data transformation [9–12]. The approach is applicable to four- and eight-band paraunitary filter banks (PUFBs) and its main advantage lies in structural losslessness. But it also turns out that regularity can be straightforwardly imposed on the corresponding lattice structures. Namely, this property is equivalent to a certain relation between their hypercomplex coefficients. Thus far, this has been proved only for one-regular four-band systems [13] so the subject is not exhausted. In this contribution, we show how the quaternionic approach can be used to obtain the first regularity in eight-band LP PUFBs. Practical design examples support theoretical derivations to prove their relevance to real implementations.

Notations: Column vectors are denoted by lower-case bold-faced characters, whereas matrices by the upper-case ones. $\mathbf{I}_m$ and $\mathbf{J}_m$ denote the $m \times m$ identity and reverse identity matrices, respectively. The superscript T denotes transposition. The following specific vectors are helpful: $\mathbf{e} = [\,1 \quad 0 \quad 0 \quad 0\,]^T$, $\hat{\mathbf{e}} = [\,0 \quad 0 \quad 0 \quad 1\,]^T$, $\mathbf{a} = [\,1 \quad 1 \quad 0 \quad 0\,]^T$, $\mathbf{0} = [\,0 \quad 0 \quad 0 \quad 0\,]^T$, $\mathbf{o} = [\,1 \quad 1 \quad 1 \quad 1\,]^T$, and $\hat{\mathbf{o}} = [\,1 \quad -1 \quad 1 \quad -1\,]^T$. The L_2-norm is used in the discussion.

2. QUATERNIONIC VS CONVENTIONAL LATTICE STRUCTURES FOR LP PUFBS

2.1 Conventional lattice structure

Linear phase filter banks are characterized by (anti)symmetric impulse responses. For even M, such an M-band paraunitary system is commonly obtained by factorizing its polyphase transfer matrix $\mathbf{E}(z)$ in the following way [2]:

$$\mathbf{E}(z) = \mathbf{G}_{N-1}(z)\mathbf{G}_{N-2}(z)\cdots\mathbf{G}_1(z)\mathbf{E}_0 \tag{1}$$

with

$$\mathbf{E}_0 = \frac{1}{\sqrt{2}}\mathbf{\Phi}_0\mathbf{W}\,\mathrm{diag}\left(\mathbf{I}_{M/2}, \mathbf{J}_{M/2}\right) \tag{2}$$

and

$$\mathbf{G}_i(z) = \frac{1}{2}\mathbf{\Phi}_i\mathbf{W}\mathbf{\Lambda}(z)\mathbf{W}, \qquad i = 1,\ldots,N-1 \tag{3}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{I}_{M/2} \end{bmatrix} \tag{4}$$

$$\mathbf{\Lambda}(z) = \mathrm{diag}\left(\mathbf{I}_{M/2}, z^{-1}\mathbf{I}_{M/2}\right) \tag{5}$$

and

$$\mathbf{\Phi}_i = \mathrm{diag}\left(\mathbf{U}_i, \mathbf{V}_i\right) \tag{6}$$

The design freedom is related to the $M/2 \times M/2$ orthogonal matrices $\mathbf{U}_i$ and $\mathbf{V}_i$. Without affecting the factorization completeness, it can be reduced by taking

$$\mathbf{\Phi}_i = \mathrm{diag}\left(\mathbf{I}_{M/2}, \mathbf{V}_i\right), \qquad i > 0 \tag{7}$$

as it has been proved in [14]. This simplifies the filter bank design process as the dimensionality of the corresponding numerical optimization task is much lower.

Coefficient synthesis is even easier in the subclass of LP PUFBs which have pairwise-mirror-image (PMI) frequency responses (i.e. symmetric with respect to $\pi/2$) [15]. This property can be expressed in terms of both the transfer functions and impulse responses:

$$H_{M-1-k}(z) = \pm H_k(-z) \quad \Leftrightarrow \quad h_{M-1-k}(n) = \pm(-1)^n h_k(n) \quad (8)$$

of the analysis filters ($k = 0,\dots,M-1$). It can be easily obtained by modifying the factorization given above. Namely, it is sufficient to have

$$\mathbf{U}_i = \mathbf{\Gamma}\mathbf{V}_i\mathbf{\Gamma}, \qquad i = 0,\dots,N-2 \qquad (9a)$$
$$\mathbf{U}_i = \mathbf{J}_{M/2}\mathbf{V}_i\mathbf{\Gamma}, \qquad i = N-1 \qquad (9b)$$

in (6), where $\mathbf{\Gamma}$ is a diagonal matrix such that $[\mathbf{\Gamma}]_{mm} = (-1)^{m-1}, m = 1,\dots,M/2$. Thus all freedom degrees are related only to the matrices $\mathbf{V}_i$.

For design purposes, the unconstrained matrices $\mathbf{U}_i$ and $\mathbf{V}_i$ can be parameterized by using Givens rotations [2,8]. The rotation angles can be tuned to maximize the coding gain of the system. This can be done by using unconstrained optimization algorithms which are extremely effective. Unfortunately, for $M \geq 2$, neither a rotation combination nor the equivalent matrix represented directly preserve losslessness after their quantization.

2.2 Quaternions

Quaternions are hypercomplex numbers that have one real and three distinct imaginary parts. The most straightforward quaternion representation is the rectangular one:

$$q = q_1 + q_2 i + q_3 j + q_4 k, \qquad q_1, q_2, q_3, q_4 \in \mathbb{R} \qquad (10)$$

It involves the imaginary units i, j, k related by the identities:

$$\begin{gathered} i^2 = j^2 = k^2 = ijk = -1 \\ ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j \end{gathered} \qquad (11)$$

which constitute quaternion multiplication rule. This operation is noncommutative ($pq \neq qp$) unless one of the operands is a scalar or they are reciprocals of each other. This distinguishes quaternions principally, as other algebraic definitions related to them are nothing more than simple four-dimensional generalizations of those for complex numbers. This is evident in conjugate

$$\overline{q} = q_1 - q_2 i - q_3 j - q_4 k, \qquad (12)$$

norm (modulus)

$$|q|^2 = q\overline{q} = \overline{q}q = q_1^2 + q_2^2 + q_3^2 + q_4^2, \qquad (13)$$

and reciprocal (which is crucial for division existence)

$$q^{-1} = \frac{\overline{q}}{|q|^2} \qquad (14)$$

calculations.

The modulus $|q|$ and three angles ϕ, ψ, and χ constitute the polar representation:

$$\begin{array}{ll} q_1 = |q|\cos\phi & 0 \leq \phi \leq \pi \\ q_2 = |q|\sin\phi\cos\psi & 0 \leq \psi \leq \pi \\ q_3 = |q|\sin\phi\sin\psi\cos\chi & 0 \leq \chi < 2\pi \\ q_4 = |q|\sin\phi\sin\psi\sin\chi & \end{array} \qquad (15)$$

It is useful if fixed-modulus quaternions are parameterized. The unit hypercomplex numbers ($|q| = 1$, and hence $q^{-1} = \overline{q}$) are important in further discussion.

We are also interested in the correspondence between quaternions and four-element column vectors:

$$q \quad \Leftrightarrow \quad \mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^T \qquad (16)$$

Namely, in such a notation, hypercomplex multiplication can be written in two equivalent forms:

$$\begin{aligned} pq \quad \Leftrightarrow \quad & \underbrace{\begin{bmatrix} p_1 & -p_2 & -p_3 & -p_4 \\ p_2 & p_1 & -p_4 & p_3 \\ p_3 & p_4 & p_1 & -p_2 \\ p_4 & -p_3 & p_2 & p_1 \end{bmatrix}}_{\mathbf{M}^+(p)} \times \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \\ = & \underbrace{\begin{bmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & q_4 & -q_3 \\ q_3 & -q_4 & q_1 & q_2 \\ q_4 & q_3 & -q_2 & q_1 \end{bmatrix}}_{\mathbf{M}^-(q)} \times \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \end{aligned} \qquad (17)$$

which utilize different multiplication matrices — the left- $\mathbf{M}^+(\cdot)$ and right-operand $\mathbf{M}^-(\cdot)$ one.

Both the matrices are orthogonal

$$\mathbf{M}^{\pm}(q)^{-1} = \frac{1}{|q|}\mathbf{M}^{\pm}(q)^T \qquad (18)$$

and form groups with respect of multiplication. Moreover

$$\mathbf{M}^{\pm}(q)^T = \mathbf{M}^{\pm}(\overline{q}) \qquad (19)$$

2.3 Quaternionic lattice structure

Block diagonal orthogonal matrices such as $\mathbf{\Phi}_i$ in (6) and (7) can be parameterized by using hypercomplex numbers. The authors have proved in [11] that the factorization:

$$\begin{aligned} &\operatorname{diag}(\mathbf{U},\mathbf{V}) = \\ &= \left(\operatorname{diag}\left(\mathbf{M}^-(s),\mathbf{M}^-(s)\right)\operatorname{diag}\left(\mathbf{M}^-(\overline{r}),\mathbf{M}^-(r)\right)\right)\cdot \\ &\cdot\left(\operatorname{diag}\left(\mathbf{M}^+(\overline{q}),\mathbf{M}^+(q)\right)\operatorname{diag}\left(\mathbf{M}^+(p),\mathbf{M}^+(p)\right)\right) \end{aligned} \qquad (20)$$

with four unit quaternions, is always possible for arbitrary 4×4 orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$. There is some freedom in factor ordering and the expression can be greatly simplified if $\mathbf{U}$ or $\mathbf{V}$ is the identity matrix.

At once, the above facts have been used to derive quaternionic variants of the conventional lattice factorizations for eight-band LP PUFBs shown in the previous section. In particular, the choice:

$$\begin{aligned} \mathbf{\Phi}_0 = &\operatorname{diag}\left(\mathbf{M}^-(s_0),\mathbf{M}^-(s_0)\right)\operatorname{diag}\left(\mathbf{M}^-(\overline{r}_0),\mathbf{M}^-(r_0)\right)\cdot \\ &\cdot\operatorname{diag}\left(\mathbf{M}^+(\overline{q}_0),\mathbf{M}^+(q_0)\right)\operatorname{diag}\left(\mathbf{M}^+(p_0),\mathbf{M}^+(p_0)\right) \end{aligned} \qquad (21)$$

and

$$\mathbf{\Phi}_i = \operatorname{diag}\left(\mathbf{M}^-(\overline{r}_i),\mathbf{M}^-(r_i)\right)\operatorname{diag}\left(\mathbf{M}^+(\overline{q}_i),\mathbf{M}^+(q_i)\right) \qquad (22)$$

for $i = 1,\dots,N-1$, is necessary to cover the whole class of LP systems.

To have PMI property in addition to LP, one must take

$$\begin{aligned} \mathbf{\Phi}_i = &\operatorname{diag}(\mathbf{\Gamma},\mathbf{I}_4)\operatorname{diag}\left(\mathbf{M}^-(q_i),\mathbf{M}^-(q_i)\right)\cdot \\ &\cdot\operatorname{diag}\left(\mathbf{M}^+(p_i),\mathbf{M}^+(p_i)\right)\operatorname{diag}(\mathbf{\Gamma},\mathbf{I}_4) \end{aligned} \qquad (23)$$

for $i = 0,\dots,N-2$ and

$$\begin{aligned} \mathbf{\Phi}_{N-1} = &\operatorname{diag}(\mathbf{J}_4,\mathbf{I}_4)\operatorname{diag}\left(\mathbf{M}^-(q_{N-1}),\mathbf{M}^-(q_{N-1})\right)\cdot \\ &\cdot\operatorname{diag}\left(\mathbf{M}^+(p_{N-1}),\mathbf{M}^+(p_{N-1})\right)\operatorname{diag}(\mathbf{\Gamma},\mathbf{I}_4) \end{aligned} \qquad (24)$$

The main advantage of such an approach is structural losslessness as the compositions of quaternion multiplication matrices maintain orthogonality regardless of coefficient quantization. The subsequent sections show that the technique is also advantageous for regularity investigation.

3. REGULARITY IN QUATERNIONIC LATTICE STRUCTURES

3.1 The concept of regularity

For an M-band filter bank, its regularity is defined as the number of zeros at mirror (aliasing) frequencies $2k\pi/M, k=1,\dots,M-1$ of the lowpass filter $H_0(z)$ [3]. To obtain K degrees of regularity, the polyphase matrix $\mathbf{E}(z)$ must satisfy the following condition [2]

$$\frac{d^n}{dz^n}\left\{\mathbf{E}\left(z^M\right)\left[\begin{array}{cccc} 1 & z^{-1} & \cdots & z^{-(M-1)} \end{array}\right]^T\right\}\Big|_{z=1} = c_n\mathbf{e} \quad (25)$$

with $c_n \neq 0$ for $n=0,\dots,K-1$. In particular, for the first regularity ($K=1$) and the eight-band case, the above expression simplifies to

$$\mathbf{E}(1)\left[\begin{array}{c}\mathbf{o}\\ \mathbf{o}\end{array}\right] = c_0\left[\begin{array}{c}\mathbf{e}\\ \mathbf{0}\end{array}\right] \quad (26)$$

It is easy to verify that this is equivalent to having zero responses of all bandpass filters $H_i(z), i=1,\dots,8$ at DC (zero) frequency. So, a constant input is entirely captured by the lowpass filter, and there is no DC leakage to other bands, which would cause the checkerboard artifact in the case of image coding application [16].

3.2 Conventional regularity formulations

If a particular factorization of the polyphase transfer matrix is assumed, (25) and (26) can be further expanded and possibly simplified. For example, the expression:

$$\mathbf{\Phi}_{N-1}\cdots\mathbf{\Phi}_1\mathbf{\Phi}_0\sqrt{2}\left[\begin{array}{c}\mathbf{o}\\ \mathbf{0}\end{array}\right] = c_0\left[\begin{array}{c}\mathbf{e}\\ \mathbf{0}\end{array}\right] \quad (27)$$

is the one-regularity condition for the eight-band PUFB based on the lattice factorization described in Sec. 2.1. The identities:

$$\frac{1}{2}\mathbf{W}\mathbf{W} = \mathbf{I}_8 \qquad \mathbf{W}\,\mathrm{diag}\left(\mathbf{I}_4,\mathbf{J}_4\right)\left[\begin{array}{c}\mathbf{o}\\ \mathbf{o}\end{array}\right] = 2\left[\begin{array}{c}\mathbf{o}\\ \mathbf{0}\end{array}\right] \quad (28a)$$

$$\mathbf{\Lambda}(1) = \mathbf{I}_8 \qquad \mathbf{G}_i(1) = \mathbf{\Phi}_i \qquad \mathbf{E}_0 = \mathbf{\Phi}_0\sqrt{2}\left[\begin{array}{c}\mathbf{o}\\ \mathbf{0}\end{array}\right] \quad (28b)$$

lie behind its derivation. As the norms of the both sides of (27) have to be equal, and that of $\mathbf{o}$ is 2, $c_0 = \pm\sqrt{2}2$.

The examination of a particular parameterization of the block diagonal orthogonal matrices $\mathbf{\Phi}_i$ allows to express the regularity conditions at the lowest level and incorporate them into filter bank synthesis procedure. In [2,16], this has been done in terms of the rotation angles of the lattice components. Unfortunately, this solution does not take into account that the quantization of rotation matrices leads to certain deviations of their angles. It is much more practical to constrain the coefficients of the computational scheme directly.

This is possible in dyadic- and lifting-based factorizations [4–7]. However, LP imposition is not straightforward in the dyadic-based design techniques, which also do not take the difficulties related to finite precision implementation into account. These problems do not concern lifting schemes but such structures are inherently biorthogonal.

Thus, the solution proposed here, which offers other compromises can be treated as a supplement or an alternative for the existing design methods.

3.3 First regularity for eight-band LP PUFB

Theorem 1. *An eight-band LP PUFB, realized according to the quaternionic approach, is one-regular if and only if*

$$r_{N-1} = \frac{1}{2}\overline{q}_{N-1}\cdots\overline{q}_0 p_0 o \overline{r}_0 s_0 \overline{r}_1\cdots\overline{r}_{N-2} \quad (29)$$

Proof. Knowing the details of the hypercomplex factorization from the previous section, we can expand (27) further. It is equivalent to two separate regularity conditions due to the block diagonal structure of all $\mathbf{\Phi}_i$. The first condition:

$$\begin{aligned}&\mathbf{M}^-\left(\overline{r}_{N-1}\right)\mathbf{M}^+\left(\overline{q}_{N-1}\right)\cdots\mathbf{M}^-\left(\overline{r}_1\right)\mathbf{M}^+\left(\overline{q}_1\right)\cdot\\ &\cdot\mathbf{M}^-\left(s_0\right)\mathbf{M}^-\left(\overline{r}_0\right)\mathbf{M}^+\left(\overline{q}_0\right)\mathbf{M}^+\left(p_0\right)\mathbf{0} = 2\mathbf{0}\end{aligned} \quad (30)$$

is obviously always satisfied, so we can focus on the second one:

$$\begin{aligned}&\mathbf{M}^-\left(\overline{r}_{N-1}\right)\mathbf{M}^+\left(\overline{q}_{N-1}\right)\cdots\mathbf{M}^-\left(\overline{r}_1\right)\mathbf{M}^+\left(\overline{q}_1\right)\cdot\\ &\cdot\mathbf{M}^-\left(s_0\right)\mathbf{M}^-\left(\overline{r}_0\right)\mathbf{M}^+\left(\overline{q}_0\right)\mathbf{M}^+\left(p_0\right)\mathbf{o} = 2\mathbf{e}\end{aligned} \quad (31)$$

This matrix identity can be easily converted into the following expression:

$$\overline{q}_{N-1}\cdots\overline{q}_0 p_0 o \overline{r}_0 s_0 \overline{r}_1\cdots\overline{r}_{N-1} = 2e \quad (32)$$

which involves quaternionic coefficients directly. If we decide to make r_{N-1} constrained, the right-multiplication by $r_{N-1}/2$ leads to (29). □

3.4 First regularity for eight-band LP PUFB with PMI respones

Theorem 2. *An eight-band LP PUFB with PMI property, realized according to the quaternionic approach, is one-regular if and only if*

$$p_{N-1} = \frac{1}{2}\hat{e}\overline{q}_{N-1}\cdots\overline{q}_0\overline{\hat{o}}\overline{p}_0\cdots\overline{p}_{N-2} \quad (33)$$

Proof. The reasoning is conducted similarly to that for general LP PUFB. Given the factorization details in (23) and (24), we can expand (27) further and concentrate on the expression:

$$\begin{aligned}&\mathbf{J}_4\mathbf{M}^-\left(q_{N-1}\right)\mathbf{M}^+\left(p_{N-1}\right)\mathbf{\Gamma\Gamma}\mathbf{M}^-\left(q_{N-2}\right)\mathbf{M}^+\left(p_{N-2}\right)\mathbf{\Gamma}\cdots\\ &\cdots\mathbf{\Gamma}\mathbf{M}^-\left(q_0\right)\mathbf{M}^+\left(p_0\right)\mathbf{\Gamma}\mathbf{o} = 2\mathbf{e}\end{aligned} \quad (34)$$

The identities $\mathbf{\Gamma\Gamma} = \mathbf{I}_4$, $\mathbf{\Gamma o} = \hat{\mathbf{o}}$ and $\mathbf{J}_4\mathbf{e} = \hat{\mathbf{e}}$ allow its simplification to the form:

$$\begin{aligned}&\mathbf{M}^-\left(q_{N-1}\right)\mathbf{M}^+\left(p_{N-1}\right)\mathbf{M}^-\left(q_{N-2}\right)\mathbf{M}^+\left(p_{N-2}\right)\cdots\\ &\cdots\mathbf{M}^-\left(q_0\right)\mathbf{M}^+\left(p_0\right)\hat{\mathbf{o}} = 2\hat{\mathbf{e}}\end{aligned} \quad (35)$$

which corresponds to the quaternionic equality:

$$p_{N-1}\cdots p_0\hat{o}q_0\cdots q_{N-1} = 2\hat{e} \quad (36)$$

To have p_{N-1} imposed by the remaining parameters, as in (33), both the sides must be subsequently right-multiplied by the reciprocals of q_{N-1} to p_{N-2}. □

3.5 Remark on finite precision arithmetic

Although the derived conditions (33) and (33) are clear, it is not so easy to maintain the regularity of quantized lattice structure. If the number N of the factorization stages increases, then higher precision in required to represent the constrained quaternion. Thus, sophisticated optimization algorithms for binary hypercomplex multipliers as well as entire filter bank implementations seem to be an appropriate subject of further research.

4. DESIGN EXAMPLES

4.1 Coefficient synthesis methodology

Filter bank synthesis was performed using polar form (15) of quaternions. The optimal angles related to the unconstrained quaternion lattice coefficients in (29) and (33) were searched using standard Matlab routines intended for unconstrained optimization i.e. `fminunc` and `fminsearch`. The goal was to maximize the stopband attenuation of the magnitude responses together with the coding gain calculated assuming an AR(1) input process with unit variance and the correlation coefficient of 0.95.

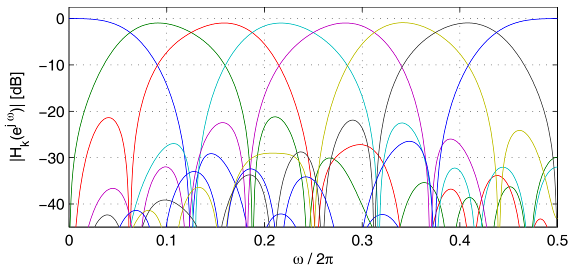

Figure 1: Magnitude responses of the designed LP PUFB.

Table 1: Quaternion coefficients giving the responses in Fig. 1.

Coeff.	Re	Im_i	Im_j	Im_k
s_0	-0.0297993	0.0669165	0.8901262	-0.4497883
p_0	-0.2784728	-0.5164164	0.6659689	-0.4607087
q_0	-0.9687160	-0.0713704	-0.2281805	0.0665523
q_1	0.2571736	-0.2085346	-0.8265446	0.4551912
q_2	0.8835067	-0.2654561	0.0588609	-0.3814242
r_0	-0.1264641	0.6558758	-0.1001648	-0.7374285
r_1	0.9928326	0.0690029	-0.0687024	-0.0692971
r_2	0.9069558	0.3701135	-0.1523974	-0.1312336

Table 2: Rational coefficients giving the characteristics in Fig. 2.

Coeff.	Re	Im_i	Im_j	Im_k	Wordlength
s_0	1/2	0	7/8	-1/8	4
p_0	-1/4	-1/2	5/8	-1/2	4
q_0	-15/16	-1/16	-1/4	0	5
q_1	1/4	-1/4	-7/8	1/2	4
q_2	7/8	-1/4	1/8	-3/8	4
r_0	-1/16	15/16	1/4	-1/4	5
r_1	15/16	1/16	-1/16	-1/16	5
r_2	$\frac{3820719}{2^{22}}$	$\frac{611699}{2^{21}}$	$-\frac{54443}{2^{19}}$	$-\frac{435031}{2^{22}}$	23

4.2 LP PUFBs

Two design examples are provided for ordinary LP PUFBs. In both the systems filter length is 24, so the corresponding factorizations consist of three stages. The difference is in the precision of coefficient representation.

In the first case, it was infinite and the achieved coding gain equals 9.37 dB at the minimal stopband attenuation of 21 dB. The magnitude response of the filter bank is shown in Fig. 1 followed by Table 1, which contains the coefficient values.

In the second example, the coefficients are the rational numbers. Such a limitation facilitates multiplierless implementations but also causes a certain deterioration of achievable system characteristics. For the values in Table 2 the coding gain decreases to 9.30 dB and stopband attenuation to 13 dB as Fig. 2a shows.

Both the considered systems are one-regular owing to the satisfaction of the condition (29) by their coefficients. For rational coefficient LP PUFB, the plots related to its regularity are shown in Fig. 2b and 2c.

4.3 LP PUFBs with PMI property

The theoretical results for LP PUFBs with PMI property are also supported with two design examples, which again differ in the coefficient domain.

For coefficients represented with infinite precision, their values in Table 3 have been obtained. Coding gain is 9.37 dB and stopband attenuation is not less than 22 dB as Fig. 3 shows.

For the rational coefficients given in Table 4, there is a noticeable change in the coding gain. It equals 9.25 dB now, whereas the stopband attenuation decreases to 19 dB, as it can be seen in Fig. 4a.

(a)
(b)
(c)

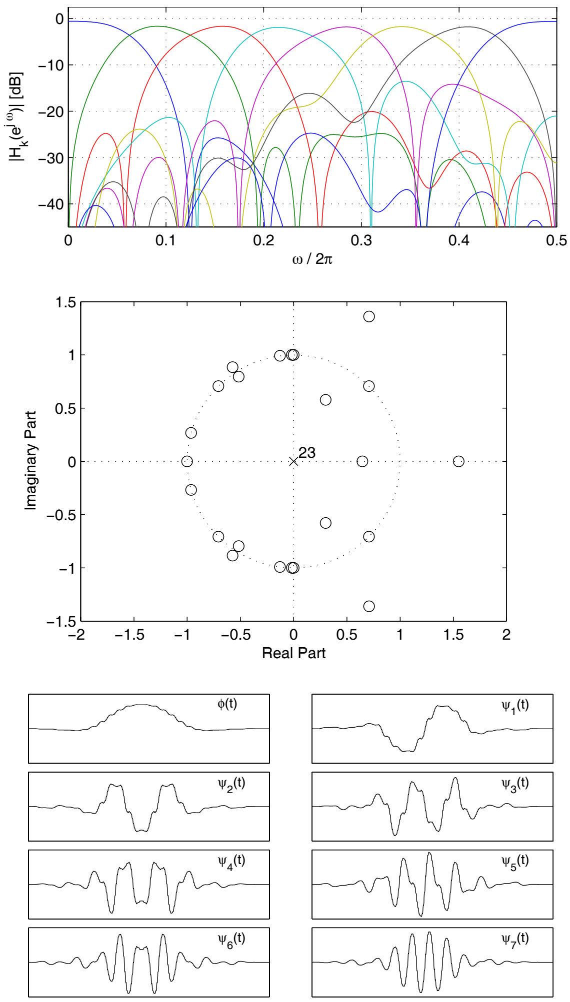

Figure 2: Designed rational-coefficient LP PUFB: magnitude responses (a), zeros of $H_0(z)$ (b), scaling function and wavelets (c).

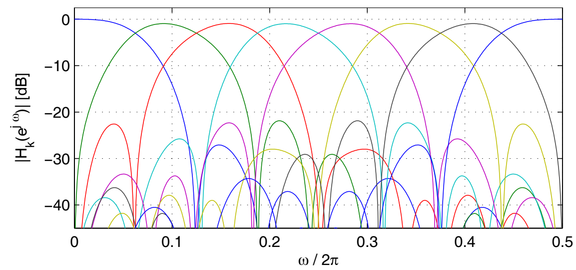

Figure 3: Magnitude response of the designed LP PUFB with PMI.

Table 3: Quaternion coefficients giving the responses in Fig. 3.

Coeff.	Re	Im_i	Im_j	Im_k
p_0	-0.0546530	-0.8381704	-0.2287760	0.4920823
p_1	0.9557711	0.0614376	0.1436822	0.2491636
p_2	-0.6224147	0.0345565	-0.5578179	-0.5479461
q_0	0.9916213	0.0606809	-0.0625301	-0.0953675
q_1	-0.0144669	0.6629532	0.1228110	0.7383774
q_2	-0.9805812	-0.0261340	0.0941266	0.1700519

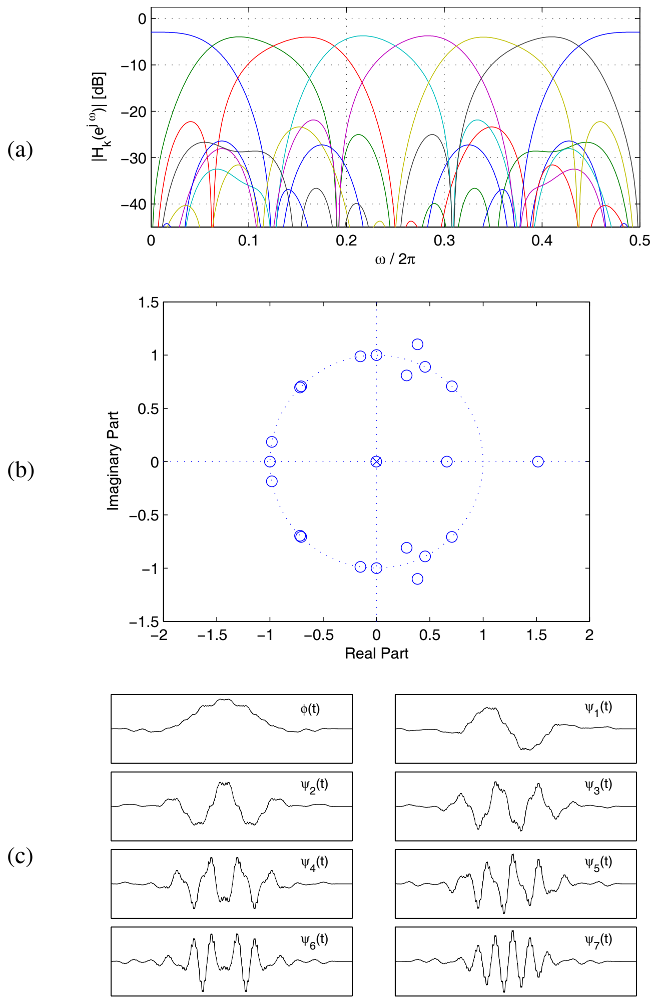

Figure 4: Designed rational-coefficient LP PUFB with PMI property: magnitude responses (a), zeros of $H_0(z)$ (b), scaling function and wavelets (c).

Table 4: Rational coefficients giving the characteristics in Fig. 4.

Coeff.	Re	Im$_i$	Im$_j$	Im$_k$	Wordlength
p_0	-1/8	-7/8	-1/4	1/2	4
p_1	7/8	0	1/8	1/4	4
p_2	$-\frac{4307}{2^{13}}$	$\frac{1349}{2^{14}}$	$-\frac{8563}{2^{14}}$	$-\frac{3261}{2^{13}}$	15
q_0	7/8	0	0	-1/8	4
q_1	0	5/8	1/8	3/4	4
q_2	-1	-1/16	1/16	1/8	5

The provided plots confirm that the incorporation of condition (33) into the synthesis routine guarantees the first regularity of both the filter banks.

5. CONCLUSIONS

An alternative application of the quaternionic approach in eight-band LP PUFB design was considered. The technique was used to structurally impose the first regularity in addition to inherent orthogonality. The concisely formulated constraints on quaternionic lattice coefficients can be easily included into a design procedure, even if a fixed-point implementation is aimed. Appropriate design examples are given to confirm the solution.

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