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SUBSPACE-BASED ESTIMATION OF DIRECTION-OF-ARRIVAL WITH KNOWN DIRECTIONS

Guillaume Bouleux and Rémy Boyer

Laboratoire des Signaux et Systèmes (LSS)
CNRS, Université Paris XI (UPS), SUPELEC
Gif-Sur-Yvette, France
remy.boyer@lss.supelec.fr

ABSTRACT

Estimation of Directions-Of-Arrival is an important problem in various applications. A priori *knowledge on the signal location is sometimes available and previous works have exploited this prior-knowledge. The principle is to "orthogonally" deflate the signal subspace and therefor to cancel the known part of the steering matrix. Our solution is based on a simple modification of the well-known MUSIC criterion by substituting the classical Moore-Penrose pseudo-inverse by the* obliquely weighted *pseudo-inverse. The later is in fact an efficient way to introduce prior-knowledge into subspace fitting techniques.*

1. INTRODUCTION

Directions-Of-Arrival (DOA) of narrow-band sources estimation is one of the central problems in passive radar, sensor sonar, radio-astronomy, and seismology. This problem has received considerable attention in the last 30 years, and a variety of techniques for its solution have been proposed. Sometimes, in practical situations, we have the knowledge of some *a priori* known signal direction (DOA) and several methods have been proposed to incorporate this prior-knowledge into estimation algorithm. Prior-knowledge of DOA can be classified into two families depending if we assume *soft* or *hard* constraints. *Soft* constraints mean that we known approximatively *all* the signal directions. This class of method is known under the name of beamspace methods [11] and has received attention as data reduction methods. The second class of approach incorporates the exact knowledge of a subset of the signal directions. This constraint is somewhat more restricting but more interesting gains can be expected. The exact knowledge of signal direction allows the deflation of the signal subspace and only the remaining directions have to be estimated. In [4], a Constrained-MUSIC (CMUSIC) algorithm has been presented. The key idea is to orthogonally project the data onto the noise subspace spanned by the steering vectors associated to the known directions. In [3], the authors show that the orthogonal deflation of the signal subspace leads to a smaller variance for highly correlated or coherent sources but cannot help for uncorrelated sources with closely-spaced DOA. In addition, they suggest to use not only orthogonal projectors but also oblique projectors.

Based on this principle, we propose to modify the MUSIC [8, 10] Least-Squares (LS) criterion in a view to tacking into account prior knowledge of $M-S$ (among M) DOA. Toward this end, we rewrite the MUSIC criterion in the context of the oblique projector algebra. The resulting LS criterion can be decomposed into the sum of two contributions. The first term is the CMUSIC criterion and the second one is a corrective function which integrates the prior-knowledge. Finally, we show that our solution outperforms the CMUSIC algorithm for uncorrelated sources with closely-spaced DOA. In addition, we show the remarkable robustness of this algorithm to a small error on the prior-knowledge. Note that our methodology is also valid for all the methods belonging to the class of subspace fitting techniques [14]. In addition, some potential applications of this algorithm can be considered in biomedical signal analysis [12] and harmonic retrieval [13].

2. ALGEBRA OF OBLIQUE PROJECTORS

We start with a brief discussion of oblique projections [1], and we recall that the only requirement of a matrix $E_{[X\ Y]}$ to be a projector is $E_{[X\ Y]}$ is idempotent, *ie.*, $E^2_{[X\ Y]} = E_{[X\ Y]}$. Let $\mathcal{R}(X)$ and $\mathcal{R}(Y)$ be subspaces of $\mathbb{C}^L$ that intersect trivially, *ie.*, $\mathcal{R}(X) \cap \mathcal{R}(Y) = \{0\}$. Then, the projector on $\mathcal{R}(X)$ along $\mathcal{R}(Y)$ is the linear operator $E_{[X\ Y]}$ satisfying:

- $\forall x \in \mathcal{R}(X),\ E_{[X\ Y]}x = x,$
- $\forall y \in \mathcal{R}(Y),\ E_{[X\ Y]}y = 0,$
- $\forall z \in \mathbb{C}^L,\ E_{[X\ Y]}z \in \mathcal{R}(X).$

The geometric interpretation of the above properties is $\forall s = x + y + z \in \mathbb{C}^L$ where $x \in \mathcal{R}(X)$, $y \in \mathcal{R}(Y)$ and $z \in (\mathcal{R}(X) \cup \mathcal{R}(Y))^\perp$ then $E_{[X\ Y]}s = x$. So, the complex Euclidean Space is decomposed according to $\mathbb{C}^L = (\mathcal{R}(X) \cup \mathcal{R}(Y))^\perp \oplus \mathcal{R}(X) \oplus \mathcal{R}(Y)$. Let V be a complex matrix having full column rank, obtaining by the concatenation of matrices X and Y according to $V = [X\ Y]$. The orthogonal projector onto $\mathcal{R}(V)$ is then defined as

$$P_V = VV^\dagger = E_{[X\ Y]} + E_{[Y\ X]} \tag{1}$$

where $(.)^\dagger$ denotes the Moore-Penrose pseudo-inverse [13] and

$$E_{[X\ Y]} = X(X^H P_Y^\perp X)^{-1} X^H P_Y^\perp. \tag{2}$$

This property is important since it highlights the link between the orthogonal projector P_V and oblique projectors $E_{[X\ Y]}$ and $E_{[Y\ X]}$. In addition, the ranges for $E_{[X\ Y]}$ and $E_{[Y\ X]}$ are $\mathcal{R}(X)$

and $\mathcal{R}(Y)$, respectively, and the null spaces for $E_{[X\ Y]}$ and $E_{[Y\ X]}$ are $\mathcal{R}(Y) \oplus \mathcal{R}(V^{\perp})$ and $\mathcal{R}(X) \oplus \mathcal{R}(V^{\perp})$, respectively. Finally, note that

$$E_{[Y\ X]}E_{[X\ Y]} = E_{[X\ Y]}E_{[Y\ X]} = 0. \tag{3}$$

3. MATRIX-BASED REPRESENTATION OF THE DOA ESTIMATION PROBLEM

In this section, we introduce the classical matrix-based representation of DOA estimation problem for Uniform Linear Array (ULA[1], [5]) and we define the notion of partitioned Vandermonde matrix.

3.1. Parametric Multi-Input Multi-Output (MIMO) model

Assume there are M narrowband plane waves simultaneously incident on an ULA with L sensors. The array response for the t-th snapshot is given by

$$x(t) = Z\alpha(t) + b(t) \tag{4}$$

where $x(t) = [x_1(t) \dots x_L(t)]^T$, $b(t) = [b_1(t) \dots b_L(t)]^T$, $\alpha(t) = [\alpha_1(t) \dots \alpha_M(t)]^T$ and $x_\ell(t)$ is the observation on the ℓ-th sensor, $\alpha_m(t)$ is m-th source and $b_\ell(t)$ is the zero-mean Gaussian white noise of variance σ^2. In addition, the thin $L \times M$ Vandermonde steering matrix is defined by $[Z]_{nm} = \frac{1}{\sqrt{L}} e^{-2i\pi(\Delta/\lambda)\sin(\theta_m)n}$ where θ_m is the m-th DOA, Δ is the distance between two consecutive sensors and λ is the wavelength. Parameter M is assumed to be known or previously estimated [9]. So, the final MIMO model for T snapshots is

$$X = [x(1) \dots x(T)] = Z\Lambda + B \tag{5}$$

where $\Lambda = [\alpha(1) \dots \alpha(T)]^T$ and $B = [b(1) \dots b(T)]$.

3.2. Partitioned steering matrix and deflated subspace

Assume that we know $M - S$ DOA among M. Without loss of generality, the Vandermonde matrix Z can be partitioned according to

$$Z = \begin{bmatrix} A & B \end{bmatrix} \tag{6}$$

where the $L \times S$ matrix A is the matrix composed by the S desired DOA and B collects the $M - S$ *a priori* known DOA. As $S \leq M$, matrix A (respectively B) is a rank-S (rank-$(M - S)$) matrix. Note that it is always possible to rewrite Z according to expression (6) since we can introduce a non-deficient permutation matrix such as $ZP = [A\ B]$. We name $\mathcal{R}(A)$ the deflated signal subspace since its dimension is $M - S$ which is smaller than the dimension of the signal subspace $\mathcal{R}(Z)$. We have $\mathcal{R}(A) \subseteq \mathcal{R}(Z)$. For simplicity, we assume that the sources associated to the known and to the unknown parts are uncorrelated. Consequently, the spatial covariance matrix is block-diagonal and is defined according to

$$\begin{aligned} R_X &= ZR_\Lambda Z^H + \sigma^2 I & (7)\\ &= R_A + R_B + \sigma^2 I & (8) \end{aligned}$$

where $R_A = AR_{\Lambda_A}A^H$ with R_{Λ_A} the spatial covariance of the unknown sources and $R_B = BR_{\Lambda_B}B^H$ with R_{Λ_B} the spatial covariance of the known sources.

4. PRIOR MUSIC-LIKE ALGORITHM

It can be seen that the MUSIC algorithm attempts to find one component at a time which is most orthogonal to the noise subspace. Let $p(\theta)$ be a generic test vector parameterized by θ which is given by

$$p(\theta) = \frac{1}{\sqrt{L}} \begin{bmatrix} 1 & e^{-2i\pi(\Delta/\lambda)\sin(\theta)} & \dots & e^{-2i\pi(\Delta/\lambda)\sin(\theta)(L-1)} \end{bmatrix}^T.$$

The basic unconstrained MUSIC-like optimization problem can be described according to

$$\arg\min_\theta f(\theta) \text{ where } f(\theta) = \left\| P_Z^\perp p(\theta) \right\|^2 \tag{9}$$

with $P_Z^\perp = I - ZZ^\dagger$ the orthogonal projector associated to $\mathcal{R}(Z)$. Now, consider a new criterion associated to the Prior-MUSIC algorithm

$$\arg\min_\theta f'(\theta) \text{ where } f'(\theta) = \left\| \left(I - ZZ^\ddagger \right) p(\theta) \right\|^2 \tag{10}$$

where $Z^\ddagger = Z^\dagger E_{[A\ B]}$ denotes the *obliquely weighted* pseudo-inverse[2] where $E_{[A\ B]}$ is the oblique projector on the unknown space $\mathcal{R}(A)$ along the known space $\mathcal{R}(B)$ defined in (2).

After some derivations and using expressions (1), (3) and the fact that all projectors are idempotent, the cost function in criterion (10) can be rewritten according to

$$f'(\theta) = \left\| \left(P_Z^\perp + E_{[B\ A]} \right) p(\theta) \right\|^2. \tag{11}$$

Now, one can easily verify that $P_Z^\perp E_{[A\ B]} = 0$ and therefor expressions (10) and (11) become

$$\arg\min_\theta f'(\theta) \text{ where } f'(\theta) = f(\theta) + \left\| E_{[B\ A]} p(\theta) \right\|^2. \tag{12}$$

As we can see, the above expression is a MUSIC-like criterion with an additional corrective term which takes into account the prior-knowledge. Note that

- $\forall p(\theta) \in \mathcal{R}(Z) = \mathcal{R}(A) \cup \mathcal{R}(B)$, $f(\theta) = 0$ and thus is minimal.
- $\forall p(\theta) \in \mathcal{R}(A)$, the corrective term, $\left\| E_{[B\ A]} p(\theta) \right\|^2 = 0$. So, criterion (12) is null.
- $\forall p(\theta) \in \mathcal{R}(B)$, the corrective term is
$$\left\| E_{[B\ A]} p(\theta) \right\|^2 = \|p(\theta)\|^2 = 1.$$
So, criterion (12) is not minimal.

[1] Note that the ULA assumption is only use to propose, in section 5, a *root* version of the PMUSIC algorithm. Otherwise, for arbitrary array geometry, we can use the *spectral* version of this algorithm.

[2] Note that solution $\alpha = Z^\ddagger p(\theta)$ is the minimal norm solution of criterion $\arg\min_{\theta,\alpha} ||p(\theta) - E_{[A\ B]} Z\alpha||^2$ subject to $p(\theta) \in \mathcal{R}(A)$.

4.1. Estimation of projectors $P_Z^\perp$ with known part B

The estimation of projector $P_Z^\perp$, denoted by

$$\hat{P}_Z^\perp = I - \begin{bmatrix}\hat{A} & B\end{bmatrix}\begin{bmatrix}\hat{A} & B\end{bmatrix}^\dagger, \tag{13}$$

is obtained by the methodology introduced in the context of the CMUSIC algorithm [4]. Hereafter, symbol ˆ denotes unknown quantities.

4.2. Estimation of oblique projectors

4.2.1. Invariant to change of basis

Let us begin by an important remark. The oblique projectors $E_{[B\ A]}$ and $E_{[A\ B]}$ are invariant to change of basis. Indeed a basis of space $\mathcal{R}(A)$ is not unique, so consider (for instance through the SVD) another basis Φ such as $\mathcal{R}(A) = \mathcal{R}(\Phi)$. We know that there exits an invertible matrix Θ such as $\Phi\Theta = A$. In that case, it comes the two following equalities:

$$E_{[B\ A]} = E_{[B\ \ A\Theta^{-1}]} \text{ and } E_{[A\ B]} = E_{[A\Theta^{-1}\ \ B]}. \tag{14}$$

This invariance property for $E_{[B\ A]}$ is a consequence of the fact that $P_A^\perp$ is essentially unique since $P_A = A\Theta^{-1}\Theta A^\dagger = AA^\dagger$. For projector $E_{[A\ B]}$, we can show this result in the following manner:

$$\begin{aligned} E_{[A\Theta^{-1}\ B]} &= A\Theta^{-1}\Theta\left(A^H P_B^\perp A\right)^{-1}\left(\Theta^{-1}\Theta\right)^H A^H P_B^\perp \\ &= E_{[A\ B]}. \end{aligned}$$

4.2.2. Estimation of projector $E_{[B\ A]}$

Knowing matrix B, we need to estimate projector $P_A^\perp$ in $E_{[B\ A]}$ to compute the corrective term in expression (12). Now, observe that

$$P_B^\perp Z = \left(I - BB^\dagger\right)\begin{bmatrix}A & B\end{bmatrix} = \begin{bmatrix}P_B^\perp A & 0\end{bmatrix}. \tag{15}$$

After that, consider the following weighted spatial covariance matrix:

$$\bar{R}_X = P_B^\perp R_X. \tag{16}$$

Note that it is possible to show that projector $P_B^\perp$ does not destroy the statistic distribution of the noise. In addition, as R_X admits a Vandermonde decomposition according to expression (7), it comes

$$\begin{aligned} \bar{R}_X &= P_B^\perp Z R_\Lambda Z^H + \sigma^2 I & (17)\\ &= P_B^\perp A R_{\Lambda_A} A^H + \sigma^2 I & (18) \end{aligned}$$

and the rank of $\bar{R}_X$ is S. The above expression shows that we can estimate projector $P_A^\perp$ according to

$$\hat{P}_A^\perp = \bar{V}^*\bar{V}^T \tag{19}$$

where $\bar{V}$ is constituted by the $L - S$ last columns of the right singular-basis V of the sample weighted spatial covariance $P_B^\perp XX^H/T$. Note that we use the right basis since the left basis is corrupted by projector $P_B^\perp$. Finally, using expression (19), the estimated oblique projector is given by

$$\hat{E}_{[B\ A]} = B\left(\hat{P}_A^\perp B\right)^\dagger = \left(\hat{P}_A^\perp P_B\right)^\dagger. \tag{20}$$

4.3. Spectral Prior-MUSIC algorithm

Let us define

$$\begin{aligned} f_{\texttt{CMUSIC}}(\theta) &= p(\theta)^H \hat{P}_Z^\perp p(\theta), \\ f_{\texttt{COR}}(\theta) &= p(\theta)^H \left(\hat{P}_A^\perp P_B \hat{P}_A^\perp\right)^\dagger p(\theta). \end{aligned}$$

Based on the above notations and expression (12), we introduce the spectral-PMUSIC criterion according to

$$\arg\max_\theta\ \mathcal{C}(\theta)^{-1} \text{ where } \mathcal{C}(\theta) = f_{\texttt{CMUSIC}}(\theta) + f_{\texttt{COR}}(\theta). \tag{21}$$

It is straightforward to see that the peaks in the pseudo-spectrum, $\mathcal{C}(\theta)^{-1}$, coincide with the unknown DOA.

5. ROOT PRIOR-MUSIC

The enumerative search procedure associated to the spectral PMUSIC criterion (21) is a costly operation. Thanks to the ULA assumption, we expose the ”root” version of the PMUSIC algorithm which has a lower complexity cost. In addition, it is well-known that the ”root” version of the MUSIC-like algorithms is superior to its spectral form [7].

5.1. Root-CMUSIC

The criterion of the root-CMUSIC is based on polynomial $f_{\texttt{CMUSIC}}(z)$ where $z = e^{-2i\pi(\Delta/\lambda)\sin(\theta)}$. Due to the ULA assumption, $p(.)$ has a Vandermonde structure and the DOA estimation problem can be formulated in term of finding the zeros of the above conjugate centro-symmetric polynomial of degree $2L-2$. This symmetry is a consequence of the Hermitian character of projector $\hat{P}_Z^\perp$ and the explicit computation of the coefficients of $f_{\texttt{CMUSIC}}(z)$ denoted by $\{q_\ell\}_{\ell\in[1-L:L-1]}$ is given by summing along the diagonal of the projector matrix. In addition, we have $q_\ell = q_{-\ell}^*$ and q_0 is real and equals to $\texttt{Tr}(\hat{P}_Z^\perp) = L - M$. Moreover, one can easily verify that $f_{\texttt{CMUSIC}}(z)$ is equal to its reciprocal polynomial [2] and therefor if z_m is a zero then ${z_m^*}^{-1}$ is also a zero, *ie.*, $(z_m, {z_m^*}^{-1})$ occur in pairs. Note that for the M desired DOA, we have constraint $|z_m| = 1$, *ie.*, DOA belong to the unit circle. In presence of noise, the DOA may be extracted (among $2L-2$ possible roots) based on their proximity to the unit circle.

5.2. Polynomial form of the corrective function

In this part, we follow the same methodology as for the root-MUSIC approach, and we associate a polynomial form to $f_{\texttt{COR}}(\theta)$ such as: for all unknown DOA, polynomial $f_{\texttt{COR}}(z)$ must be zero. Note that due to the fact that $\left(\hat{P}_A^\perp P_B \hat{P}_A^\perp\right)^\dagger$ is Hermitian, the coefficients of $f_{\texttt{COR}}(z)$, noted $\{p_\ell\}_{\ell\in[1-L:L-1]}$, are conjugate centro-symmetry, *ie.*, $p_\ell = p_{-\ell}^*$, $p_0 = \texttt{Tr}\left(\hat{P}_A^\perp P_B \hat{P}_A^\perp\right)^\dagger$ and therefor $(z_m, {z_m^*}^{-1})$ occur in pairs. Consequently, the Root-PMUSIC is based on the following result.

Theorem 1 *The S roots of the polynomial form*

$$\mathcal{C}(z) = f_{\texttt{CMUSIC}}(z) + f_{\texttt{COR}}(z)$$

are the DOA without the subset of the known DOA.

Proof: *As we know that $(z_m, z_m^{*\,-1})$ occur in pairs, we can give the factorized forms of polynomials $f_{\texttt{CMUSIC}}(z)$ and $f_{\texttt{COR}}(z)$ according to:*

$$f_{\texttt{CMUSIC}}(z) = \prod_{m=1}^{S}(z-z_m)\left(z-\frac{1}{z_m{}^*}\right)\prod_{m=S+1}^{M}(z-z_m)\left(z-\frac{1}{z_m{}^*}\right) \quad (22)$$

$$\prod_{m=1}^{L-M-1}(z-z'_m)\left(z-\frac{1}{z'_m{}^*}\right) \quad (23)$$

and

$$f_{\texttt{COR}}(z) = \prod_{m=1}^{S}(z-z_m)\left(z-\frac{1}{z_m{}^*}\right)\prod_{m=1}^{L-S-1}(z-z''_m)\left(z-\frac{1}{z''_m{}^*}\right) \quad (24)$$

where $\{z_m\}$ are the desired (known or unknown) DOA and $\{z'_m\}$ and $\{z''_m\}$ are the extraneous DOA. Based on expressions (22) and (24), $\mathcal{C}(z)$ admits the following factorization:

$$\mathcal{C}(z) = I(z)Q(z) \quad (25)$$

where

$$I(z) = \prod_{m=1}^{S}(z-z_m)\left(z-\frac{1}{z_m{}^*}\right)$$

$$Q(z) = \prod_{m=S+1}^{M}(z-z_m)\left(z-\frac{1}{z_m{}^*}\right)\prod_{m=1}^{L-M-1}(z-z'_m)\left(z-\frac{1}{z'_m{}^*}\right) + \prod_{m=1}^{L-S-1}(z-z''_m)\left(z-\frac{1}{z''_m{}^*}\right).$$

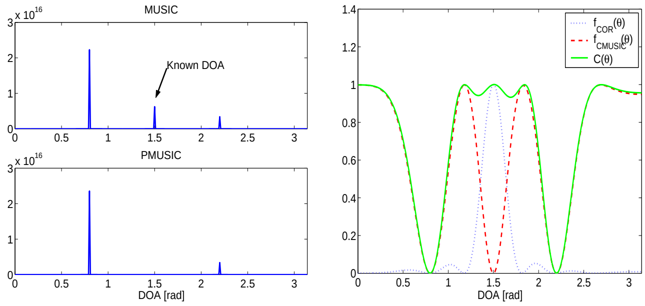
Clearly, $Q(z)$ has no trivial roots, ie.*, any known or unknown DOA are solution to $Q(z) = 0$. Inversely, we only have $I(z) = 0$ for the unknown DOA. So, according to expression (25), zeros of $\mathcal{C}(z)$ are only the DOA which annulate $I(z)$,* ie.*, the unknown DOA.* ■

6. NUMERICAL SIMULATIONS

6.1. Illustration

In this part, we consider a numerical example to illustrate the PMUSIC algorithm. On Fig. 1-a, we have drawn the pseudo-spectrums of the MUSIC and the PMUSIC algorithms for three DOA where one is known and two others have to be estimated. First, note on the PMUSIC pseudo-spectrum that the known DOA has been efficiently cancelled from the MUSIC pseudo-spectrum without altering the unknown one. In contrast to the classical MUSIC algorithm, we can note on Fig. 1-b that $\mathcal{C}(\theta)$ has only two null values at 0.8 and 2.2 rad.

On Fig. 2, we have drawn the zero location with respect to the unit circle for the root-MUSIC and root-PMUSIC algorithms. Note that the zeros occur in pairs, as expected. However in presence of noise, selecting the zeros (with unit modulus constraint) based only on $f_{\texttt{COR}}(z)$ is a difficult task due to their proximity to the unit circle. So, a decision only based on $f_{\texttt{COR}}(z)$ seems ineffective. Inversely, note that a decision on criterion $\mathcal{C}(z)$ is a more practicable task.



(a) (b)

Fig. 1. (a) MUSIC and Prior-MUSIC pseudo-spectrums for three AOA (one known and two unknown), (b) $f_{\texttt{CMUSIC}}(\theta)$, $f_{\texttt{COR}}(\theta)$ and $\mathcal{C}(\theta)$ for $L = 20$ sensors and $T = 100$ snapshots.

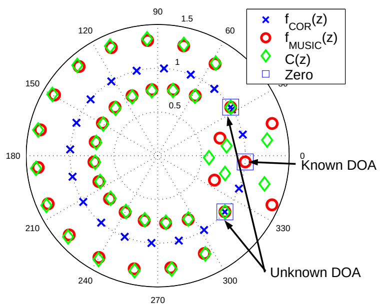


Fig. 2. Zero location with respect to the unit circle.

6.2. Performances

6.2.1. Accuracy of the proposed method

The context of these simulations is an Uniform and Linear Array. The performance of the proposed method is compared to the classical root-MUSIC [7] and to a "root" version of the CMUSIC algorithm presented in [4]. The accuracy of the estimation of the DOA of interest is measured thought the RRMSE (Root Relative Mean Square Error). Each simulation is based on 1000 Monte-Carlo trials. On Fig. 3-a, we consider well separeted DOA, *e.g*, $\theta = [0.8\ \ 1.7]$. In this situation, all the tested algorithms are equivalent. On Fig. 3-b, we choose closely spaced DOA, *e.g*, $\theta = [0.8\ \ 0.75]$. In this scenario, we can note that the root-PMUSIC algorithm shows a RRMSE close to the root-MUSIC algorithm for a single DOA and is more efficient than the root-MUSIC for two DOA. By the light of this example, we can say that most of the influence of the known DOA has been efficiently cancelled. This is not the case for the CMUSIC algorithm. Despite of the fact, that this algorithm performs slightly better than the root-MUSIC for two DOA, its accuracy is close to the two DOA root-MUSIC algorithm than the single one.

6.2.2. Robustness

For this example, the scenario is the same as for Fig. 3-b but we perturb the known DOA in the range $[\theta_2(1-\frac{20}{100}),\ \theta_2(1+\frac{20}{100})]$ and we compute the RRMSE of the DOA of interest. Fig. 4 shows

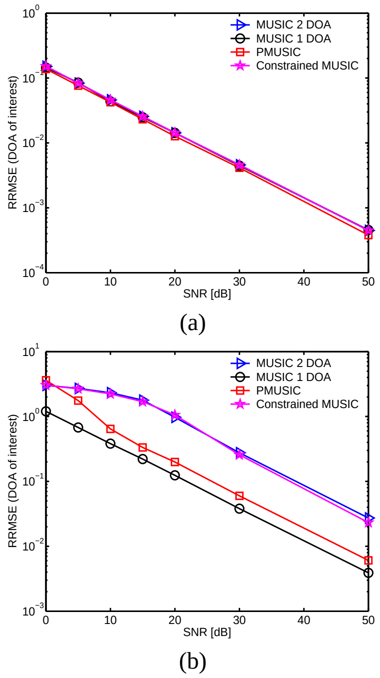


Fig. 3. RRMSE Vs. SNR, (a) Non-close spaced DOA with $L = 10$ sensors and $T = 10$ snapshots, (b) Close-spaced DOA with $L = 15$ sensors and $T = 100$ snapshots.

that for the CMUSIC algorithm, a small error on the known DOA affects drastically the estimation of the DOA of interest, in particular for high SNR. Inversely, we can note the remarkable robustness of the root-PMUSIC algorithm for several SNRs. This leads to think that a multi-stage version of the PMUSIC algorithm can be a valuable solution. This direction is the subject of current researches [6].

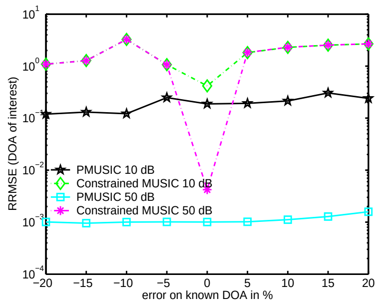


Fig. 4. RRMSE Vs. Error on the known DOA with $L = 15$ sensors and $T = 100$ snapshots.

7. CONCLUSION

In this work, we have presented a subspace-based solution to estimate S DOA among M using the knowledge of $M - S$ known DOA. Our solution is based on an oblique deflation of the signal subspace. We show that the proposed algorithm, called Prior-MUSIC, efficiently mitigates the influence of the known DOA on the DOA of interest in particular when the DOA are closely spaced. In addition, this algorithm is remarkably robust when an error corrupts the known DOA.

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